Advanced Finite Element Analysis

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Lecture – 8

Yeah, we were talking about normality flow rules in the last class and we said that normality flow rule is very important for us to define what we called as delta epsilon ij P, ij P or in other words, epsilon dot ij P.

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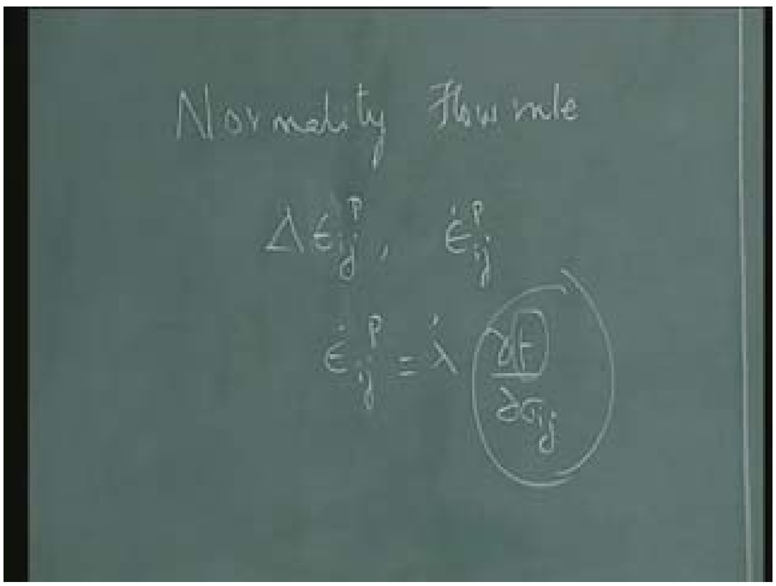

I said already that they are very similar and we wrote down that epsilon dot ij P can be written as lambda dot dow f by dow sigma_{ij}. Remember that actually this is the yield function either capital F or small f, because the second part of the small, of the capital F does not have sigma_{ij}, so it does not matter. So, our whole idea first is to calculate, what this particular part of this equation gives us. That is where we stopped and we said that that is a constant which we have determined and see what the physical meaning of delta lambda dot is, that also we will see in a minute.

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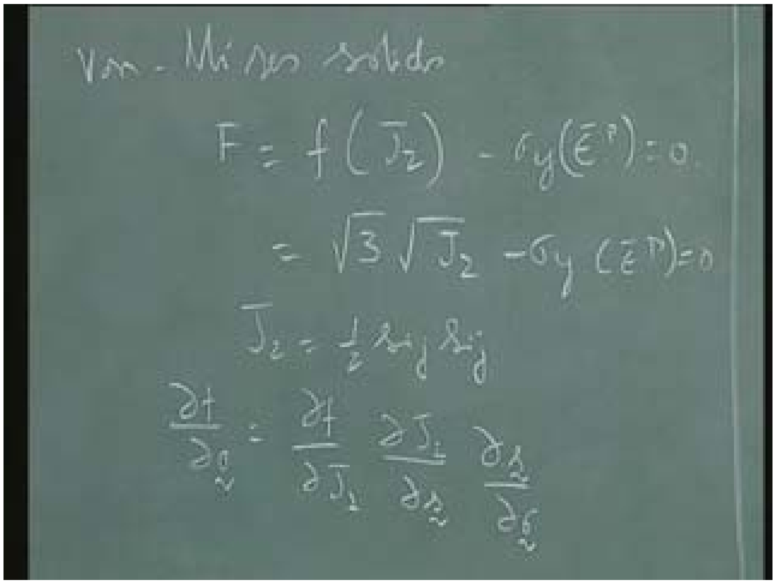

So, in order to calculate this, this particular part of the equation especially for Mises, Von-Mises solids, let us recognize that the first part of the Mises solid is written actually, though we said it is written in terms of sigma, it is actually written in terms of J_2 and that we wrote that as function of J_2 minus sigma_y epsilon bar P is equal to zero. I am sure now you know what is epsilon bar P. It is the equivalent plastic strain. More specifically this becomes root 3 root J_2 minus sigma_y epsilon bar P is equal to zero. Remember that J_2 is equal to half s_{ij} s_{ij}, where s is the deviatoric part of the stress tensor.

In other words, in order to calculate dow f by dow sigma, I have to use the chain rule. So, how do I do that? I say that then, in this case it is actually dow f by dow J_2. I should, I could have started it with dow capital F; it does not matter, so, dow f by dow J_2 dow J_2 by dow s dow s by dow sigma. So, that is what will give me dow f by dow sigma.

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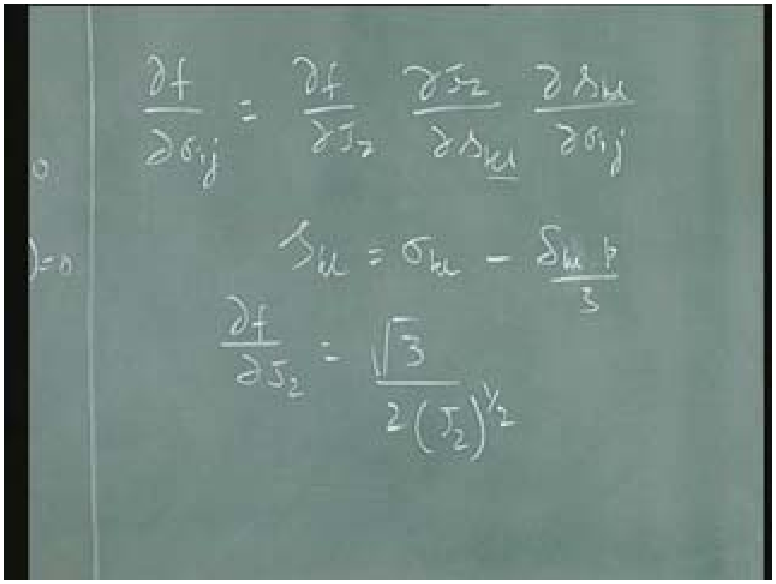

To write this in an indicial notation, which will make our job easier, you can write that dow f by dow sigma$_{ij}$ is equal to dow f by dow J_2 dow J_2 by, how do I write that? dow s_{kl} dow s_{kl} by dow sigma$_{ij}$. Look at the way we have written. So, we know that when I repeat this, then there is a summation involved in k and l and one more relationship, I know of course I have to use that and I am sure all of you know it that s_{kl} in terms of sigma$_{kl}$, I have to write it in terms of sigma$_{kl}$ and that is what is that? That is equal to sigma$_{kl}$ minus delta, yeah, that how do I write that? Say, delta$_{ij}$, delta$_{ij}$ into P, into P divided by, sorry delta$_{kl}$ into P divided by 3. Instead of writing that, I can also write this as in a slightly different fashion that I can write this to be, what? In terms of delta, so, how do I write that?

Let us keep it like that; we will come to it in a minute. Now the first part is to differentiate dow J_2 by, dow f by dow J_2. So, what do I get when I differentiate dow f by dow J_2? So, dow f by dow J_2 is equal to what? dow f by dow J_2 here is that; so, root 3 is there; root 3 into, divided by 2 into J_2 power half. So, that is the first thing. Then, I have to differentiate dow J_2 by dow s and dow s by dow sigma. So let us, dow J_2 by dow s happens to be s_{ij} and let us now differentiate dow s by dow sigma.

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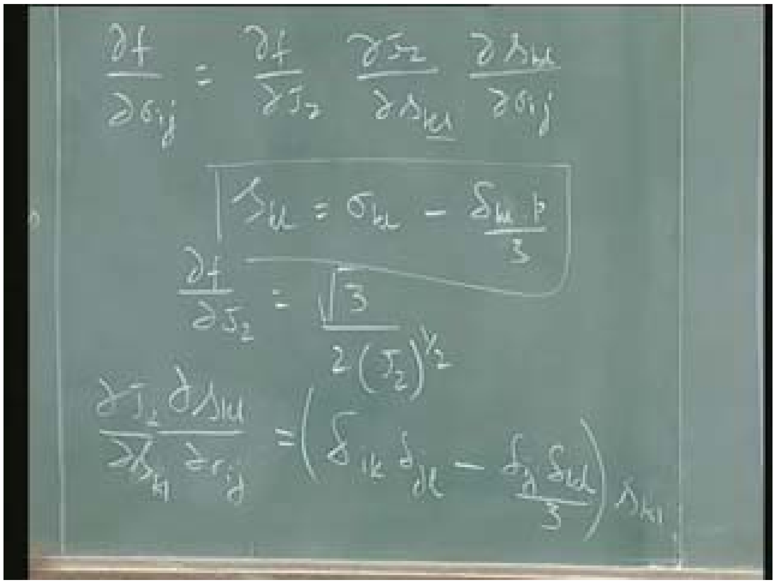

Now, when I differentiate, look at that carefully dow s by dow sigma; what, what is the result of this? What is the result of differentiating this tensor s with respect to another tensor sigma? What will you get? How many terms will you get? Yes; so, that is correct. So s_{ij}, it will be something like, you can say that it is dow s_{ij} by dow $sigma_{kl}$ or dow s_{kl} by dow $sigma_{ij}$. That means that what you will get here is a fourth order tensor and in fact this dow s_{kl} by dow $sigma_{ij}$ is fourth order tensor. In fact, it is very similar like dow $epsilon_{kl}$ by dow $sigma_{ij}$ if you can write, the result will be our …. So, I have to now calculate dow s_{kl} by dow $sigma_{ij}$. So, how do I get that? Look at this expression here and tell me what is that I will get? So, I have to get, let me write that clearly; dow s_{kl} by dow $sigma_{ij}$. What is the, what is the first term? What happens to my first term? Delta, what will be the first term?

So, when i is equal to k and j is equal to l, this term will become 1. So, what will be in terms of delta? $Delta_{ik}$ $delta_{jl}$ ik jl minus P is actually, what is P? $Delta_{ij}$ $sigma_{ij}$; that is this term $delta_{ij}$ $sigma_{ij}$. So, I will get minus $delta_{ij}$ $delta_{kl}$ by 3. Now delta, this dow J_2 by dow s_{kl}, from this expression, J_2 being half s_{ij} s_{ij} and fine, both of them are dummy, so, you can write it as s_{kl} s_{kl}, so, will become s_{kl}. So, dow J_2 by dow s_{kl} into dow s_{kl} by dow $sigma_{ij}$ will be this multiplied by s_{kl}. So, what will be the result?

The result will be, this will be s_{ij}, because k has to be equal i, j has to be equal to l, so, the first term will give rise to s_{ij} and what will be the second term? Delta$_{kl}$ s_{kl}, delta$_{kl}$ into s_{kl} that happens to be s_{11} plus s_{22} plus s_{33} that happens to be, no, s_{kl}, delta$_{kl}$ into s_{kl}, delta$_{ij}$ delta$_{kl}$ into s_{kl}. So, delta$_{kl}$ s_{kl} is the trace of the deviatoric tensor and that happens to be zero. So the result is, result of this is, equal to s_{ij}. So note that, note this carefully that it is just not, you cannot just differentiate this and say that this term here, what I have here is equal to s_{ij}, you have to really go through a very detailed process in order to say that the result is s_{ij}. In other words, the result is s_{ij}, because when you multiply these two terms, it so happens that the result is s_{ij}. Is it clear?

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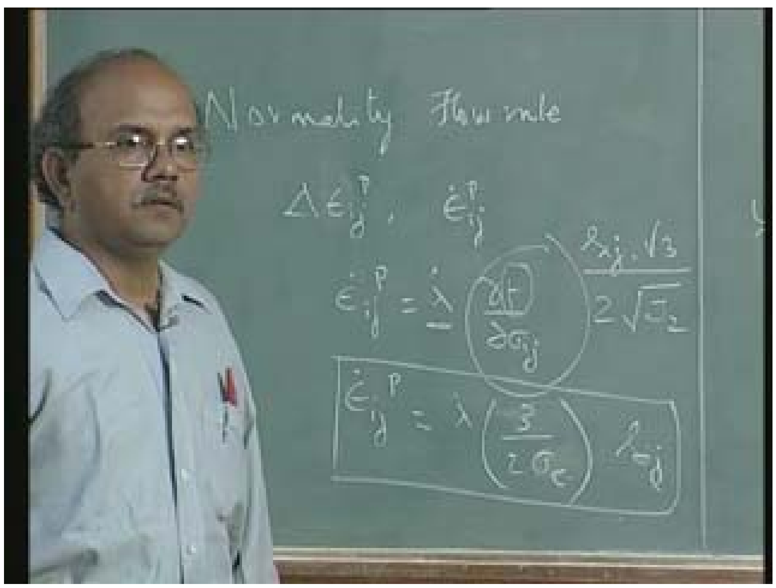

So, this term now happens to be s_{ij}. Then, I have some more terms. What are the terms, into, look at those terms. Those are the other terms that are there. s_{ij} into root 3 divided by 2 root J_2. In other words, epsilon dot ij P can be written as lambda dot into root 3 by 2 root J_2 into s_{ij}. What is this in terms of equivalent stress? You can write this in terms of equivalent stress, yeah, equivalent stress. So, equivalent stress is what? Root 3 into root J_2; so, multiplying the numerator and the denominator by root 3, root 3 so, this will become 3. When I put root 3 here, root 3 into root J_2 that will become equivalent stress. Remember that equivalent stress was written as root of 3 by 2 s_{ij} s_{ij}. So, this is how we wrote the equivalent stress. So, this is how, this is what you get when I substitute properly the dow f by dow sigma$_{ij}$ in the normality flow rule.

What does this equation now tell you? Look at that. What does this equation tell you? No, there is no power here. We are talking about the epsilon dot ij P. In other words, look at these terms. You know these terms are scalar terms. These tensorial terms are here and here. The strain, plastic strain tensor or plastic strains are coaxial with deviatoric stress, both of them or in other words both of them are in the same direction.

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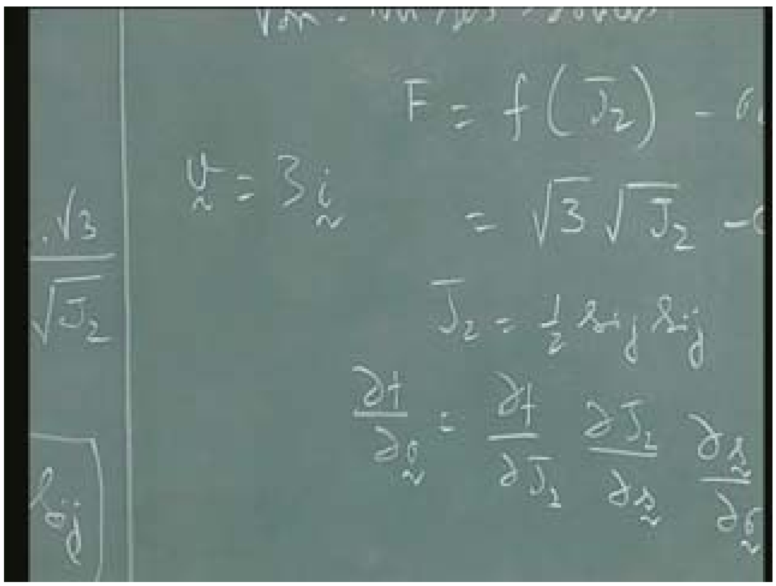

It is very simple, it is like saying that if I say v is equal to, I mean velocity vector, v is equal to say, I say, 3i, what does it mean? It means that my velocity is in the direction of i. Both of them, both the vectors are coaxial; they are in the same direction. That is what it means when I say v is equal to 3i. In the same fashion here you see that it is a very important result that this and this are in the same direction and that proportionality I had 3 there here; on the other hand, I have in this case, I have this …… here. Of course, this derivation is valid only for my Mises, Von-Mises criteria, but it would, this equation would change with, if the, if the, my yield function is different. Is that clear, any question?

Having done this, now let, let me just recaptulate what all we did. So, we had lot of plasticity theory going in the last class. We talked about yield criteria, we talked about hardening rules - isotropic and kinematic hardening and then we talked about loading unloading criteria and now we talked about normality flow rule and we derived the relationship for normality flow rule as well. We also talked about equivalent plastic strain and of course the Mises stress as result of, what? The result of Mises yield criteria. Having done all this, what is my goal? What am I doing or why am I doing all this? Remember that when I want to calculate the plastic strains and deformations in a one dimensional element, what we essentially did was to calculate, what was that?

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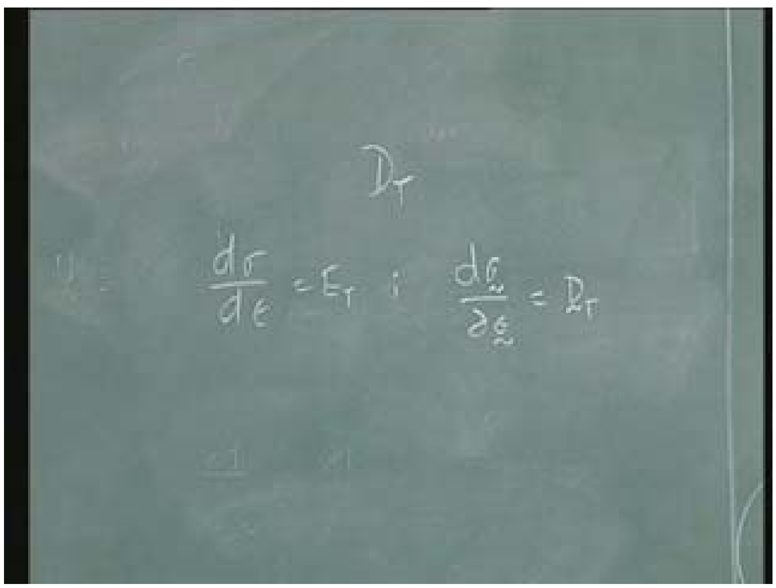

E_T; remember that we had A E_T by L. So, my whole idea here is to calculate an equivalent E_T or in other words, for a multiaxial case, this we called it as D or in other words, we have to calculate D_T. So, how do I get D_T? Now, what is D_T? Remember that d sigma by d epsilon is what we called as E_T and hence D_T is nothing but d sigma by d epsilon is equal to D_T. Is it clear? Look at that; d sigma by d epsilon is equal to D_T. Now, what is D_T, what is the, what is the order of this tensor? Yeah, so, dow sigma$_{ij}$, similar to what we did in the case of s_{kl}. So, it is, it is that. Now, in order to derive D_T, we go to what is called consistency condition. Yeah, that is one of the things which we said as a part of our yield function that a point should lie on the yield surface. So, we go over to the consistency condition.

How do I write consistency condition?

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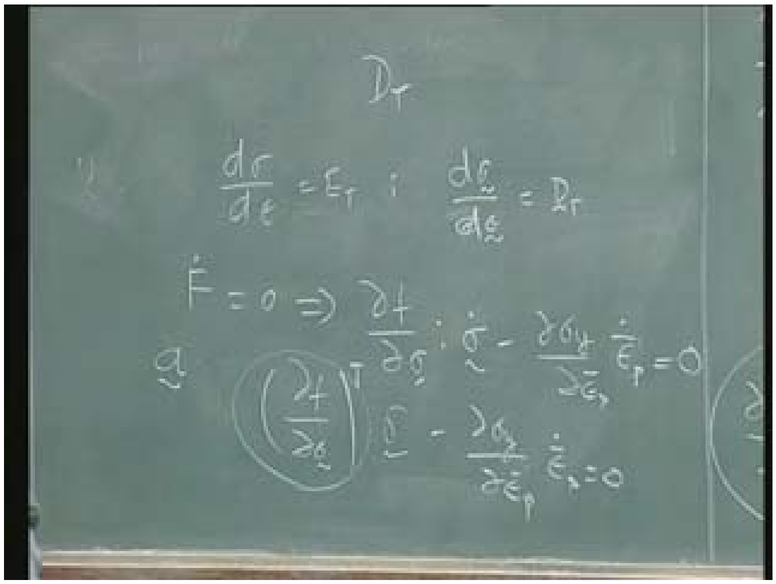

I write consistency condition stating that F dot, F dot, which is the result of the change of stress as well as the internal material variable is equal to zero and that implies in our case an equation of the form dow f by dow sigma sigma dot plus or in this case it becomes minus, minus dow sigma$_y$ by dow epsilon bar P into epsilon dot P, sorry, yeah, P is equal to zero; epsilon bar dot P, epsilon bar dot P equal to zero. What we did was only calculating df and df by dt is what we did here. If you want to write this in a matrix form, this can be done; dow f by dow sigma, of course these things are tensorial form, sigma minus dow sigma$_y$ dow epsilon bar P epsilon bar P is equal to zero. Most of the, you know, papers and books and literature call this as a matrix. It is customary to call this as a matrix or a vector, rather. People also call that as flow vector. So, the symbol if you see, a, it means that it is dow f by dow sigma. We know how to calculate dow f by dow sigma; we did all those exercises only now.

Now, ultimately what is that I want? I want a relationship between d sigma by, sorry. I think it should have been d sigma by d epsilon; I want a relationship between d sigma by d epsilon or in other words, sigma dot and epsilon dot.

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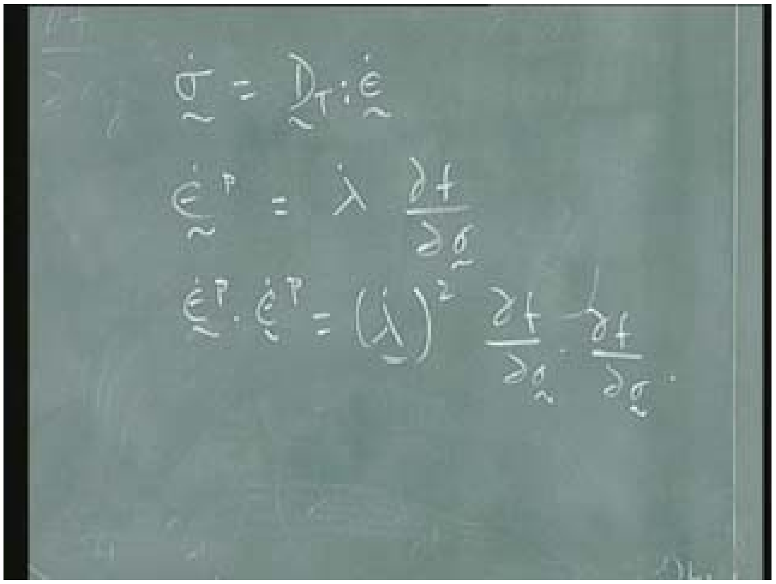

How do I get, my whole point is how do I get the relationship between sigma dot is equal to say D_T epsilon dot; that is the whole idea. Now, having defined this epsilon bar dot P, I can calculate that from now my definition of epsilon dot P is equal to lambda dot, let me call that as dow f by dow sigma, so that epsilon dot P, taking the inner product with itself, epsilon dot P is equal to lambda dot squared into dow f by dow sigma into dow f by dow sigma. In a matrix notation what it means is that we, we say this is a column vector; you can treat it as a column vector. In a matrix notation, this inner product simply means that we take epsilon dot P transpose epsilon P is what we take and write like this. Now, situation is very simple. What I am, actually first step, as a first step what I am trying to do is to find out what is this proportionality constant. Remember there was a proportionality constant; my first job is to get if possible some sort of an idea of what this proportionality constant is.

What I am going to do now is to substitute for, from this result I am going to substitute it there and see what I get. So, let me substitute for dow f by dow sigma from here, see what is that you get.

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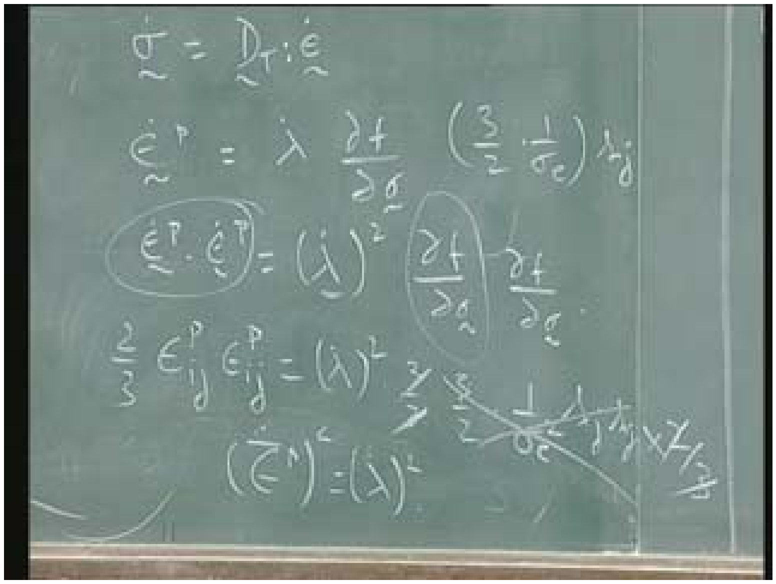

So, this is equal to lambda dot squared, why not you try, it is very simple. I would like you to try that and tell me, because that will give you some sort of a revision for what all you have done. Substitute this and tell me what that expression is. Look at this and then substitute it from 3 by 2 sigma$_e$ s_{ij}. So for this, substitute 3 by 2 sigma$_e$ s_{ij} and see what you get. So, I will get two times 3 by 2 sigma$_e$ squared. So, I will get 3 by 2 into 3 by 2 into 1 by sigma$_e$ squared into s_{ij} s_{ij}. Now let us see, let us see how you, yeah, because there is a dow f by, how do I, how did I get this value?

Yes, dow f by dow sigma is this, we just calculated it previously. We said lambda dot, see, this we calculated it. Remember that we, we did that in the previous step. So I am, I am just substituting it now, in terms of S for dow f by dow sigma which I have there. That is all I be doing, nothing else. So, I am substituting it here and I get, then what else? It is very simple. Look at these terms, 3 by 2, it is, it is a sitting duck; 3 by 2 s_{ij} s_{ij} by sigma$_e$ squared. What happens, what is the definition for sigma$_e$? That is J_2; root 3 by root

3 of 3 by 2 s_{ij} s_{ij}; so, 3 by 2 s_{ij} s_{ij} by $sigma_e$ squared. What happens to that term? Whole thing goes off; they cancel out, so, this whole thing cancels out. So, you have lambda dot squared into 3 by 2 is equal to epsilon dot epsilon dot P. Now, what is the equivalent plastic strain? What is this? This is nothing but, how do I write that in indicial notation? I can write that in indicial notation as like that. Now, what is the equivalent plastic strain? 2 by 3; correct, 2 by 3 epsilon. So, multiplying 2 by 3 on either side that is multiplying that and this side also by 2 by 3 that goes off and remember that this was root of the equivalent plastic strain. So, that becomes epsilon bar dot P squared is equal to lambda dot squared. In other words, what is lambda dot?

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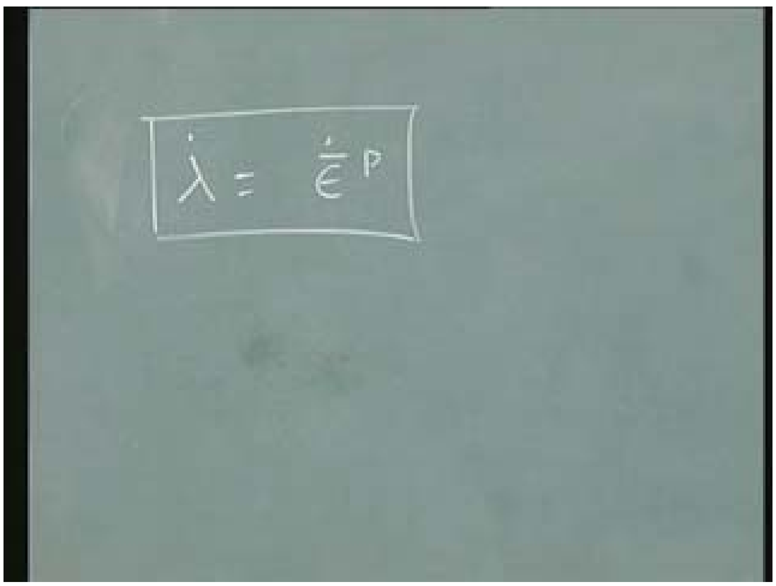

Yeah, simple that is all. Lambda dot has a meaning and lambda dot is equal to epsilon bar dot P. So that proportionality constant now clearly comes out to be the rate of increase of the equivalent plastic strain. So, that is the meaning for lambda dot. Yes, any question? Yeah, have a look at it. May be I will give you a minute to see all the steps; any confusion, let me know. Look at those steps what we did. No confusion, fine. So, now let us come back to, what I am going to do is to replace what I have here with lambda dot, so that I can get lambda dot. So, let us come back to this equation which is my consistency condition. Now remember that we had used this dow $sigma_y$ by dow epsilon bar P or strictly speaking it is d sigma$_y$ by d epsilon bar P, because sigma is only a function of epsilon bar P.

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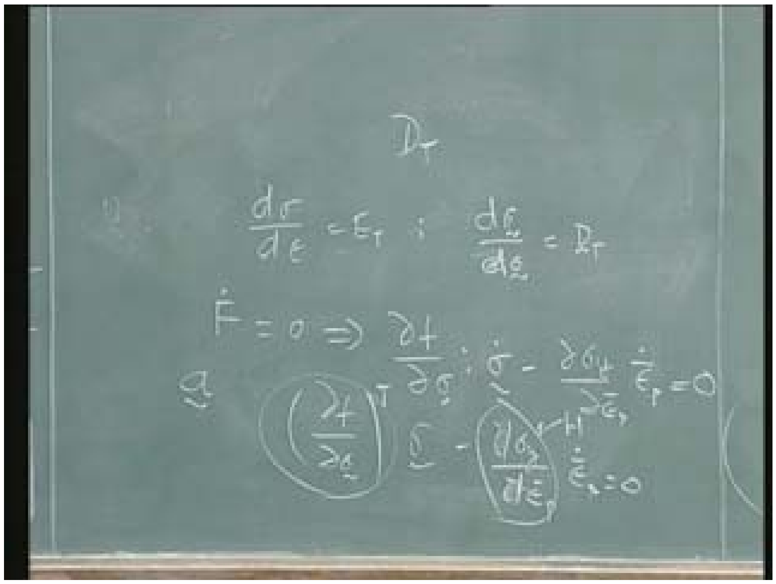

Sigma can also be a function of some other, you know damage parameters, we said; in this case since it is just a function, so you can also write it also as d, does not matter and what is that, what is this? That is the plastic modulus. So, this can be written also as, what is the symbol we use? H.

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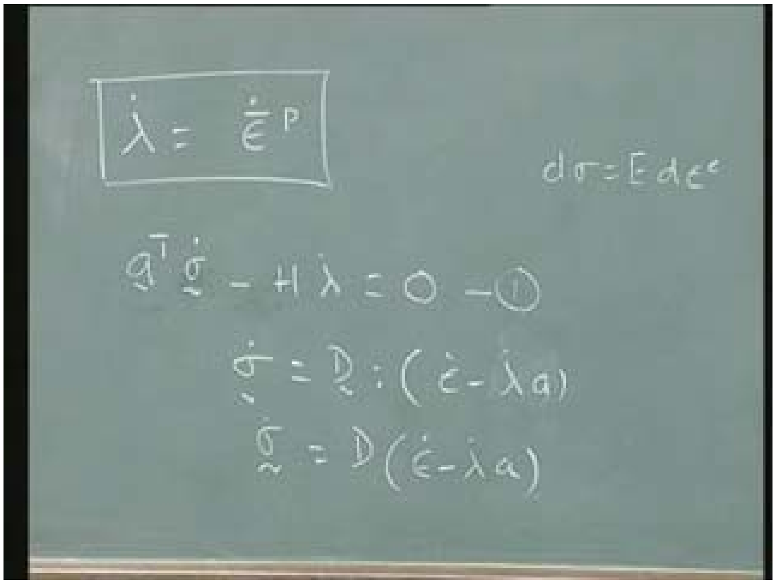

So, this equation can be written also as a transpose sigma, if you want it in a matrix form, minus H into what, epsilon bar dot P which is equal to lambda dot that is equal to zero. Is it clear? Now, this is one equation we have, consistency condition we have. Now we said if you remember, that of the total strain in the elastoplastic region, we can split this into an elastic part and a plastic part. This is additive decomposition. We said it is perfectly all right for small strain cases. From that, we also said that the elastic strains keep increasing and so on. We saw the phenomenological approach as well for this. So, in a mutliaxial situation, how do I write this? I can write sigma dot, which is actually D sigma, equivalent to that, is equal to D into epsilon dot minus epsilon dot P.

So, what is essentially this? This equation is the multi dimensional equivalent of saying that d sigma is equal to E into d epsilon e. That is it is the multiaxial equivalent of that. No, no, no, no. What is this D? D is the elastic, please note that this is D not D_T. We had, remember we had E E_T and H. So, the equivalent of E in the multiaxial case is D. Remember, this is the same as B transpose D_B. Material elastic property, what you get in B transpose D B. So, this is the elastic part of it. Is that clear?

Now, what is epsilon dot P? Epsilon dot P, from this expression I can write it as lambda dot a. So, I will replace epsilon dot P to be lambda dot a, sorry, this should be a, a transpose sigma dot, both of them are dot quantities. Dow f or yeah, dow f by dow sigma into sigma dot, this dot, that is, that is how it should be, so, lambda dot a. Now, let me substitute this equation into that equation and see what I get? Actually this is a matrix form. I should not have written it as a matrix form; it does not matter. If you want to write this also in the matrix form, then I can write this as D epsilon dot minus lambda dot a.

I am switching from matrix to a tensorial form. I mean this since you are now familiar with, so it does not matter which form I use. I wanted to say these both, because papers in this area use both these notations. That is the reason why I am using both of them. All these things are matrix, this and this. It is a D matrix, the familiar D matrix that you write.

Now my idea is to get what is lambda dot? So, from here I want to get, substituting inside, I want to get lambda dot. Let us see, please substitute it. Look at this expression, please substitute it and let us see what you get for lambda dot. Look at that expression.

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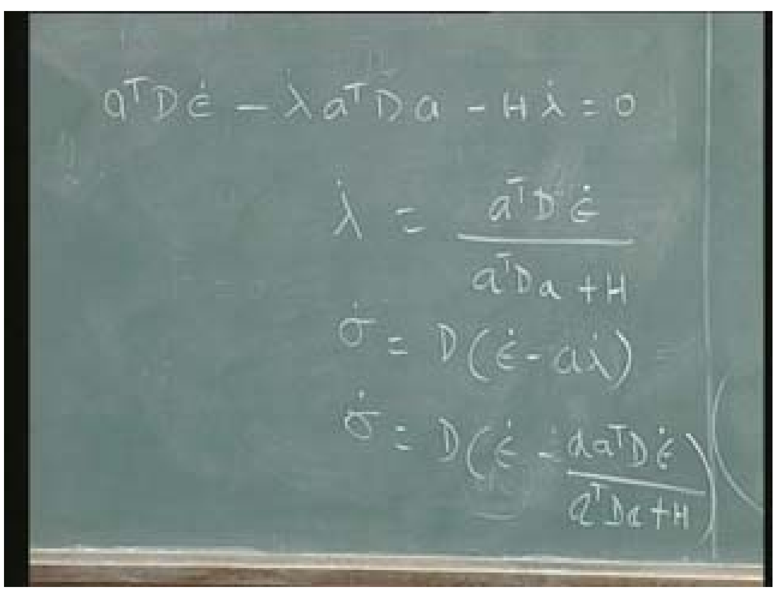

Substituting there, so I get a transpose, sigma dot is equal to D epsilon dot, so, a transpose D epsilon dot that is my first term minus second term, lambda dot is there; so, lambda dot a transpose D a, D a that is my second term minus H lambda dot is equal to zero. From this it is very clear that lambda dot, what I did was very simple, I just substituted it, nothing else. From this, it is very clear that lambda dot is equal to a transpose D epsilon dot divided by, divided by a transpose D a plus H. Is that clear? Any question on this?

Now let us see, I want to get sigma dot in terms of epsilon dot. How do I get it? How do I get it? Very simple, go back to this expression here, oh, I have just removed it. That is this expression. Let me write that expression back here, sigma dot is equal to D into epsilon minus sorry epsilon dot minus epsilon dot P and epsilon dot P is lambda dot a; lambda dot a. That is what we had written. Now, what you do is to substitute for lambda dot. In fact, small trick there, I will write it as a lambda dot, because just a scalar quantity, lambda dot is a scalar quantity. So substituting, sigma dot, lambda dot here, this becomes D epsilon dot minus a a transpose D epsilon dot divided by a transpose D a plus H. Is that clear and this term can now be written as, a slightly different fashion. How do I write that?

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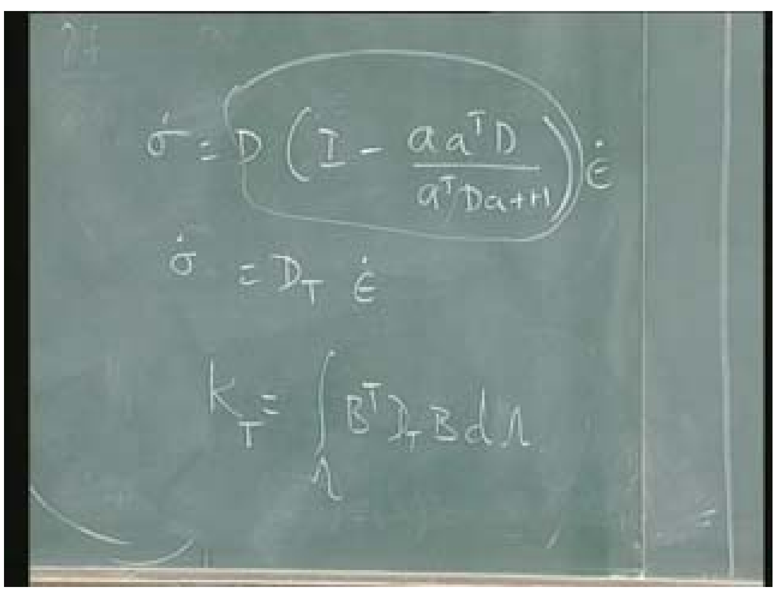

I write this as sigma dot is equal to D into I minus a a transpose D divided by a transpose D a plus H the whole thing epsilon dot. Look at that terms carefully. Does it ring a bell, this equation? So, this, this is a very, very crucial equation, so, this rings a bell and say that that is the D_T epsilon dot. Note how we wrote our E_T matrix before. AE by L, we had taken it outside and said that 1 minus, you remember we wrote that, 1 minus, what is it? There are two terms; in similar fashion, one term and another term. We said that with time the stiffness matrix actually becomes less stiff or in other words E_T assumes a smaller value and that the plasticity effects are brought in through this. In the same fashion this, we write this, and that is the D_T. Now, this has to be used where? This has to be used for stiffness matrix. So, K now can be written as or K tangent, tangent stiffness matrix, K tangent is written as B transpose D_T B d omega. Is that clear?

Yes. No; so, is this D_T valid for elastic region? In the elastic region, it is very simple. This will not be there; this term will not be there. So, obviously D will be there. Same way the problem is done, exactly in the same fashion as we did one-dimensional case, there is no difference. You always check whether you are in the elastic region or plastic region. So, how do I implement this? It is very simple. So, I have an element; I had, I had commented on this before, but now I think it will be very easy. So you can, before I talk about more difficult issues, let me finish comment on this. See, you can calculate this by for example numerical integration; gauss quadrature rule.

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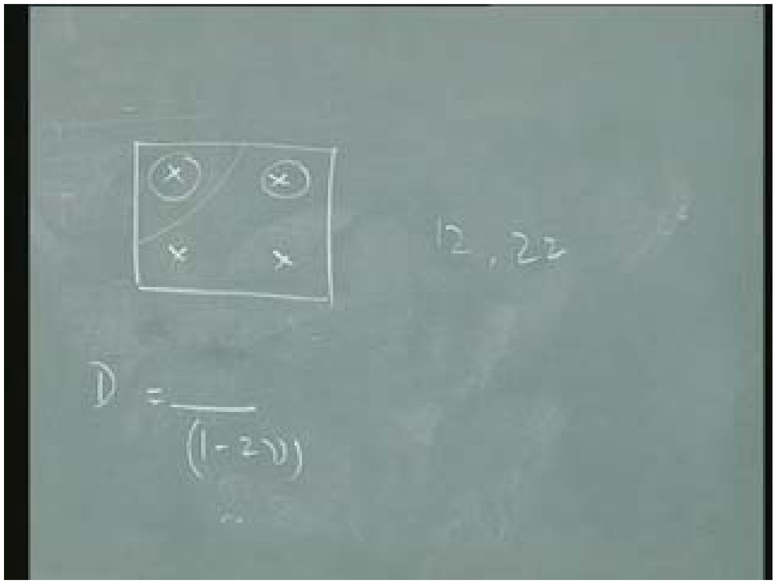

So, what you do is suppose I have an element and I use a 2 by 2 Gauss quadrature rule, which is very common for this, then I calculate at each of the Gauss quadrature point, I calculate this K or in other words, I calculate, multiply this, calculate this and multiply this by a weight and then add this out. It may be possible that all in an element, not all nodes, sorry all Gauss points have yielded. It is possible that one has yielded, other may not or might not have yielded and so on. In fact, plasticity gives rise to jump discontinuity within an element. So, it is possible that a part of the element might be elastic and other one may be in elastoplastic. So, this may be in elastoplastic, this may be in plastic. So, it is customary to hold also in every Gauss point what are called as history variables and one of the history variables that you hold is a yield switch.

You say whether this Gauss point has yielded or not, you will have a yield switch; it is customary to do that. It is also necessary that as history variables we hold plastic strains. So, if you really look at it, every Gauss point has a series of history variables. In small strain case this may be say around 12 and in finite deformation case, it can be 22. So, in every Gauss point, you will have, you will hold 22 variables. So, for every element you will hold 4 into 22. In fact, I am going to expand that in a minute, 4 into 22. So you will, you will consume enormous space in non-linear analysis, basically because of this history variables which have to be kept track of in the, at the Gauss points. Is that clear? It is very important also that you update these history variables; history in the sense that, the term history comes up because it is the one which follows the history of loading of that element. Is that clear?

Having done that, I have to back track a bit and then look at certain problems that may arise at this place. What is the problem that is going to arise? Can I use it straight away? Yes, certain problems you can use it, but certain other problems you cannot use it. What are those problems where you can use it and where I cannot use it directly? Exactly, so what happens? You know we, we were told in our earlier classes that the Poisson's ratio in the plastic region happens to be 0.5. We were told that this is because of incompressibility of the material, but I already commented that the material here is nearly incompressible, because the elastic strains or elastic part of the strains are not incompressible. So, in other words, in the elastic region you have only Poisson's ratio of 0.3, in the plastic region you have 0.5. So, if you consider plastic strain, strictly it is 0.5, but if we consider elastic strains it is only 0.3. That is what we meant. So, it is nearly incompressible material. So, what does this incompressibility give you? What are the problems that it gives?

Remember that our old D matrix, go to my previous, you know classes on linear elastic case; you will see that in D matrix we had a term, outside that big matrix we had a term, which involves 1 minus 2 nu. Now, if nu goes to 0.5, you will see automatically that D matrix goes to infinity. What really happens when the D matrix increases? When the D matrix increases tremendously, the stiffness matrix K also becomes highly ill-conditioned with the result that you have a condition called mesh locking.

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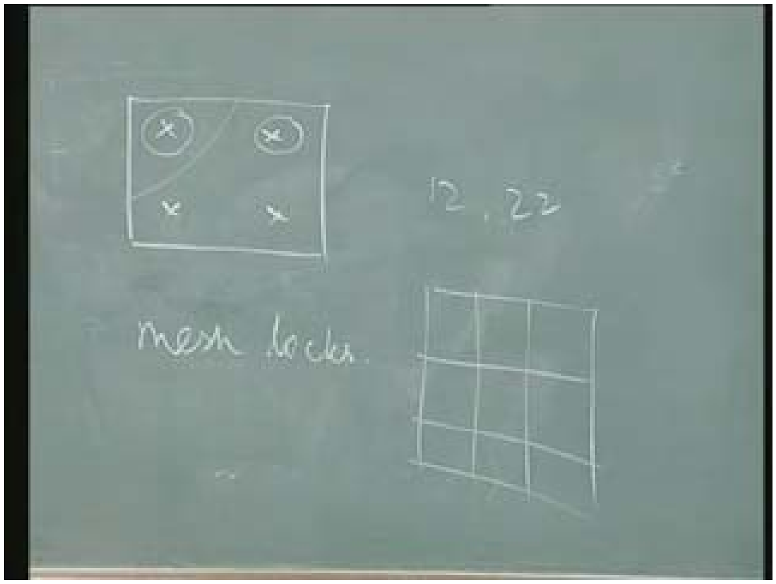

There is no deformation, there is a propagation of stiffness and so the mesh locks. Under this condition, if you actually do a problem you will see that there will not be any displacement of this whole mesh. This is typical not only in, in this kind of elastoplastic problem but also in incompressible elasticity, where say for example, if you are analyzing rubber you will again have the same problem where the material is incompressible or nu is nearly 0.5 and you will get into difficulty. What do I mean by mesh locking? So, if you look at a mesh, then the mesh just will not move. We will see in more detailed fashion what really happens. If I have a mesh like this and apply the load, mesh will not move or in other words, there will not be any displacement of these nodes. This is called as mesh locking. The whole of efforts of various researchers in the past decade or decade and half has been directed to look at this problem of mesh locking.

How would you now handle this incompressibility condition? Now can I do something here, play with it and then can I remove incompressibility? Not remove, in the sense that, remove the problems of incompressibility. There have been various approaches over the years and this can be broadly be classified as mixed variational approach. I am not going to talk immediately about mixed variational approach, because I need to do some more things before we can talk about that. But, I am going to talk about a very interesting effect that comes up when I do not fully integrate the terms that are involved in the stiffness matrix, but do what is called as selective reduced integration.

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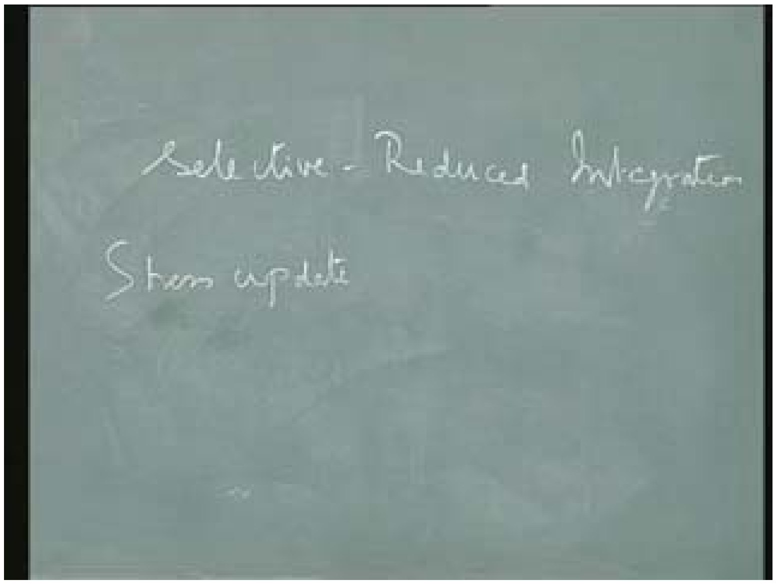

We will just define what it is and then we will continue it in the next class. What we mean by selective reduced integration? We know what full integration is. Now, reduced integration is that we reduce the order; if I, if I need 2 by 2 for full integration, then I need to reduce this order to say 1 by 1. That is what is called reduced integration. We again know all the problems that would come up in the reduced integration. Yeah, I cannot do reduced integration, because that would give rise to hour glassing and other issues. So, I have to use then stabilization procedures and all that. But, in this case what I do is I make use of this problem, problem within quotes, to solve the issue here. So, instead of completely involved in reduced integration, I split these terms that are involved here into two parts and select a part and then do a reduced integration on that part and another part I leave it and do full integration. That is what is called as selective reduced integration.

How is this done, we will see that in the next class; before we close if there is any question on what we did. That is the next issue we have to solve. Once I solve or once I know how to calculate the stiffness matrix by selective reduced integration, the only thing that remains after that for me to do small strain plasticity is my stress update. So, once we learn something about, I am not going to completely cover selective reduced integration; that means we will be deviating away from plasticity, but I am going to, I am go to talk about only one popular technique and then we will go to stress update and finish small strain plasticity. Is that clear? Please revise what we, what we meant by numerical integration and isoparametric element formulation in the next class, so that what I am going to teach on reduced integration will be useful to you. So, we will meet in the next class.