Advanced Operations Research

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Lecture No - 31

Single Server Queueing Models

In this lecture, we continue our discussion on Queueing Models.

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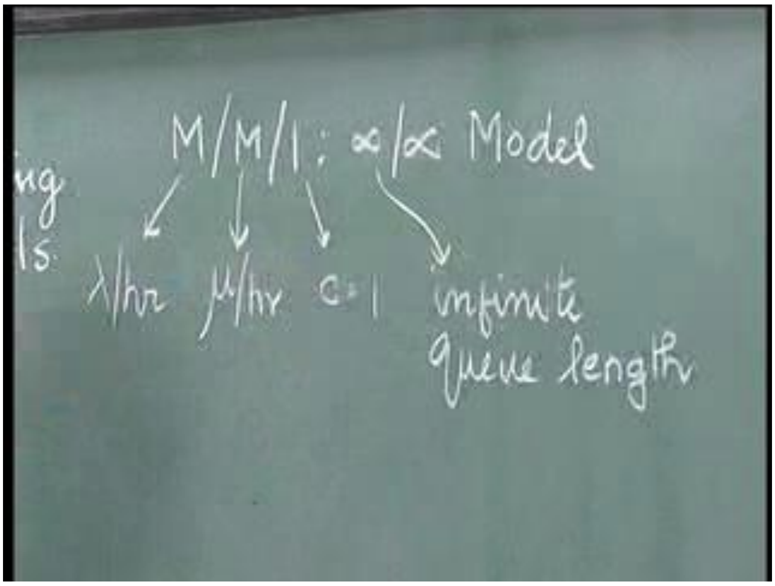

In the previous lecture, we derived some expressions for the M/M/1: infinity infinity model, where this M (Refer Slide Time: 00:36 min) stands for Poisson arrivals with lambda per hour. This M stands for exponential service with mu per hour. This is the number of servers - C equal to 1. This is infinite queue length and this is infinite population.

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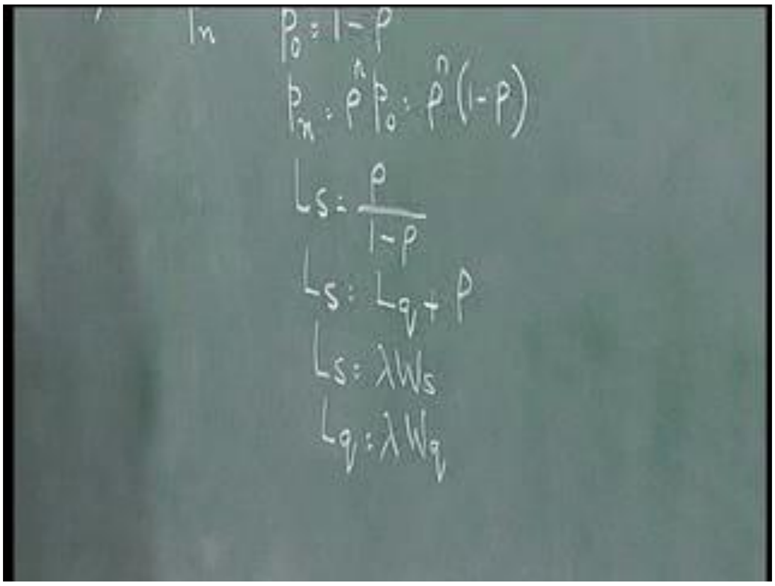

We also used the notation where p_n is the probability that at steady state there are n people in the system. We derived p_0 is equal to 1 minus rho, where rho is equal to lambda by mu. We also indicated that lambda by mu should be less than 1; so, rho should be less than 1, particularly when we have infinite queue length.

So, the most important equation that we derived is p_0 is equal to 1 minus rho. We also derived p_n is equal to rho power n p_0, which is rho power n into 1 minus rho. We also derived expressions particularly for L_s is equal to rho by 1 minus rho; we also said L_s is equal to L_q plus rho. We also used W_s. We also used the other relationship where we have L_s equal to lambda W_s and L_q is equal to lambda W_q.

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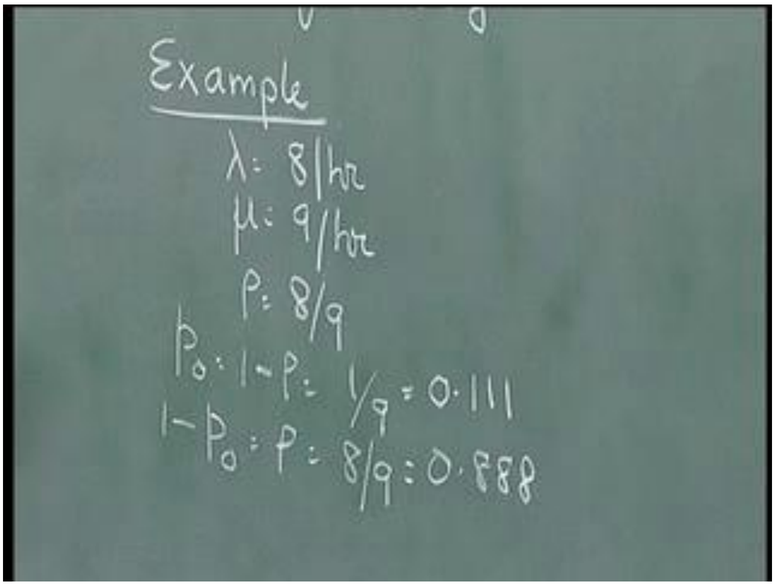

Now, with all these equations, let us work out a numerical example to understand what is meant by each of these equations. Let us look at a case where, for example, lambda is equal to 8 per hour and mu is equal to 9 per hour. This would give us rho is equal to 8 by 9; rho is lambda by mu, which is 8 by 9. Now, from this equation p_0 is equal to 1 minus rho, which is 1 by 9, which is 0.111. Now, p_0 equal to 0.111 tells us that there is a probability of 0.111 that there is no person in the system. So, p_0 is the probability that at steady state there is no one in the system; that is 0.111 or 11.1 percent of the times there is nobody in the system.

When there is nobody in the system it means that the server is free. So, the probability that the server is not free is 1 minus p_0; so 1 minus p_0 is equal to rho which is equal to 8 by 9, which is equal to 0.888 or 0.889. Now, this is the probability that there is at least one person in the system, which means, this is the probability that the server is not free or the server is busy or the server is utilized. So, the probability that the server is utilized is in this example 1 minus p_0 which is 0.888.

Now, let us try to find out, what is the probability that there is no queue?

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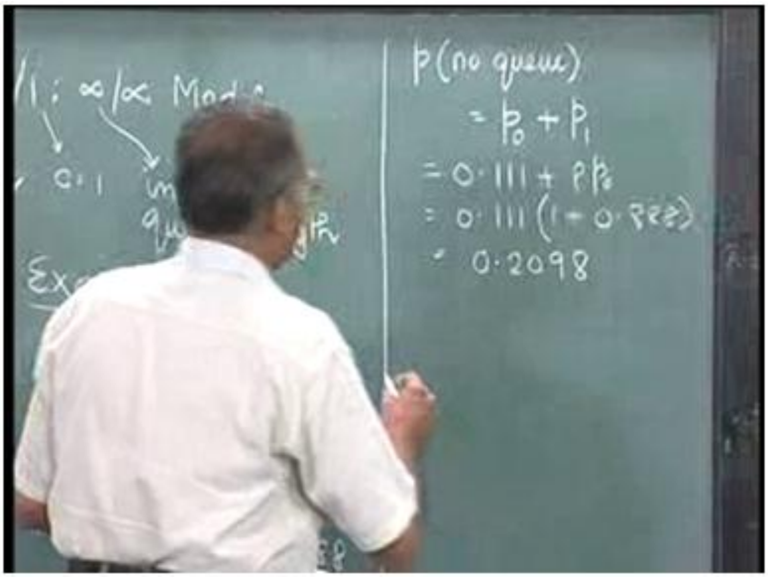

What is the probability that there is no queue? So, there is no queue when there is no one in the system - p_0 or there is one person in the system and that person is getting served; so there is no queue. So p_0 plus p_1 would tell us that there is no queue. So p_0 is 0.111 plus p_1 is rho p_0, which is 0.111 into 1 plus 0.888 and this would give us 0.2098. So, for our example, probability that there is no queue is 0.2098, which means about 21 percent of the times there will be no queue; a person who comes, joins the queue as the first person. Now, if it turns out that out of these 21 percent of the times, there could be sometimes when the server is free, so the person who comes in will straight away go to the server and get the service. There could be sometimes where a person who comes in, joins as the first person in the queue and waits for service.

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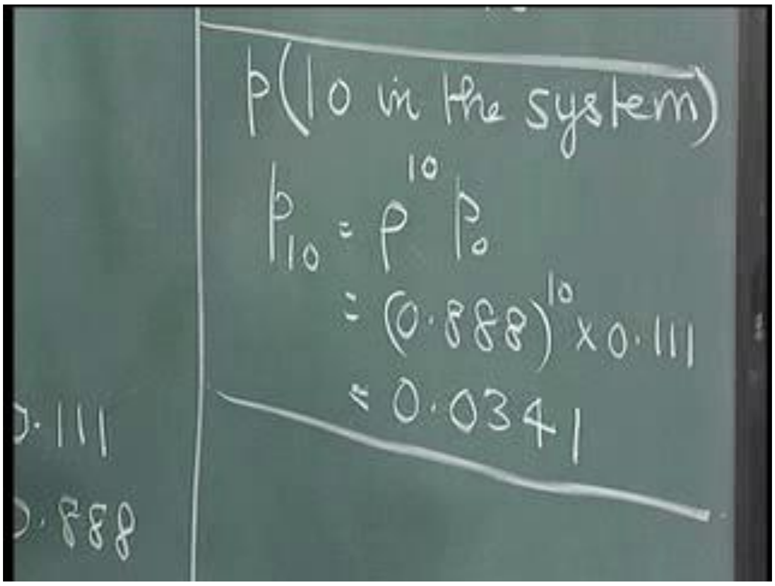

Now, let us try to find out what is the probability that there are 10 people in the system. Probability that there are 10 people in the system is given by p_{10}, which is rho to the power 10 p_0, which is 0.888 to the power 10 into 0.111 and this will be given as 0.0341. So, this computation tells us that the probability that there are 10 people in the system is 0.0341, which means roughly 3.4 % of the times there will be 10 persons in the system, 1 person getting served and the remaining 9 waiting for service.

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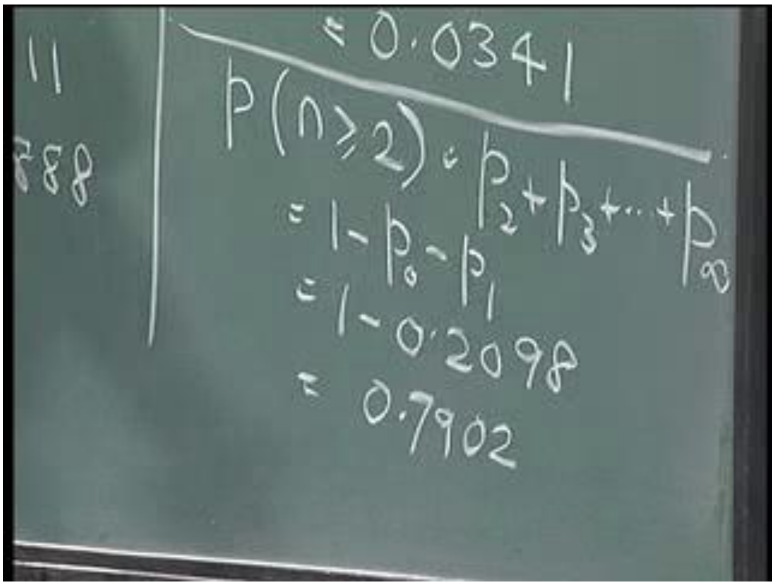

Now, if we wish to find out the probability that there are at least 2 or more people in the system, which means probability of n greater than or equal to 2, the probability that there are at least 2 or more people in the system will be equal to p_2 plus p_3 plus etc., plus $p_{infinity}$. Now, this will be equal to 1 minus p_0 minus p_1, because, p_0 plus p_1 plus p_2 plus etc., up to infinity is 1.We have already found out that p_0 plus p_1 is 0.2098. So, 1 minus 0.2098, this will be 0.7902. So 0.7902 which means 79 % of the times there will be 2 or more people in the system.

So, this way we can find out the probabilities of estimating or finding out the probabilities that a given number are there in the system; more than a given number are there in the system; up to a given number are there in the system and so on. Also, the server utilization; server utilization is 1 minus p_0.

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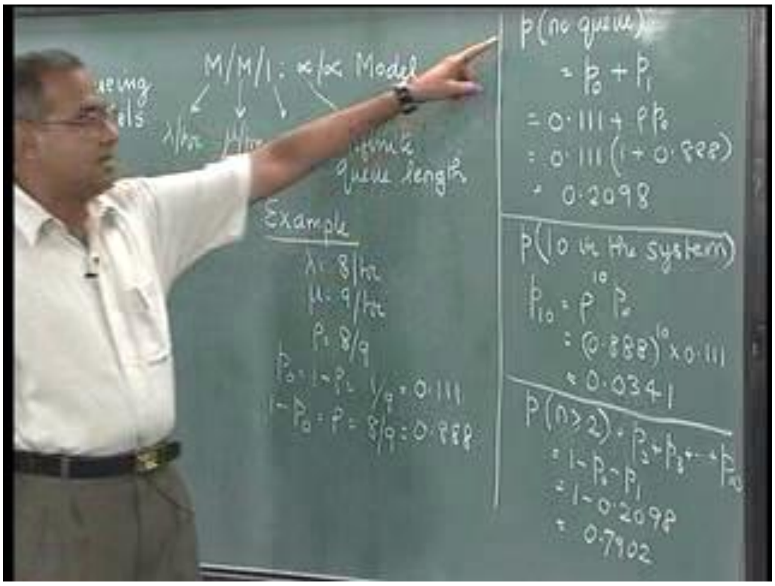

Probability that there is no q is this; probability that there are exactly 10 is given here. Probability that there are at least 2 is given here (Refer Slide Time: 09:33 min). Like this; we can have our computations to find out the probabilities of so many number of people in the system.

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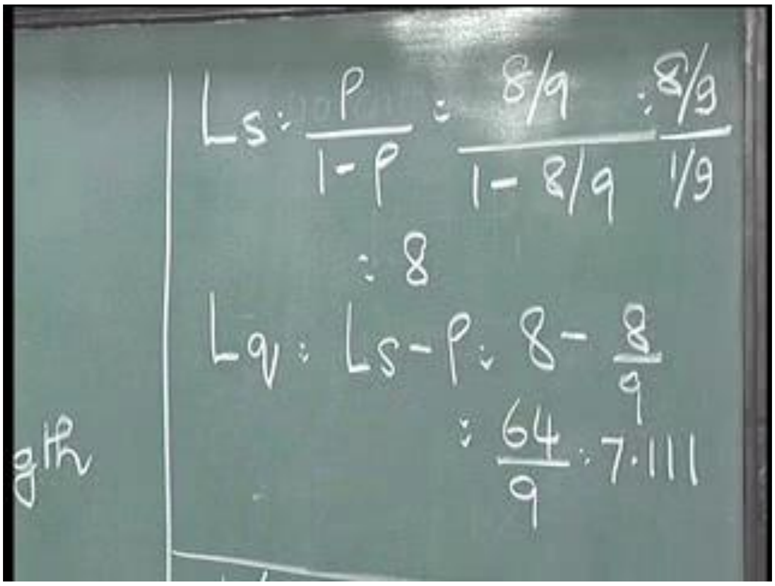

Now, for this expected number in the system, L_s is equal to rho by 1 minus rho, which is 8 by 9 divided by 1 minus 8 by 9, which is 8 by 9 divided by 1 by 9, which is equal to 8. Now expected number in the system is 8, which means that when a person walks into this system, then on an average the person can expect 8 people in the system; 1 person is getting served and the remaining 7 are waiting in the system.

Now, L_q is equal to L_s minus rho which is 8 minus 8 by 9, which is 64, 72 minus 8, 64 by 9, 9 times 7 is 63, 7.111. So the expected number of people in the queue is 7.111 in this. Now we can also find out W_s and W_q.

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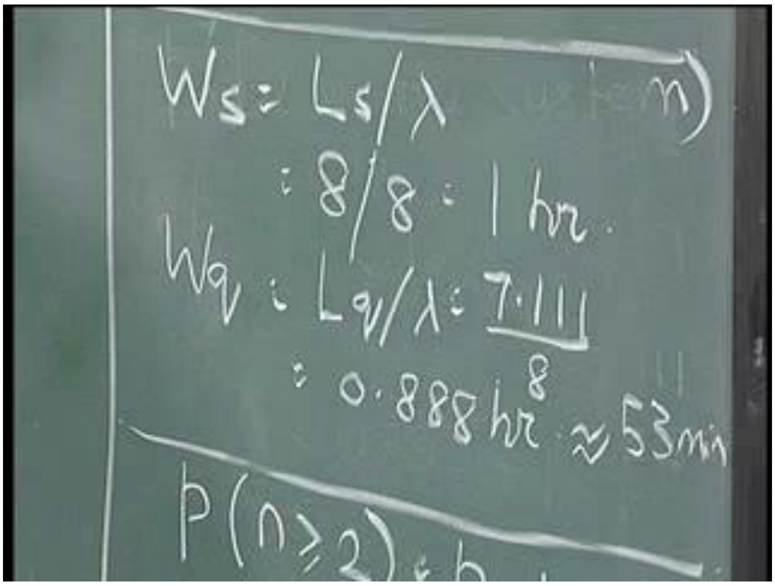

W_s is equal to L_s by lambda, which is 8 divided by 8, which is 1 hour. A person that comes into this system will require to wait for about 1 hour, both to wait in the queue and to get the service, and then leave the system. So expected waiting time in the system is 1 hour; W_q is expected waiting time in the queue is L_q by lambda, which will be 7.111 by 8; this is 0.8 times 8 is equal to 64, 71; 8 times 8 is equal to 64; 0.888 hour which is approximately 53 minutes. The person would have to wait on an average for about 53 minutes before the person gets his or her turn for service; then there is a service time and then the person comes out of this system.

We also need to understand a little bit more about this L_s, L_q, W_s and W_q. Now, the L_s, L_q, W_s and W_q are all expected values. L_s is the expected length of the system, which means the expected number of people who will be there in the system, which happens to be 8. Now it does not mean that every time a person comes in, the person will see exactly 8 people. So it does not hold for a particular instance. There could be times when you come and join this line there can be about 3 people. There could be times when there are about 10 people or 12 people and so on. But if you keep on computing the number of people who are there in the system and then average it out, the average will come out to 8. So, that is what this actually means. This does not say that for every person who comes in, the person has to wait for 1 hour in the system or 53 minutes in the queue. There could be instances where it is less; there could be instances where it is more, but on an average this will be 1 hour in the system and 53 minutes in the queue. So we have to understand that these are only expected values. And as more and more people come in, if we take the individual values and average them out, we will get these kinds of numbers: 8, 7.111, 53 minutes, 1 hour and so on.

Now, obviously we can do this the moment different values of lambda and mu are given or even if it is enough to give rho, one can compute all this (Refer Slide Time: 13:55 min).

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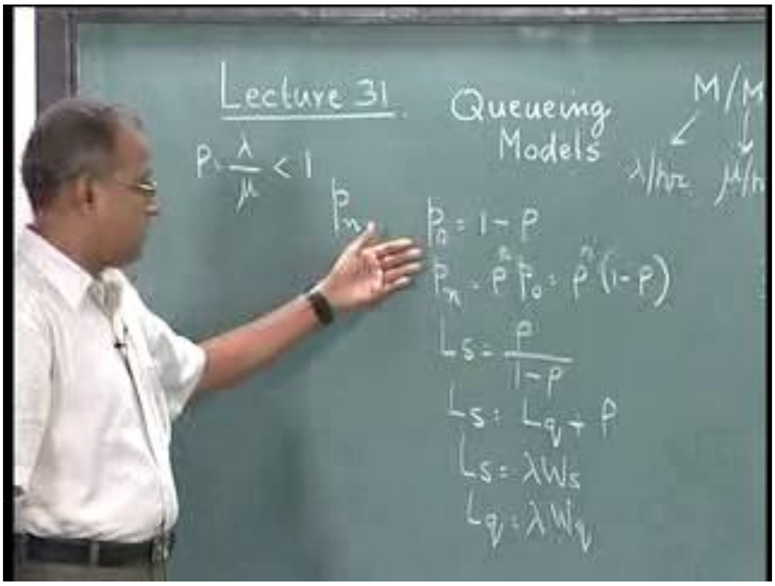

So, we know the equations of p_n, we know the equations for L_s, L_q, W_s and W_q. These are the expected outputs which are normally looked at (Refer Slide Time: 14:04 min). These are outputs or these are parameters which are of importance from a customer view point. These are internal equations that relate the input that is rho, to the probabilities and using this rho we can get expressions for each one of them.

Now we move on to the second model. The second model is called M/M/1: N infinity model, which means that the queue length now is not infinite, but the queue length is finite. So, it becomes a finite queue length model. So, some of those equations will still hold, except that when we have N plus 1 people in the system or if we have N people in the system, the person who is coming in does not join the line and leaves. So whenever a person comes in, the person can see either 0 or 1 or 2 up to n; the person does not see the N plus 1. So the system will not have more than capital N number of people.

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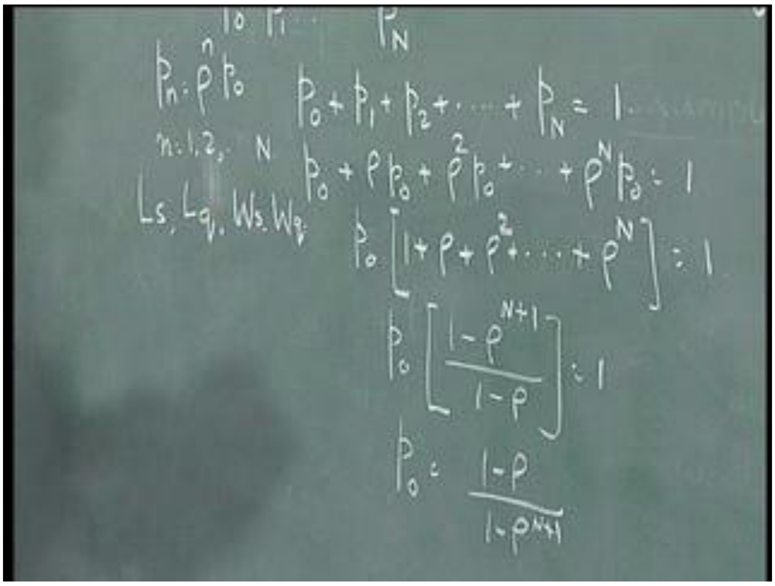

So, the probabilities will now become p_0, p_1, up to p_N where capital N is the maximum number that this system will hold. So, one of the equations will change and we will have p_0 plus p_1 plus p_2 plus etc., plus p_N will be equal to1. Now we know the relationship between this. So this is p_0 plus rho p_0 plus rho square p_0 etc., plus rho power n p_0 is equal to 1. Now if we take p_0 outside into 1 plus rho plus rho square plus rho power n is equal to 1.

Now, in this case, rho can even be equal to 1 or greater than 1, because we have this forced baulking that actually happens. So this is a finite geometric progression that has n plus 1 terms with the first term equal to 1 and the common ratio being equal to rho. So p_0 into the summation of this will be 1 minus rho to the power n plus 1 by 1 minus rho is equal to 1; from which p_0 is equal to 1 minus rho by 1 minus rho to the power N plus 1. So, the equation or the expression for p_0 changes; earlier, it was only 1 minus rho; now, it is 1 minus rho divided by 1 minus rho to the power N plus 1. So, moment the expression for p_0 changes, expression for a general p_N is equal to rho to the power np_0 up to n equal to N or up to N is equal to 1 to N; we can use this to find out the probabilities that there are small n number of people in this system. The rest of the equations are quite similar and we derive them and we need to derive the expressions for L_s, L_q, W_s and W_q. Now, let us derive the expression for L_s.

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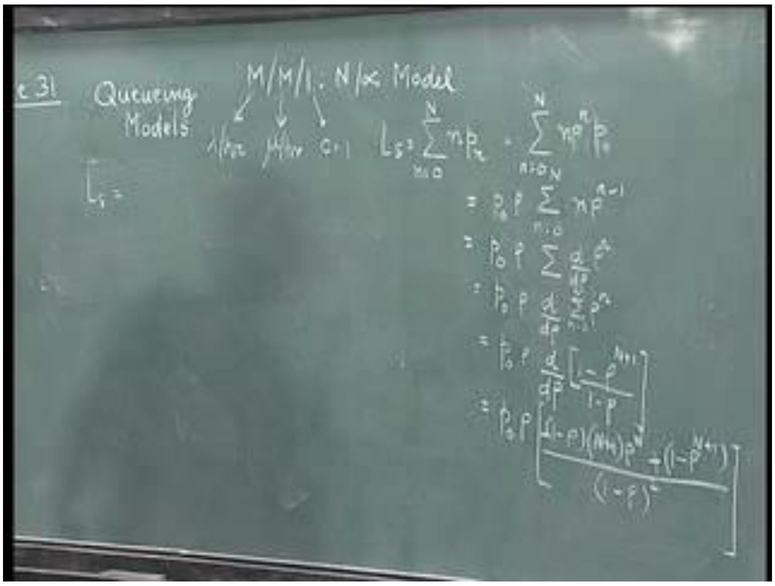

Now, L_s is the expected number of people in the system. So it is an expected value from 0 to capital N, because we may have 0 people in the system 1, 2, etc., up to capital N. So, this will be like 0 into p_0, plus 1 into p_1, plus 2 into $p_{2,}$ so normal expected value. So we write the expression here. So this is sigma n equal to 0 to N, n rho power np_0. So, now we take p_0 outside and rho outside; so, p_0 into rho sigma n equal to 0 to N n rho power n minus 1. Now, this is p_0 into rho into sigma d by d rho of rho to the power n. So, this will be p_0 into rho and as usual, we switch the differentiation and the summation to get d by d rho sigma rho power n.

So, this will be p_0 into rho into d by d rho of rho power n. This is summation up to n equal to 0 to N. So this is 1 minus rho to the power N plus 1 by 1 minus rho. Now this has to be differentiated to get p_0 rho into, this is 1 minus rho the whole square, we use the formula d by d rho U by V is VDU minus UDV by V square.

So, we have 1 minus rho the whole square. This is 1 minus rho into differentiation of this which is VDU. So, this is minus N plus 1 rho to the power N with a minus sign. This comes because this will give a 0, the minus will stay, here rho power N plus 1 would give us N plus 1 rho power N minus UDV, which is 1 minus rho to the power N plus 1 into differentiation of this, which is minus 1. So this will become a plus by this (Refer Slide Time: 21:21).

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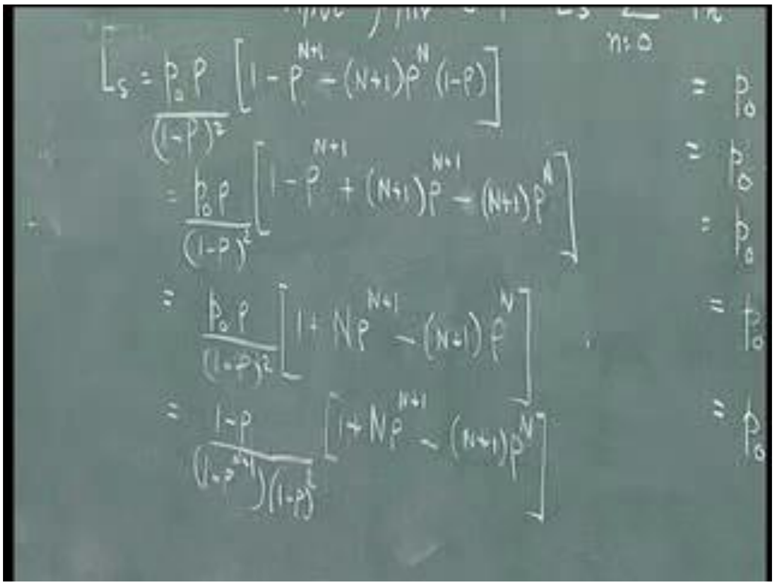

Now this one is further expanded to L_s is equal to p_0 rho by 1 minus rho the whole square into 1 minus rho to the power N plus 1 plus N plus 1 rho to the power N into 1 minus rho. This will become p_0 rho by 1 minus rho square into 1 minus rho to the power N plus 1 minus (N plus 1) rho to the power N plus 1. Now what I am doing is I am multiplying this N plus 1 into rho to the power N into minus rho will give me minus N plus 1 rho to the power N plus 1.

Now, I am multiplying the other one; so plus N plus 1 rho to the power N. This on further simplification will give me p_0 rho by 1 minus rho square 1 minus. Now, this is minus rho to the power N plus 1. So when you expand this I get this: N plus 1 rho to the power N and what do I get here? Minus of N plus 1 rho to the power (Refer Slide Time: 25:00).

So, this when I differentiate, I get V square VDU 1 minus rho into N plus 1 rho to the power N with the minus sign plus UDV 1 minus rho to the power N plus 1 into minus 1. So I get plus 1 minus rho to the power N plus 1. So I write this as 1 minus rho to the power N plus 1 minus N plus 1 rho to the power N 1 minus rho. I am writing this term first. So 1 minus rho to the power N plus 1 and I am writing this term later. So I get minus 1 minus rho N plus 1 rho to the power N. So this will become 1 minus rho to the power N plus 1. Now I am expanding this. There is a minus sign here (Refer Slide Time: 26:05 min). N plus 1 rho to the power N into minus rho will become N plus 1 rho to the power N plus 1 with a plus sign here. This 1 will become minus, minus N plus 1 rho to the power N.

So, further I get minus rho to the power N plus 1 plus N plus 1 rho to the power N plus 1. So when I expand this I will get, this term will become, there is a minus rho to the power N plus 1, there is a plus rho to the power N plus 1 which will get cancelled.

So I get plus N rho to the power N plus 1 minus N plus 1 rho to the power N. This is what I get. Now expand this p_0. Now, this p_0 is 1 minus rho by 1 minus rho to the power N plus 1 and I have 1 minus rho square already; so this into 1 plus N rho to the power N plus 1 minus N plus 1 rho to the power N. Now, again one of these will get cancelled here, with this (Refer Slide Time: 27:32 min), this will go, this will remain.

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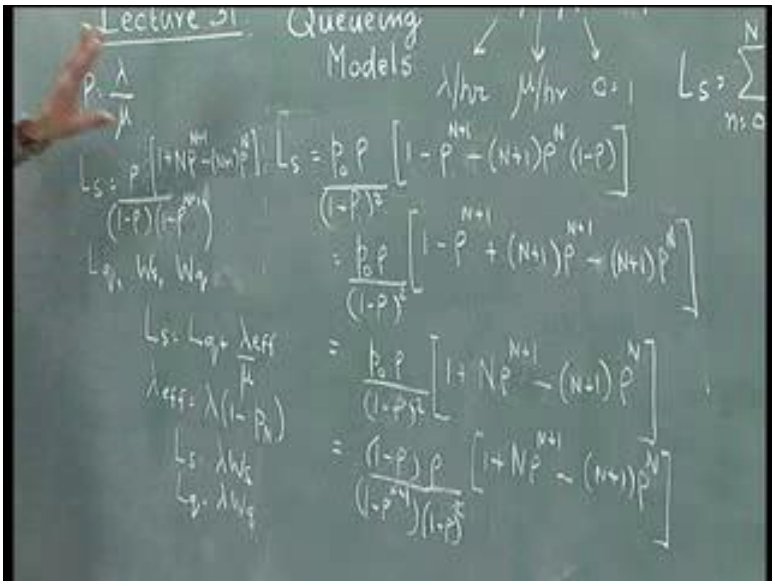

So from which I get the expression, I have a rho also here. So L_s is equal to rho by 1 minus rho into 1 minus rho to the power N plus 1 into 1 plus N rho to the power N plus 1 minus N plus 1 rho to the power N. So we get an expression like this for L_s when we put a limit on this M. So the expression gets a little more complicated. Earlier, it was only rho by 1 minus rho. Now we are multiplying this by a term which is like this: 1 plus N rho to the power N plus 1 minus N plus 1 rho to the power N by 1 minus rho to the power N plus 1.

Now, this term we multiply with the older one. Similarly p_0 was 1 minus rho. Now p_0 has become 1 minus rho divided by 1 minus rho to the power N plus 1. This 1 minus rho to the power N plus 1 keeps coming, because, we have a restriction on this M. Once we know L_s we can get the other expressions; we need to get expressions for L_q, W_s and W_q. Now, there is a slight difference in these relationships and that happens like this. Earlier we had L_s is equal to L_q plus lambda by mu. Now we have something called $\text{lambda}_{effective}$ is equal to lambda into 1 minus p_N. Now what is this $lambda_{effective}$? Because, we have a finite restriction on the queue length, we are going through what is called a forced baulking. For example, if a customer comes and the customer finds that there are already N people in the system, then the person baulks; which means, everyone who is coming in does not join the line. So some people leave. So the actual arrival rate even though it is lambda, not all lambda come in; there is an effective rate, because, the moment we have N people in the system with a probability p_N people leave the system. So people who actually enter the system, now they have a probability of 1 minus $p_{N.}$ So the effective arrival rate is lambda into 1 minus p_N which is actually less than lambda. The moment we have this $lambda_{effective}$, then we write, L_s is L_q plus $lambda_{effective}$ by mu. Then we have the usual relationships: L_s is equal to lambda W_s; L_q is equal to lambda W_q.

So we have these equations: L_s is equal to lambda W_s and L_q is equal to lambda W_q. Now this is in hours. This is people per hour. So the multiplication would give expected number of people in the system. So, these are the governing equations when we look at M/M/1 N infinity model. Now the expression for p_0 changes; p_0 now becomes 1 minus rho by 1 minus rho to the power N plus 1.

Now, the expression for L_s changes to rho by 1 minus rho into this expression: 1 plus N rho power N plus 1 minus N plus 1 rho to the power N by 1 minus rho to the power N plus 1. Then we also have an effective lambda because some people leave the system and then the L_s, L_q, W_s, W_q are now based on the effective lambda.

Now what we will do now is we will take a numerical example to understand what happens when we put a capital N into this system.

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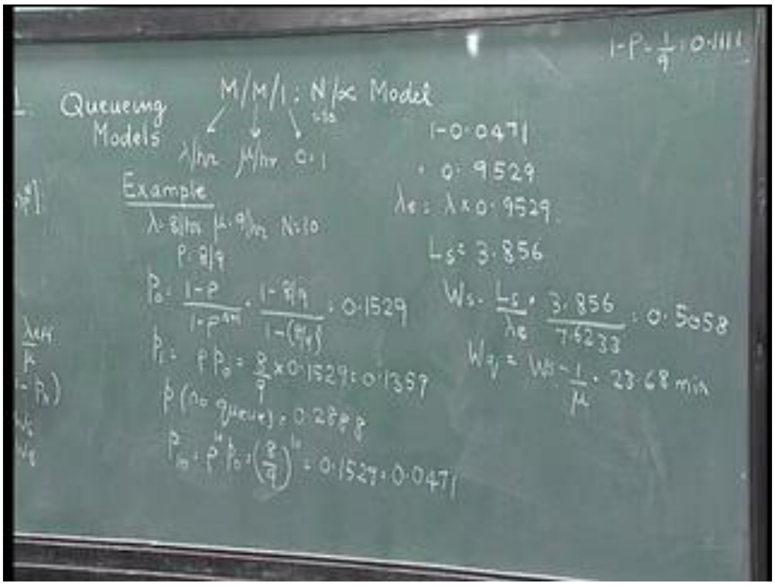

Now, if in the same previous example, let us say, we have lambda equal to 8 per hour, mu equal to 9 per hour and we restrict say N is equal to 10, which means when the system has 10 people the 11th person who comes in does not join the system. So, rho is equal to 8 by 9; now p_0 which was earlier 1 minus rho, now p_0 will become 1 minus rho by 1 minus rho to the power N plus 1, which is 1 minus 8 by 9 divided by 1 minus 8 by 9 to the power 11. This on simplification would give us 0.1529. Now the probability that there are 0 people in the system is given by 1 minus rho by 1 minus rho to the power N plus 1. So N is equal to 10. So N plus 1 becomes 11 and this on computation gives 0.1529.

So, this means that about 15 % of the times, this system will have 0 people in the system or 15 % of the times the server will be unutilized or idle. If we compare with what happened to p_0 in the earlier model, where this N was infinity, p_0 was 1 minus rho. So, 1 minus rho was 1 by 9, which is 0.1111, which means about 11.1 % of the times the server was free. Now 15 % of times of server is free. When we restricted this N equal to 10 from N equal to infinity, now it is not very good from the system point of view, in the sense that the server is idle for more times in this model, than in the earlier model.

Later, we will have to see that this is beneficial from a customer point of view because the corresponding L_s and W_s will come down. So, now when we have p_0 equal to 0.1529 and if we now try to find out what is the probability that there is no queue, it means there is no queue when we have either 0 people in the system or 1 person in the system. So, p_1 is equal to rho $p_{0,}$ which is 8 by 9 into 0.1529 and this is 0.1359. So probability of no queue is p_0 plus

p_1 which becomes 0.2888. So, about 28 % of the times there will be no queue and a person who comes will either directly go to the server, if the server is free or if 1 person is being engaged, this person will come in as the first person waiting.

Now, in the earlier model the probability that there was no queue - we had a probability of 0.209; so about 20 % of the times there was no queue, here 28 % of the times there is no queue. So, this is kind of beneficial from the user point of view that when the person comes in, there is a higher probability that there is no queue. There is a higher probability that the server is idle, which means the person can easily get to the server with a higher probability.

Now continuing - probability that there are 10 people in the system, p_{10} is rho power 10 $p_{0.}$ This will be 8 by 9 to the power 10 into 0.1529. This on simplification would give us 0.0471. So, probability that there are 10 people in the system is 0.0471. In the earlier example, probability that 10 people are in the system was 0.0341. Now this p_{10} is extremely important from this point of view, because now from p_{10}, we are going to say what is the probability that a person who comes into the system does not join the system?

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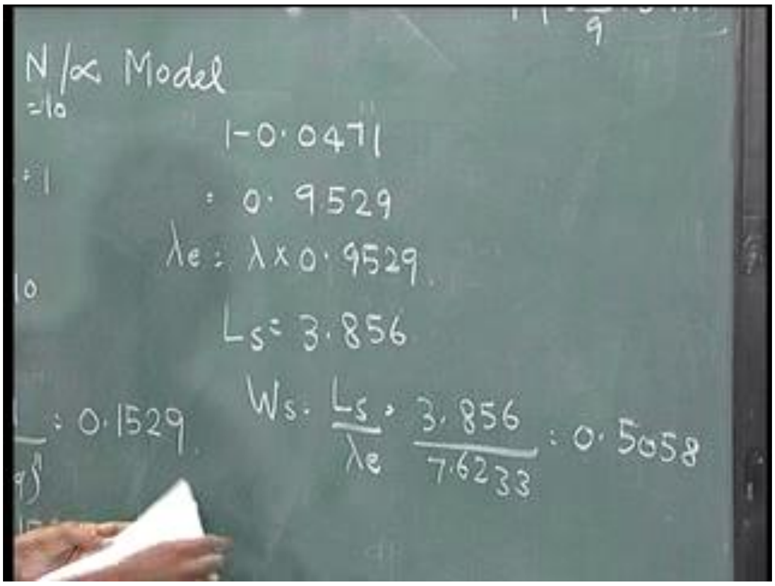

So the person who comes into the system does not join the system when there are 10 people in the system. So probability that someone does not join the system is 0.0471. So probability that a person coming in joins the system is 1 minus 0.0471, which is 0.9529; this is the probability that somebody comes into the system. So the effective arrival rate $lambda_e$ will be lambda into 0.9529, which will be less than 8.

Now continuing expected number in the system, L_s given by this formula (Refer Slide Time: 38:43 min). Now substituting the values here, rho is equal to 8 by 9, capital N is equal 10, so that N plus 1 is 11. So we can do this computation to get L_s. L_s on computation would give us 3.856. So number of people in this system now is 3.586 and in the earlier example we had L_s equal to 8. The expected number in the system in the earlier model when N is infinity was 8. Now with lot of people leaving the system, L_s - the expected number people in the system now comes down to 3.856, when we put a restriction that a maximum of 10 people will be there in this system.

Now again we can calculate the rest of them. Now W_s is equal to L_s by $lambda_{effective}$ ($lambda_e$) and this will be 3.856 divided by 7.6233. Now 7.6233, is nothing but 0.9529 into 8; because, 0.9529 is the probability that a person comes in. So the expected arrival rate is 8, which is lambda into 0.9529 which becomes 7.6233. Now this will become 0.5058, which is the waiting time in the system, which was earlier 1 hour has now come down to half an hour, because, we put this restriction.

Now, the last one to compute is W_q. W_q is equal to W_s minus 1 by mu, which will become 23.68 minutes after the substitution. Earlier it was about 53 minutes when we used N equal to infinity. So this is how we work out all these numbers. The important learning from these is that, when we put a restriction on this capital N, so p_0 will increase, which means the probability that 0 people in the system will go up, which means the system utilization or the server utilization will come down.

Now this is not a very healthy thing from a system point of view, (Refer Slide Time: 41:20 min) because, the system would like to have its server utilized as much as possible. But from a customer or a consumer point of view, you realize that probability that certain number of people are there in the system is actually increasing. So the arrival person is happy about it.

Now, in this, only 95 % of the people who actually come in to the system join in the queue. The expected numbers in the system come down drastically; expected time in the system comes down drastically. So, from a customer or a consumer point of view, this N is useful because you may come in and when you come in, this is the performance, but the customer also has to be aware that there is chance of about close to 5 % that a person coming in is not able to join the line, which again is something that the system has to worry about.

Now those who join the system are very happy because the waiting times and the expected number have come down, but obviously we will lose about 5 % of the business, because 5 %

of the customers do not come and join the line. So the N equal to 10 or the finite queue length model has these characteristics. From a system point of view, the utilization is coming down, because not all the people are joining. So, the system has to worry about the fact that about 5 % in this case do not join. From the point of view of the customer, we have a set of customers who do forced baulking, so they may not come back into this system again. They may choose not to or those who come in are very happy because their performance measures have come down drastically.

Now, we look at the third queuing model. Third model is M/M/C infinity infinity model, which is a multiple server model. So we have more than one server. We again look at the Poisson arrival and exponential service. We have more than one server. We have infinite queue length; so everybody who comes in will join the line.

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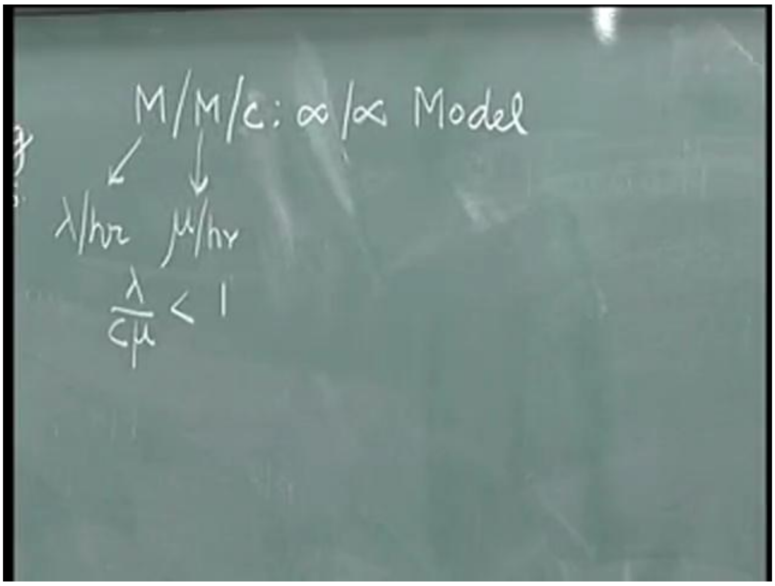

If we look at the case of say 2 servers, we may say that there is a common line and people joined this line; depending on which server is free they will go to that server. We also assume that the service rate mu per hour is the same irrespective of the server. So in this case, we should actually look at lambda by c mu, because mu is a service rate of 1 server. Since there are c servers, c mu is the effective service rate; lambda is the effective arrival rate. So in this case, lambda by c mu will have to be less than 1, because, we already said that in some form, lambda by mu has to be less than 1. Now it is lambda by c mu will have to be less than 1 or rho by c has to be less than 1.

Now, let us look at some steady state derivations for this kind of a queuing system. In this type of a queuing system, we can assume that $lambda_n$ is equal to lambda. Irrespective of the number of people in the system, the arrival rate is only lambda, but mu_n is equal to n_{mu}. Then we have n less than c and it is equal to c mu, when n is greater than or equal to c. Now, what do these equations tell us?

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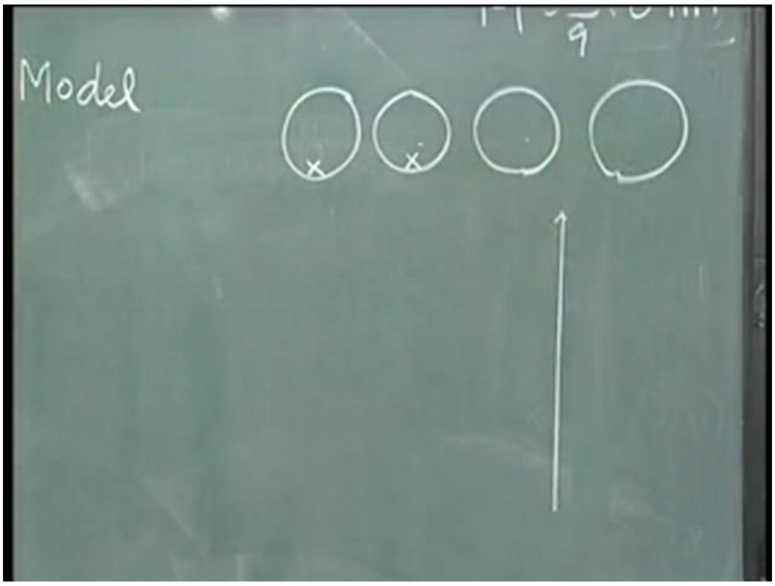

Let us assume that there are 4 servers here. So there are 4 servers. If we have 4 or more people in the system, the service rate is 4 times mu because all the servers are busy. So the service rate will be 4 times mu or c mu, when n the number in the system is greater than or equal to 4 which is what is given here.

Now, the effective service rate if this is q, if we have only 2 people in the system, there is a person here (Refer Slide Time: 46:37 min); there is a person here. The other 2 people are idle; so the effective service rate is only 2 mu.

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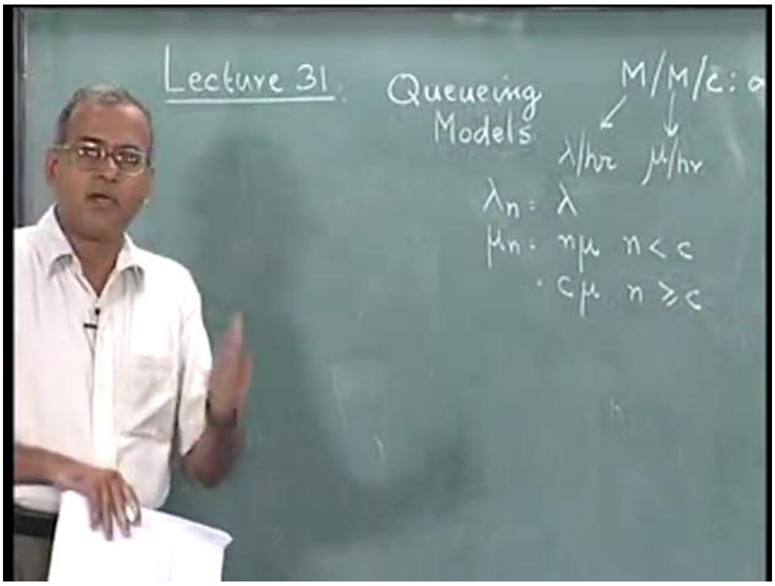

So it is equal to n mu if n is less than c and it is equal to c mu and n is greater than or equal to c. Now, with this, we also observe that in this model every person who comes will join the line. So we will have p_0, p_1, p_2, up to $p_{infinity}$ in this model.

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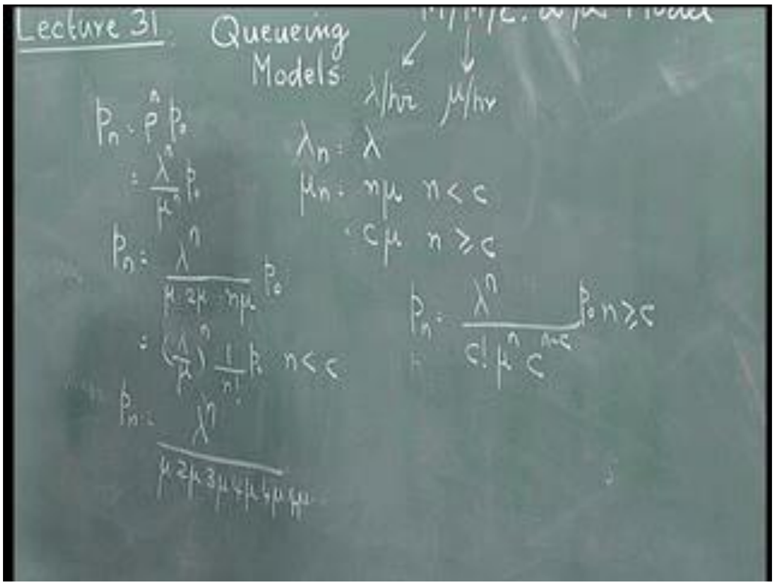

So now we write the expression for a general p_n. Now p_n has a general expression rho power np_0 or a general expression lambda power n by mu power n p_0. This is the general expression for p_n, n can go up to infinity. Now, because lambda changes or because mu changes depending on this n and c, this has to be rewritten (Refer Slide Time: 47:44 min). So this will be rewritten as p_n is equal to, if we have number of people n less than c, we have lambda power n which is the arrival rate; mu to the power will be mu, 2 mu, up to n mu into p_0. Now this comes because n is the less than c. If you look at a case where c equal to 4 and say n equal to 3, then when I have 1 person in the system I will have mu, 2 people is 2 mu, 3 people is 3 mu. So, you will have mu, 2 mu up to n mu into p_0.

So this will become lambda by mu to the power n, 1 by n factorial p_0, where n is less than c and the same p_n will be lambda power n by.... If the number of people in the system is say 6 and c is 4, we will then have mu, 2 mu, 3 mu, 4 mu. If we have 5 or 6 or 7, the service rate is always going to be 4 mu, 4 mu and so on. So, this will become p_n is equal to lambda power n by, this is 1 into 2 into 3 into 4, now c is 4; so we have c factorial mu power n. And for the first 4, it will be c factorial, for the remaining n minus c, it is c, c, c; so c factorial into c to the power n minus c into p_0. So p_n will now have two distinct expressions which is lambda by mu to the power n, 1 by n factorial p_0, where n is less than c and p_n will be lambda by mu to the power n, c factorial c to the power n minus c into p_0 and n is greater than or equal to c. So, we have both these expressions which have to be used.

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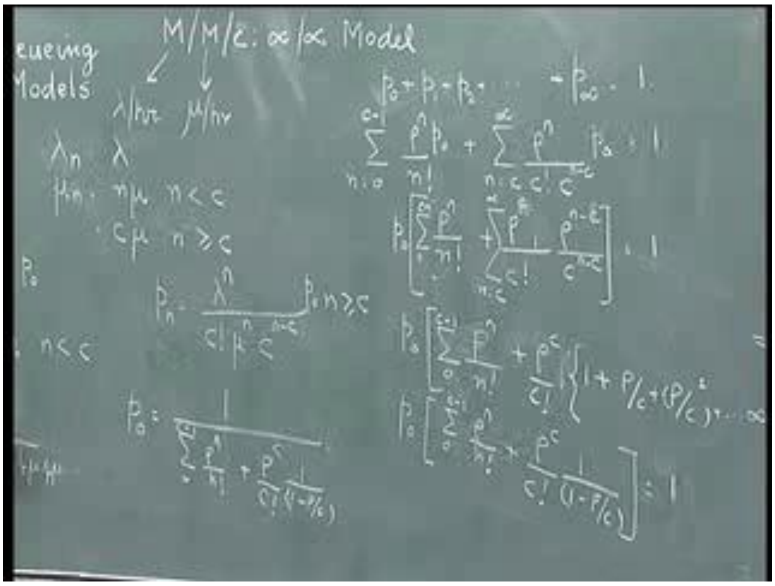

Now using these two expressions we also have p_0 plus p_1 plus p_2 plus up to $p_{infinity}$ is equal to 1. So a general p_n will now be like this, which is rho power n p_0, because lambda by mu is rho. So rho power n p_0 by n factorial; this is sigma n equal to 0 to c minus 1 because this is applicable for n less than c; so n equal to 0 to c minus 1. This is applicable for n less than c; so n equal to 0 to c minus 1 rho power n by n factorial p_0 plus sigma n equal to c to infinity rho power n by c factorial c to the power n minus cp_0 is equal to 1.

Now this will become, p_0 can be taken out; so p_0 into rho power n by.... So this n is equal to 0 to c minus 1, rho power n by n factorial. So this term will become 0 to c minus 1. So, rho power n by n factorial. So what I will do now is I will write it as p_0 into sigma 0 to c minus 1 rho power n by n factorial plus, this is expanded now (Refer Slide Time: 52:29 min). So, rho power n by rho power n p_0 c factorial. Now this rho power n is expanded as rho power c, rho power n minus c by c power n minus c; so you have sigma n equal to c to infinity is equal to1.

So this will become p_0 into sigma 0 to c minus 1 rho power n by n factorial plus expanding this, this will give us rho power c by c factorial into; p_0 is taken outside; so, we do not have to write p_0 again here; so rho power c by c factorial into... Now, this is when n is equal to c, this is n minus c; so when n is equal to c, because n minus c comes as this is rho power n is written as rho power c into rho power n minus c. So when n is equal to c, this term is 1; when n is equal to c plus 1, this term is rho by c plus rho by c the whole square up to infinity is equal to 1. Now, this is an infinite geometric series with first term as 1, the common ratio r as rho by c and we also know that rho by c will have to be less than 1. We mentioned here, that rho by c or lambda by c mu should be less than 1. Therefore, we can use the summation formula for the infinite geometric series to get: p_0 into sigma 0 to c minus 1 rho power n by n factorial plus rho power c by c factorial. Now, this is a by 1 minus r, so1 by 1 minus rho by c is equal to 1, from which we can write the expression for p_0.

So we can write the expressions for p_0 as: p_0 is equal to 1 by sigma 0 to c minus 1 rho power n by n factorial plus rho power c by c factorial into1 minus rho by c. Now this is the expression for p_0.

Now having derived this expression for p_0, p_n can be written as, if n is less than c, then we can use this expression as rho power n by n factorial p_0 when n is less than c and when n is greater than or equal to c, then p_n will become rho power n by c factorial c to the power n minus c into p_0; for p_0, we can use this expression. So once we know the expressions for p_0 and in general for a p_n the other thing that we have to do is to calculate L_s, L_q, W_s and W_q using these expressions.

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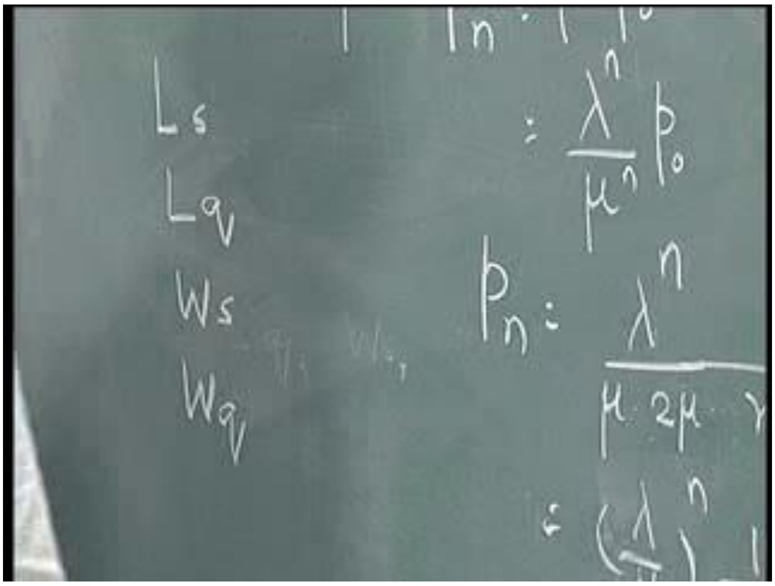

Now, the computation of L_s, L_q, W_s, W_q, we will address in the next lecture.