Advanced Operations Research

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Lecture - 33

Game Theory

In this lecture, we continue our discussion on Game Theory.

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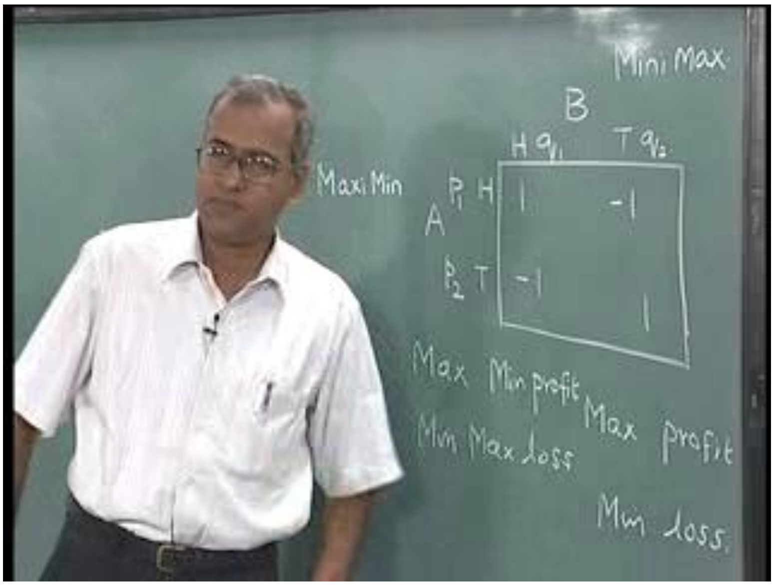

In the earlier lecture, we introduced the game theory problem using an example. Here we said the two people, A and B, are playing a game. The game is like, each person has a coin and the person can show either a head or a tail. If both of them show the same, which is, both of them show heads or both of them show tails, A gets 1 here. If it happens that there is either a head tail or tail head, it is a minus 1 for A, which means it is a 1 gain for B.

It is a zero sum game; because, whatever is A's gain is B's loss and whatever is A's loss is B's gain. This is called a two person zero sum game. We should also be aware that these two are strategies on which the person playing the game has control. In this particular type of a game, A can choose to show either a head or a tail; B can choose to show either a head or a tail. Now the question is when this being game is repeated several times and it is a zero sum game, which means some person's gain is the other person's loss and so on, at end of the game, there will be an expected profit for A or it can become a loss for A. Now, we are interested in finding out what this expected profit or loss for A is going to be knowing fully well, that the profit for A is loss for B and so on. Let us also try and simulate how A and B would play the game.

Let us assume that A decides to show a head and B also decides to show a head. In the first trial both of them will be showing a head and A will be winning. Let us assume that A decides to continue showing the head because A thinks that B will be trying to show a head. This happens for a few trials and then B understands that A is consistently playing this strategy, so B will shift to this strategy, so that B wins.

After a while, once A understands that B is consistently playing this, A will start playing this, so that A wins. Which means after a while B will go back to this, so that B wins. Then A will come to this and the game will proceed. The decision that both of them have to take is - what is the proportion of times A plays this strategy and what is the proportion of times A plays this strategy (Refer Slide Time: 03:15 min). Similarly, what is the proportion of times B plays this and B plays this. We call these proportions also as probabilities. A plays this strategy with probability p_1 and here probability p_2 or p_1 proportion of times plays this and p_2 proportion of times plays this. Similarly, for B, we call it as q_1 and q_2.

What is the objective function? In fact, one of the things we did was we said that if A is playing here, assuming that B will play this, which means A will try to maximise A's profit. A will always try to play in such a manner that A is trying to maximise A's profit and B who also wants to win, would obviously like to minimise B's loss, which means B maximises B's profit; because, the pay off matrix is written for A, we would say that A would like to maximize A's profit and B would try to minimise his or her loss.

When A plays this consistently thinking that B will play this, A is going to get this 1. Immediately after a while, B will switch because B would not like to lose and B will always try to minimise A's gain, while A would like to maximise A's gain. So, B is going to allow A to get minimum gain, which A will try to maximise. A is fully aware that B is an intelligent player and B is not going to allow A to get more and more money. A's objective will be to maximise the minimum profit, while B's objective will be to minimise the maximum loss, because, B knows that A is an intelligent player and given a chance, A will try to maximise his own profit, which means A will try to maximise B's loss. So B will play in such a manner that B will try to minimise B's loss.

A's strategy is called a Maxi Min strategy, which means A will try to maximise the minimum profit that B will allow A to get. B's strategy is called Mini Max strategy, where B will minimise the maximum loss that A is trying to inflict on B. And assuming that both of them are intelligent players and it is a two persons zero sum game, the objective shifts to what is called a Maxi Min profit for A and Mini Max loss for B. How do we solve? So the problem now becomes what is a proportion of times A will play this and this p_1 and p_2 such that A maximizes the minimum profit? What is the proportion of times B will play this, such that B will minimize the maximum loss? We have to find out this p_1 and p_2.

We have also said that both A and B are intelligent players. Therefore, there can be situations where they need not play a particular strategy. It is not necessary that this p_1 and p_2, both should be numbers strictly between 0 and 1. There could be a situation where somebody will play only this and the other person may play only this. We will now see an example where such a thing is possible and learn a little more about these proportions or probabilities.

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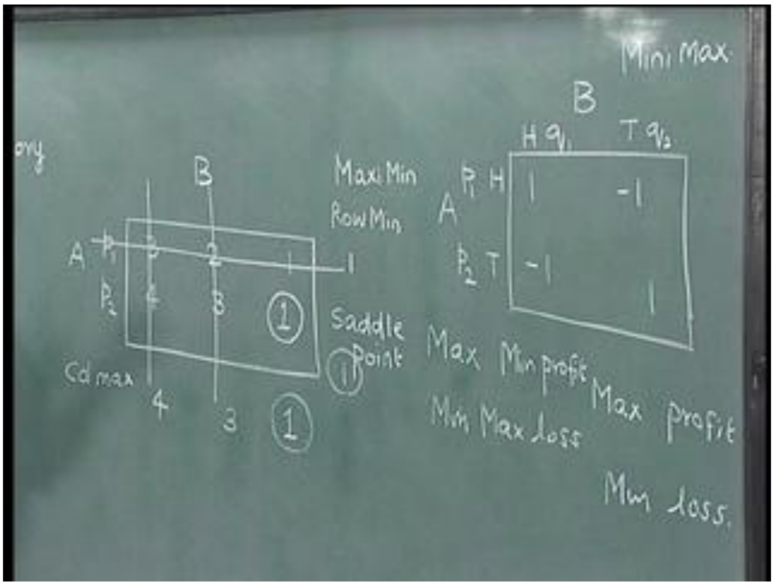

Let us look at another game where this is A, in the usual notation, this is B. This is the pay off matrix for A. A has 2 strategies and B has 3 strategies. It is not necessary that the number of strategies will have to be equal. Now, if we look at this, let us assume we are interested in saying A wishes to find out what is the proportion of times A will play this and A will play this, such that A will maximise the minimum profit. If you look at these numbers, see 3 is less than 4, 2 is less than 3, minus 1 is less than 1. A will look at these rows or these strategies and now A will understand that no matter what B plays, let us say B plays this, then A, between the two will only play this, because this gives more profit (Refer Slide Time: 08:21 min). Similarly, if B plays this A will again play this. If B plays this A will again play this. Therefore, no matter what proportion of times B plays the 3 strategies, A will not choose to play this at all, because by playing this, A will get more money than by playing this. This strategy will dominate the earlier strategy and therefore, A will leave out this strategy. A will continue to play always this second strategy with proportion p_2 equal to 1 and p_1 equal to 0.

The moment that A decides to play only this, we have also said that B is an equally intelligent person and therefore, B would say that if A is going to play this all the time, I will not play this, I will not play this (Refer Slide Time: 09:15 min) because A's profit is higher, which means my loss is higher. So B will choose to play only this strategy; therefore, this is cut down, this is cut down and we have only one that is active, which is this. As far as A is concerned, a row will dominate another if every element of the second row is greater than or equal to the corresponding element of the first row. Then, the second row will dominate the first and the first will go. As far as B is concerned, if there are two columns, such that every element of a column is less than or equal to every other element of another column, the column with a bigger element will go and will be dominated by the column with the smaller element. Therefore, given a pair of matrix, the first thing we have to do is to try and use the dominance conditions and try to reduce the size of the problem. In this example, the size of the problem finally becomes only 1, which means that in this game, A will play this strategy always (Refer Slide Time: 10:29 min); B will play this strategy always. The pay off for A is 1 and the pay off for B is minus 1. This is 1, this is called a saddle point, which means that at the end of applying all the dominance conditions, there is only one point which both A and B freeze and that is called the saddle point.

There is another way of identifying whether there is a saddle point or not. That is done as follows. Here, always find the column max because A would like to see what the column max for every column is. That is 4, 3 and 1. Here, you see the row minimum - row min. This happens to be 1 and 1. Now, you find the minimum of the column max is equal to the maximum of the row min, then it has a saddle point. That is a simple way of identifying whether this matrix has a saddle point or not. Once we know that this coincides, then this is a saddle point.

The other way to do is to apply all the dominance conditions and see at the end that there is a saddle point. If a game is given, then the first thing is to verify whether it has a single saddle point, by applying the dominance conditions. If it does not have a single saddle point, then the problem shifts to finding out the proportion of times A would play this and proportion of times B would play this and so on. In fact, if we apply the dominance condition here (Refer Slide Time: 12:30 min), we realize that we do not have a single saddle point because column maximum are 1 and 1, row minimum are minus 1 and minus 1. And minimum of the column max and maximum of the row min do not coincide. Therefore, there is no saddle point in this. Also, if there is no saddle point, then the game is at least a 2 by 2 game. It can be 2 by 2, it can be a 2 by n, it can be a m by 2, it can be a m by n. First, let us consider a 2 by 2 and then try to find out how we solve to get the proportions of the times A and B play these games. Let us look at the other example.

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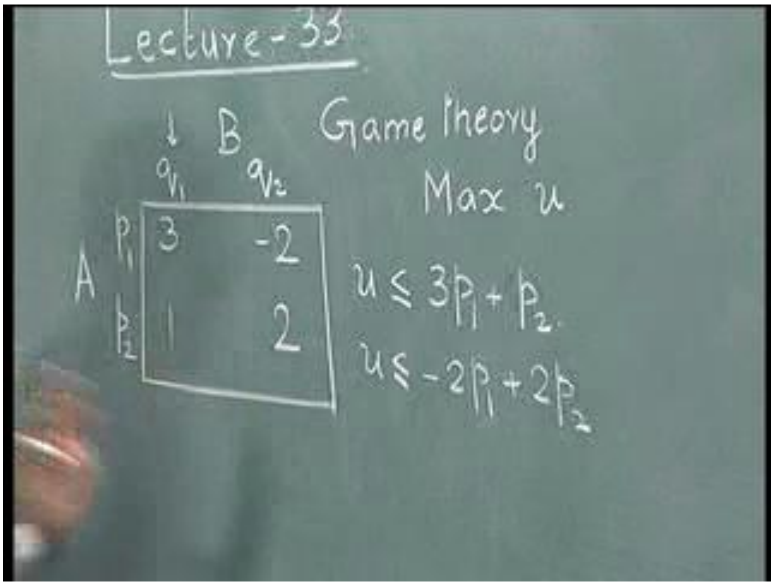

Let us look at this particular example of a 2 by 2. A has these two strategies; B has these two strategies (Refer Slide Time: 13:28 min). Therefore, what they will do is A would like to find out the number of times that A should play this strategy and the proportion of times A should play this and B will look at the proportion of times B will play these strategies. We also said that A will follow a Maxi Min strategy and B would follow a Mini Max strategy. What A would do is if I play this strategy p_1 proportion of times and this p_2 proportion of times and if B consistently plays this strategy, then A's gain will be $3p_1$ plus p_2; because, B consistently plays this, so A will get $3p_1$ plus p_2.

If B consistently plays the second strategy, A will get minus $2p_1$ plus $2p_2$. A knows that B is an intelligent player and therefore, A would like to maximise the minimum that A will get. The minimum that A will get, because it is a linear, will lie in either this or this, or the part of intersection of these two; because, this is a linear system, A would like to maximise the minimum of this and that can be given as A would like to maximise some u, such that u is less than or equal to this and u is less than or equal to this.

Now, u being less than equal to this, u being less than equal to this (Refer Slide Time: 15:13 min), would mean u is the minimum of these two functions and A would like to maximise that u. At the moment we are not going to expand this idea further, but at the moment we would like to say that if it does not have a saddle point, then the maximum actually occurs when we solve these two to get - $3p_1$ plus p_2 is equal to minus $2p_1$ plus $2p_2$; this would give us $5p_1$ is equal to p_2.

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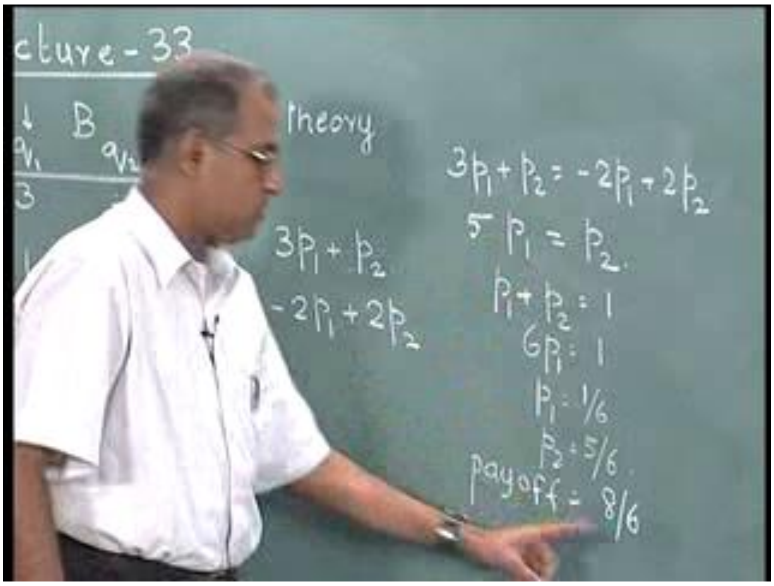

We also know that p_1 plus p_2 is equal to 1, which would give us $6p_1$ equal to 1; p_1 is equal to 1 by 6, p_2 is equal to 5 by 6. A's pay off is equal to either $3p_1$ plus p_2 or minus $2p_1$ plus $2p_2$; so $3p_1$ plus p_2 will be 8 by 6 or 10 minus 2. We get the same 8 by 6.

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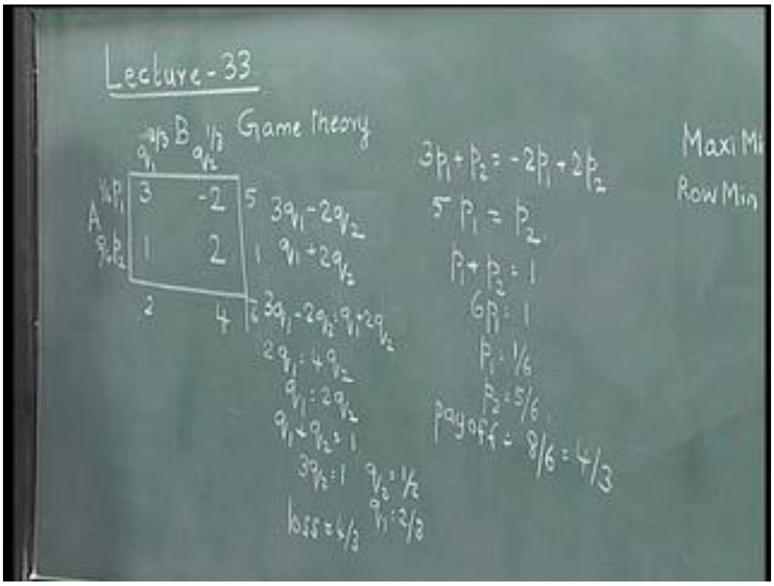

Now, if we try to do this for B, then what would B do? B would say that if I play my strategies q_1 and q_2 proportion of times and if A plays this consistently, then B's loss will be $3q_1$ minus $2q_2$. For this one, it will be q_1 plus $2q_2$. If there is no saddle point, then we can actually show that when we can equate these two, we can get the values of q_1 and q_2. We get $3q_1$ minus $2q_2$ is equal to q_1 plus $2q_2$. This would give us $2q_1$ is equal to $4q_2$ or q_1 is equal to

$2q_2$. Now q_1 and q_2 being proportions and probabilities, q_1 plus q_2 is equal to 1. So $3q_2$ is equal to 1; q_2 is equal to 1 by 3; q_1 is equal to 2 by 3.

B's loss or pay off will be q_1 plus $2q_2$ or $3q_1$ minus $2q_2$. So q_1 plus $2q_2$ is 2 by 3 plus 2 by 3, which is 4 by 3 or $3q_1$ is 6 by 3 minus 2 by 3, which is 4 by 3. This 4 by 3 is the same as 8 by 6. At the end of the game, we realise that A's expected profit will be 4 by 3; B's expected loss will be 4 by 3. Then A will play these two in the proportion 1 by 6, 5 by 6 and B will play these in the proportion 2 by 3 and 1 by 3. The pay off for this game is 8 by 6 or 4 by 3. This is how we solve a 2 by 2 game, if there is no saddle point.

There is a simple short cut to do this, which is nothing but a very simple way of solving these equations. One of which is find the absolute difference here between them. So, we get 5 and here the difference is 1, here the difference is 2 and here the difference is 4; so this adds up to 6. Now we realise that A will do 1 by 6; B will do 5 by 6. Here p_1 is 1 by 6, p_2 is 5 by 6, q_1 is 4 by 6 or 2 by 3, q_2 is 2 by 6 or 1 by 3. What you find here is the value corresponding to p_2 and what you find here is the value corresponding to q_2. This is the simple short cut to do exactly this. This is based on the same principle that you equate these two and then put q_1 plus q_2 equal to 1 or equate this and put p_1 plus p_2 equal to 1. We could either use this extended way of solving these equations or simply do a numerical short cut, by which we can get the proportion of times A plays p_1, p_2 and B plays q_1, q_2.

The next thing that we will do is we will look at a problem which does not have a saddle point; at the same time has more than 2 strategies for A or B.

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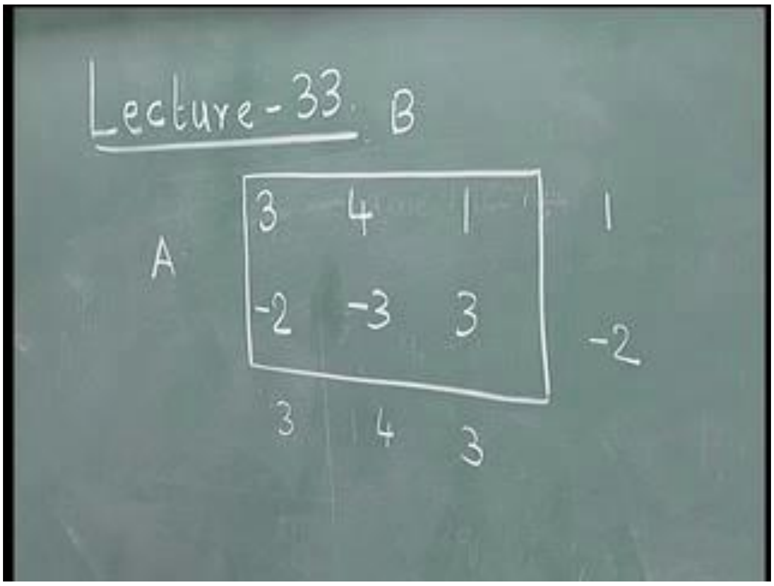

Let us look at another example which is like this. Look at the game where A has 2 strategies and B has 3; so, 3 and minus 2, 4 and minus 3, 1 and 3. Let us look at this game and try to see what happens.

We can first verify whether there is a saddle point or not. To verify whether we have a saddle point or not, we first look at the column maximum for this, which happens to be 3, 4, 3. Look at a row minimum 1 and minus 2. Minimum of this is 3; maximum of this is 1; so there is no saddle point. Another way of looking at it is also to see whether there is any dominant strategy.

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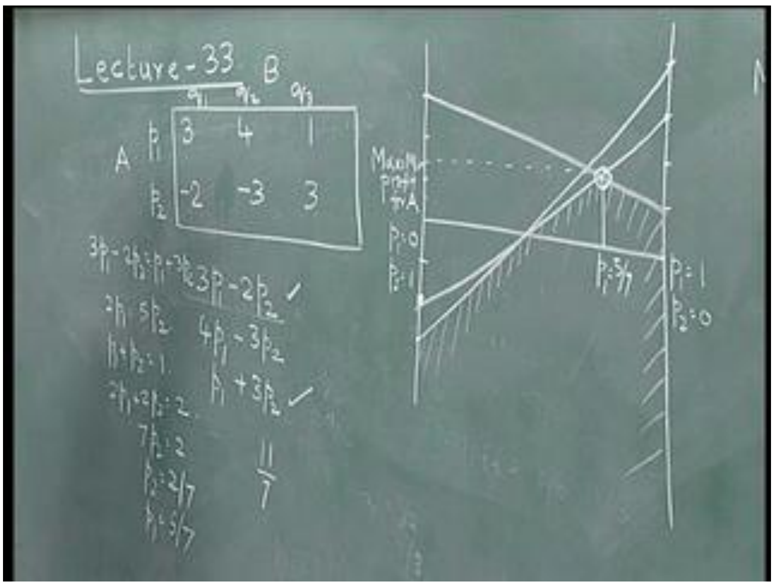

3 is greater than this, this is greater than this but this is less than this (Refer Slide Time: 21:28 min). Therefore, we do not have a dominant strategy for the rows. Again, for the columns: 4 is bigger than 3, minus 3 is smaller than minus 2, so we do not have; 4 is bigger than 1, minus 3 is smaller than 3, so we do not have; 3 is bigger than 1, but minus 2 is smaller than 3, therefore, you do not have. The resultant thing is a 2 by 3 matrix, where A has 2 options, B has 3 options.

A would like to find out p_1, p_2 and B would like to find out q_1, q_2 and q_3. We have already seen some kind of relationship between this problem and the linear programming problem, even though formally we will introduce the relationship a little later. We have to solve a problem that involves two variables, if you are solving for A. So in any two variable linear programming problems, you can follow or use the graphical method. We will try to use the graphical method and try to get a solution; so that is done by this.

This is A, this is p_1 equal to 0 and this is p_1 equal to 1, which is p_2 equal to 0 and p_2 equal to 1. A would say that if B is consistently playing this, then A's gain is $3p_1$ minus $2p_2$. This is $4p_1$ minus $3p_2$ and p_1 plus $3p_2$. Let us try and plot all this.

Let us try and plot $3p_1$ minus $2p_2$. When p_1 is equal to 0, A's gain is minus 2. When p_1 is equal to 0, this is the point for A. When p_1 is equal to 1, A's gain is 3. This is the point that relates A's gain and so this is given by the line that joins these two points. $4p_1$ minus $3p_2$, when p_1 is equal to 0, you get minus 3. When p_1 is equal to 1, you get 4. We now need to join these two lines. For the third one, when p_1 is equal to 0, we get plus 3 with 1, 2 and there is a plus 3. When p_1 is equal to 1 and p_2 is equal to 0, we get a plus 1; so plus 1 is here, so we get this (Refer Slide Time: 24:54 min). A's objective is a Maxi Min strategy; so to maximise the minimum profit that A will get. This is the profit line and anything below is the profit that A will get. Therefore A's profit line which includes all the constraints is actually given by something like this and the Maxi Min happens to be somewhere here.

If we have drawn this on the correct graph sheet, then this point will tell what is the value of p_1 at which A can get the maximum profit, but we have not drawn this to scale. So, the easier way of trying to find out p_1 is actually the solution of these two equations. Those two equations that we should be solving are $3p_1$ minus $2p_2$ and p_1 plus $3p_2$.

To get the value of that p_1, we will get $3p_1$ minus $2p_2$ is equal to p_1 plus $3p_2$. This would give us $2p_1$ is equal to $5p_2$. We know that p_1 plus p_2 is equal to 1. So $2p_1$ plus $2p_2$ is equal to 2, $2p_1$

is equal to $5p_2$, so $7p_2$ is equal to 2, p_2 is equal to 2 by 7, p_1 is equal to 5 by 7, which is this point (Refer Slide Time: 26:52 min). This will be p_1 equal to 5 by 7, which is p_2 equal to 2 by 7. The trade off will be p_1 plus $3p_2$ which is 5 by 7 plus 6 by 7, which is 11 by 7. This will be 11 by 7, which is the Maxi Min profit or pay off for A.

In this instance, it turns out that the Maxi Min pay off or profit for A is positive. It is not absolutely necessary that pay off for A should always be positive. Pay off for A can be negative, which means pay off B can be positive. Once we have found this out, we can find out the values of q_1, q_2, q_3 for B. While we can do this for A getting p_1 and p_2, in order to get it for B it will be a little more difficult to get it from this graph. Then we may have to go to the linear programming formulation based on which we can actually get the values for q_1, q_2 and q_3. Now, we generalize this for a general m into n and then we come back to this example to try and get the values for that of q_1, q_2 and q_3. Let us go back and try to formulate this as a linear programming problem.

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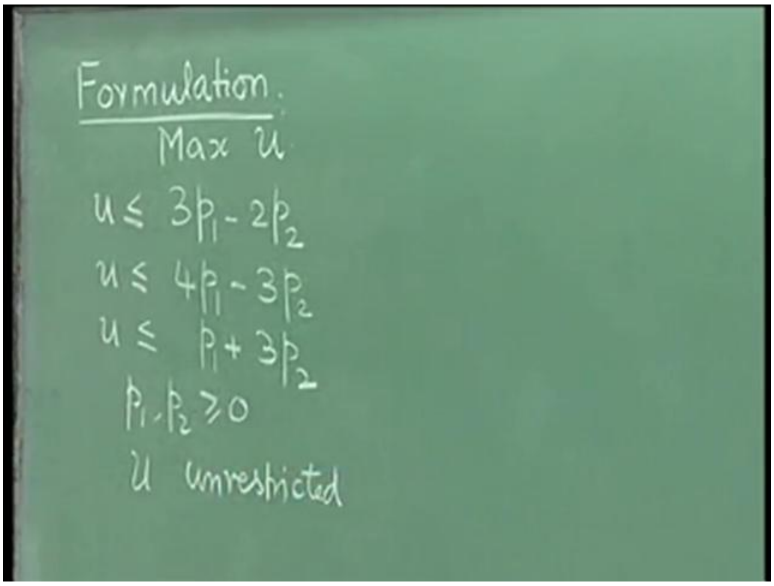

We can actually formulate it for a general m into n or 3 into 4 and so on. But let us do it for this example, so that we can come back and show the solution for q as well from the same example. We said that A would like to find out what p_1 and p_2 are. We also said these are the three profits for A. A's three profits are $3p_1$ minus $2p_2$, $4p_1$ minus $3p_2$ and p_1 plus $3p_2$.

We also said that A will follow a Maxi Min strategy. Let the minimum of these 3 profits be u. A will maximise u, such that, u is less than or equal to this. p_1, p_2 are greater than are equal to

0 and u is unrestricted in sign. Unrestricted in sign comes because at the moment we do not know, at the end of the game whether A will make a profit or A will make a loss. u can be positive, 0 or negative.

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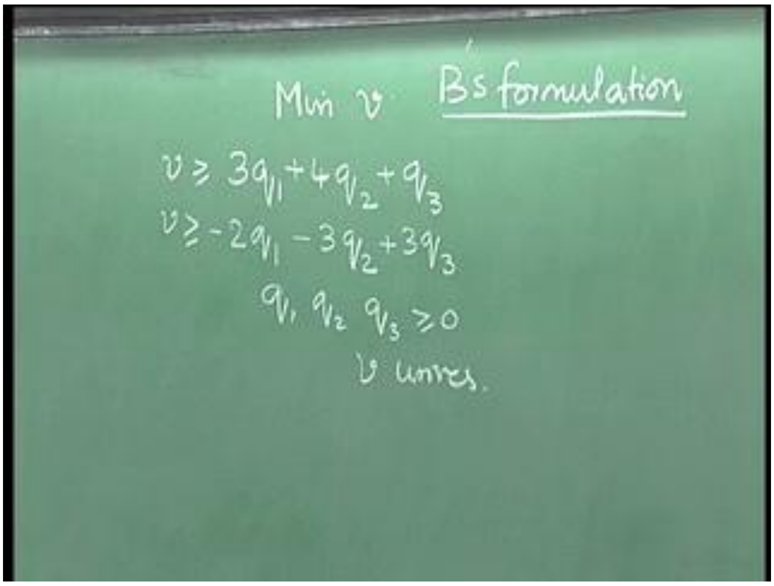

Let us also write B’s game somewhere else. B would say that if B plays this consistently (Refer Slide Time: 30:13 min), and if B wants to play this with proportions q_1, q_2 and q_3 and if A plays this consistently, B’s loss will be $3q_1$ plus $4q_2$ plus q_3. For the other one it is minus $2q_1$ minus $3q_2$ plus $3q_3$. This is B’s two losses.

B will do a Mini Max strategy. So B will try to minimise the maximum loss that A will inflict on B. B will try and do a minimise v, such that v is greater than or equal to this, and v is greater than or equal to this (Refer Slide Time: 31:14 min). Minimise, the maximum of these two. So q_1, q_2, q_3 is greater than or equal to 0, v is unrestricted in sign; now this is B’s formulation. Let us go back, this is A's formulation.

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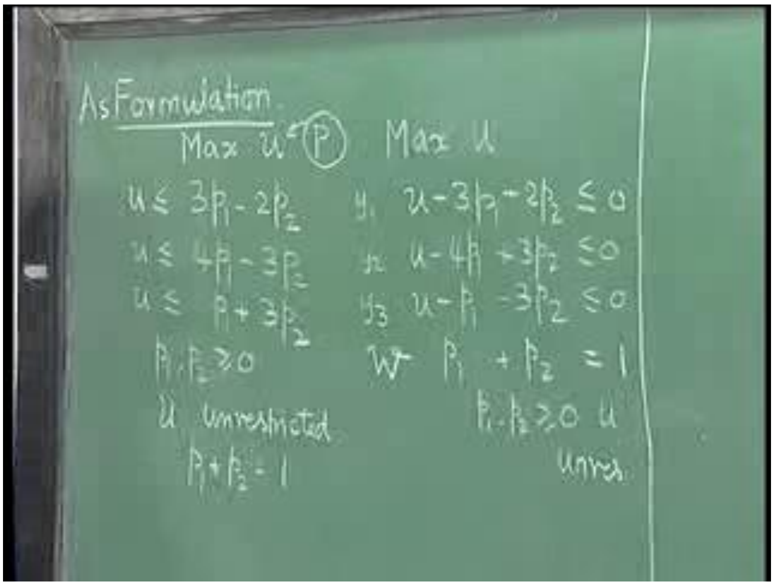

Let us go back to this problem and then try to find out the dual of A's problem. We call this as the primal. In order to write the dual of A's formulation, we need to first rewrite this primal. We also have to introduce here that p_1 and p_2 also have a relationship that p_1 plus p_2 is equal to 1. Similarly, q_1 plus q_2 plus q_3 is equal to 1. Here the primal has 4 constraints, so the dual will have 4 variables. Let us call these 4… first we have to write this in the standard form. So we are rewriting the primal as maximise u. It is desirable to have a problem less than or equal to constraints. We would have u minus $3p_1$ plus $2p_2$ is less than or equal to 0; u minus $4p_1$ plus $3p_2$ is less than or equal to 0; u minus p_1 minus $3p_2$ is less than or equal to 0; p_1 plus p_2 is equal to 1; p_1, p_2 greater than or equal to 0 and u is unrestricted. Let us write the dual to this problem. Let us define dual variables y_1, y_2, y_3, for these 3 constraints and let us define another dual variable w for this.

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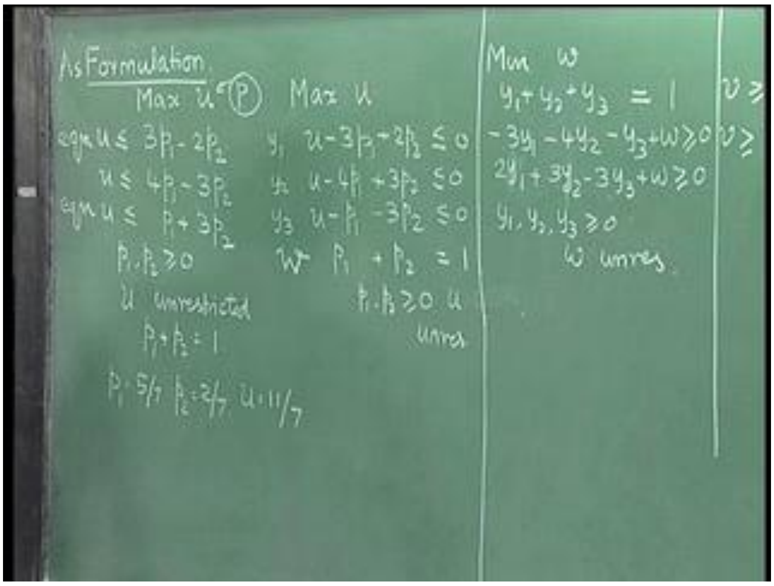

Now, the dual will be to minimise w because, 0 y_1 plus 0 y_2 plus 0 y_3 plus 1 into w. The first constraint will be y_1 plus y_2 plus y_3, corresponding to the variable u. The right hand side is 1 because u is unrestricted, this will be an equation.

Corresponding to this (Refer Slide Time: 34:44 min) it is minus $3y_1$ minus $4y_2$ minus y_3 plus w for p_1. It is a maximization problem with greater than or equal to variable; it is a minimization problem with greater than or equal to constraint. Then we will get for this $2y_1$ plus $3y_2$ minus $3y_3$ plus w is greater than or equal to 0. We have y_1, y_2, y_3 greater than or equal to 0 and we have w unrestricted. Now, we can rewrite this as minimize w, y_1 plus y_2 plus y_3 is equal to 1, w greater than or equal to $3y_1$ plus $4y_2$ plus y_3, w greater than equal to minus $2y_1$ minus $3y_2$ plus $3y_3$. y_1, y_2, y_3 greater than or equal to 0, w unrestricted.

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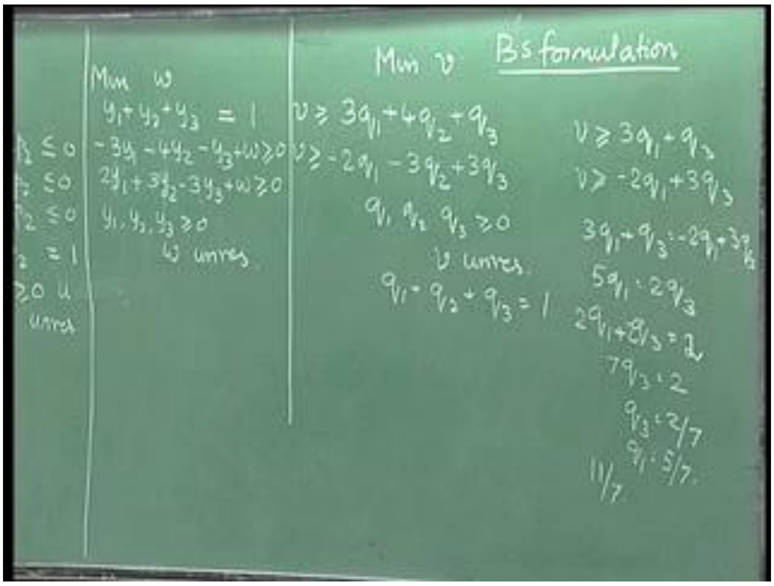

And then if we compare this with this (Refer Slide Time: 36:12 min), we understand the dual of A's problem is actually the B's problem, where y_1, y_2, y_3 or the q_1, q_2, q_3 and w becomes v. There is a primal-dual relationship between A's problem and B's problem. If we solve one of them, we can apply the duality principle to solve the other one. In this case we had 2 strategies for A and 3 for B. Any 2 into m or n into 2 can be solved using the graphical to get it for one of them, whichever person has 2 strategies; the other can be obtained by using the duality principle. If it is a general m into n, then we can formulate the linear programming problem for A or B, solve it and then using principles of dual to get the other. If we are solving it by the simplex method, it would automatically give the solution to the primal as well as to the dual. Here, because we have solved it by the graphical method, we will now come back and use the duality principles to get the quantities for B, which means to get q_1, q_2 and q_3.

To do that, we now realise that for the solution, p_1 equal to 5 by 7, p_2 is equal to 2 by 7. We have a solution p_1 is equal to 5 by 7; p_2 is equal to 2 by 7. Let us go back and substitute this. The first constraint will be and we have the solution u is equal to 11 by 7. So $3p_1$ minus $2p_2$ is solved as an equation; this is solved as an equation. $4p_1$ minus $3p_2$ is 20 minus 6, 14 by 7, is not solved as an equation and p_1 plus $3p_2$ is solved as an equation equal to 11 by 7.

These two are solved as equations, which means, the corresponding variables will be the basic variables. The one corresponding to this is q_1. The one corresponding to this is q_3. We go back and use the equation for q_1 and q_3. q_1 and q_3 will be in the solution. From this we get v is greater than or equal to $3q_1$ plus q_3 and v is greater than or equal to minus $2q_1$ plus $3q_3$. So we solve for these two. $3q_1$ plus q_3 is equal to minus $2q_1$ plus $3q_3$. This would give us $5q_1$ is equal to $2q_3$. We also know that q_1 plus q_3 is equal to 1. This comes because q_2 is non-basic. Therefore, q_2 is 0, q_2 has been left out. We have $2q_1$ plus $2q_3$ is equal to 2. So $7q_3$ is equal to 2; so q_3 is equal to 2 by 7; q_1 is equal to 5 by 7. This would give us a solution $3q_1$ plus q_3 is 6 by 7 plus 5 by 7, which is 11 by 7. $3q_3$ minus $2q_1$ is 15 minus 4, which is 11 by 7. We get the same 11 by 7 here. It is only incidental that these two values are 5 by 7 and 2 by 7. It turns out for B that q_1 is 5 by 7, this is 0 and this is 2 by 7. B will be playing these two strategies. B will not play this strategy. B will be playing these two strategies at proportions 5 by 7 and 2 by 7 respectively (Refer Slide Time: 40:40). This is how we solve the game theory problem. We looked at what the game theory problem was, we introduced the problem; we introduced the idea of Maxi Min and Mini Max.

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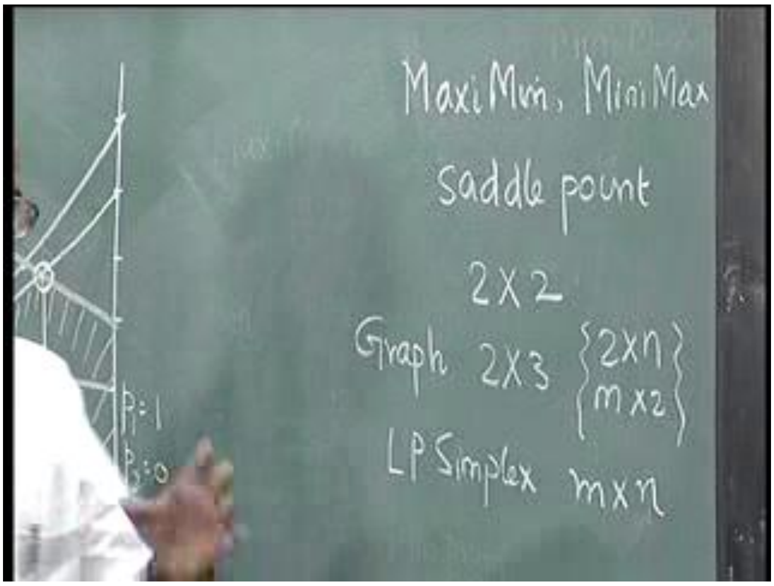

Then we said whether if the game has a saddle point, then we should able to find out the saddle point. If the game does not have the saddle point, we solved a 2 by 2 and then we solved a 2 by 3 or in general a 2 by n or an m by 2 using a graphical method. We also said that we could solve using the simplex or using linear programming simplex if we have a general m into n. We also showed that if we take A's problem and B's problem, B's problem is actually the dual of A's problem.

With this, we come to the end of our discussion on game theory. Game theory, we wish to show in this lecture series, as a good enough application of linear programming, one of the many applications of linear programming. Game theory has its own theories and algorithms. In addition, the purpose here is to show how tools of linear programming can be used to solve game theory problems.

We are now going to look at another application of linear programming, which is called the critical path method.

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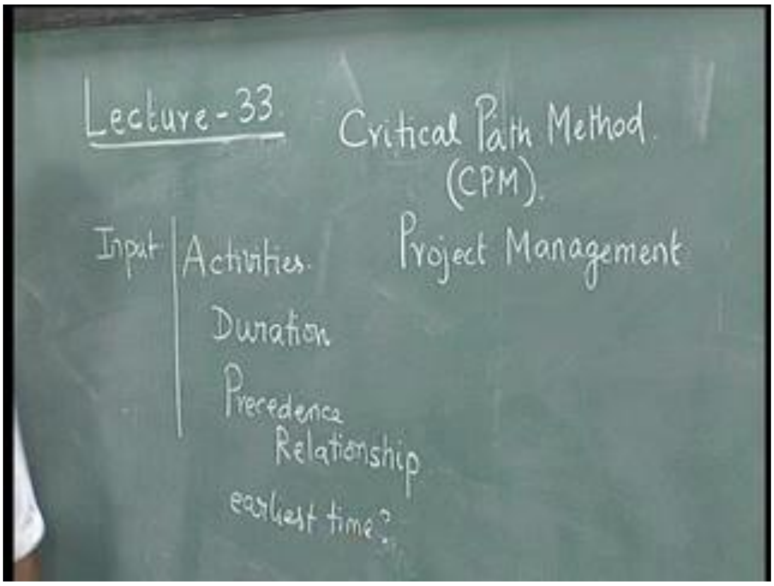

It is also called CPM. The critical path method is used in project management. Here, we wish to show critical path method once again as an application of linear programming towards real life kind of problems. The critical path method talks about, if there is project that has to be executed, how quickly this project can be executed. How fast or what is the earliest time in which this project can be completed? Every project that is to be executed will be defined in terms of certain activities that constitute the project.

The project will have a certain set of activities and each activity has a certain duration, which is the time taken to perform that activity, which could be in days or which could be in hours; it is a time unit. In the critical path method, we are going to assume that these durations are known and deterministic. Then there could be a precedence relationship among these activities. Precedence relationship among these activities which would say that unless a certain activity is finished this activity cannot be taken.

If we take a simple example in construction and say if somebody is building a house, a simple precedence relationship would be, that the house walls are built only after the pillars are raised or the pillars are raised after the foundation is made and so on. There is a precedence relationship between this. So these three are the inputs to the critical path method and the output is when earliest or what is the earliest time in which the project can be completed. This is the output that is expected from the critical path method.

We take a simple example to illustrate the activity, duration and the precedence and we try to solve this using the critical path method. Then we show the relationship between the critical path method, the algorithm to solve the critical path problem and a network model that we have already seen in this lecture series. We also show a linear programming relationship between the critical path algorithm and the linear programming methodology.

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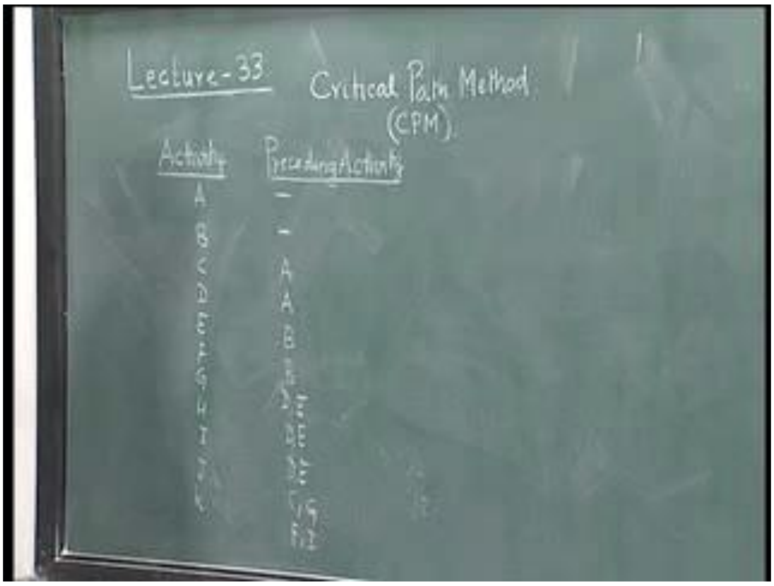

Let us take a simple example where we define a project in terms of Activity and Preceding Activity. Let us say this project is made up of several activities, say 11 activities. These activities are called: A, B, C, D, E, F, G, H, I, J and K. Let us say the precedence relationships are as follows. A does not have any precedence; B need not have any precedence; C and D have A as a preceding activity, which means activity C can start only when activity A is completed; same with D; E and F have B as the preceding activity; G has D and E; H has D and E; I has D and E; J has C and G; and K has F and I.

When we have these kinds of relationships, it is customary to draw a network that captures these relationships among the activities. Let us first draw a network representation of this data. Now, we are going to define an activity and an event. First let us draw this network and then we will start defining.

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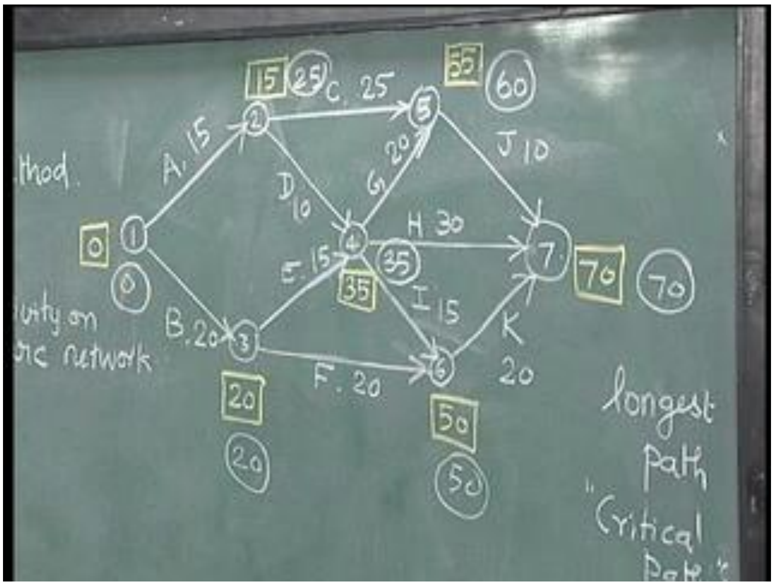

Let us first assume that we start at some 1 and activities A and B do not have any precedence; therefore, we say this is activity A and this is activity B. We number this as 2 and 3. Now we say that C and D can start only after A is completed. We are going to do C and D. This is called C, this is called D. This point indicates that A is completed. This is an event which says the ending of A. This is an event which could be the beginning of A. Here A is completed. C and D, I will call this as 3 and this as 4. I will call this as 5. Now E and F with B, I am going to call this as E and this as F with B. For G, I need both D and E to be completed. This would now indicate the completion of both D and E. Therefore, I would do G and H. G goes like this, H goes like this. I also has D and E to be completed; so I goes like this. For J, it is C and G; so this is C and G. So J comes here because C and G are over here. For K it is F and I. This is K with F and I and finally, this is where the project ends. I call this as 6 and I call this as 7.

This captures the relationship between these activities and the preceding activities. I have to give durations for this. Let us say these durations are given here: A is 15, B is 20, C is 25, D is 10, E is 15, F is 20, G is 20, H is 30, I is 15, J is 10 and K is 20. Now, we have captured the activities, precedence and the durations on this network.

Now, the first stage in any of this CPM is to actually draw the network and we have shown how to draw the network using a simple example. Each of these activities are now shown as the arcs of this network. This type of network is called activity on arc network. We could also draw something called activity on node network, where the activities are shown as nodes. Here, events are shown as nodes; an event essentially indicates the beginning or end of either an activity or a set of activities. We could have activities on arc network. We could have activity on node network.

In this lecture series, we would consistently have activity on arc network. We also need to put arrows here because, this is A, B would start somewhere and end with this. This signifies the end of B, after which E can begin with and so on. You have arrows. It is a directed network, which is like this. Now, we need to find out what is the earliest time this project will be complete. What we are going to do is we are going to have a labeling algorithm on this network. Let me first complete the algorithm and then try and explain to you that there are some things that are familiar, that we have actually done using a network of this type.

Let us now start what is called the critical path method. Let us assume that we start here at time equal to 0 (Refer Slide Time: 52:24 min). Now this is complete. This takes 15 time units; so this will be over at time equal to 15. This takes 20; so this will be over at time equal to 20. When we come to this 4, 4 earliest can be completed as 15 plus 10, 25. Here it is 20 plus 15, 35. Now, here 4 represents an event, where actually both these are completed, so that G, H and I can begin. It is not about when earliest can I reach 4, it is more about when latest I will reach 4 such that latest in the sense to minimise that late quantity. When this 4 indicates that the event, which symbolizes the completion of both D and E, then the label that we put here should be the maximum of 15 plus 10, 20 plus 15, so we get 35 here.

We take the maximum because 4, here, indicates the point where both D and E are over, such that G or H or I can begin. So with 4, we put 35 here. Similarly, for 5, this indicates the point where C and G are completed. It is 15 plus 25 is 40; 35 plus 20, 55. So earliest C and G can be completed is the maximum of 40 and 55. So we get 55. Similarly, this indicates I and F completed. It is a maximum of 40 and 50, which is 50. This represents the point at which J, H and I are completed, which is the maximum of 65, 65 and 70; so 70 comes here. 70 is the time where this project will be completed and 70 represents the earliest completion time of this project.

What we have done is we have used the labeling algorithm that essentially finds the longest path of the network, which is also called the critical path of the network. A general problem of finding the longest path in a network is a hard problem, but network with these kind of structures, where there are always arcs from I to J, J greater than I, certain topological arrangement of the arcs, then the problem of solving the longest path is actually easy and it is a modification of trying to solve the shortest path on the network. We had actually seen the shortest path on the network using the Dijkstra's algorithm, where we actually followed a very similar labeling procedure, except that for every label we found out the minimum of the times to reach that point. In the critical path method, we would always label it as the maximum of the times to reach because each node represents an event which is a completion of all the activities that terminate in that node.

This is how the longest path is computed. We can actually do a backward pass as well. We can start with a 0 here (Refer Slide Time: 56:04 min). This will be 0 plus 10, 10. We could use a circle here to show a different representation. You could use a 0 here. You could use a 10 here. This would become 20 here. This is 20 plus 15, 35. 0, 30; 10 plus 20, 30; so you set the maximum which is 35. 10 plus 25, 35; 35 plus 10, 45; here, it is 20 plus 20, 40. 20 plus 15, we have 35.

We could do a backward pass, which goes like this. This represents the forward pass. We start with a backward pass of 70. This becomes 60. This becomes 50, 70 minus 20 is 50. This is 50 minus 15, 35; 70 minus 30, 40; 60 minus 20 is 40. The smaller one comes here as 35. This is 60 minus 25 which is 35; 35 minus 10, we get 25; this is 50 minus 20, 30; 35 minus 15 is 20; here we get 25 minus 15 is 10; 20 minus 20 is 0.

When we do the backward pass, we start with this 70 and then move backwards till we get this 0. We will see more about the forward pass and the backward pass in the next lecture.