Advanced Finite Element Analysis
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Lecture – 6

Yeah, in the last class we were talking about what we called as stress update algorithm.

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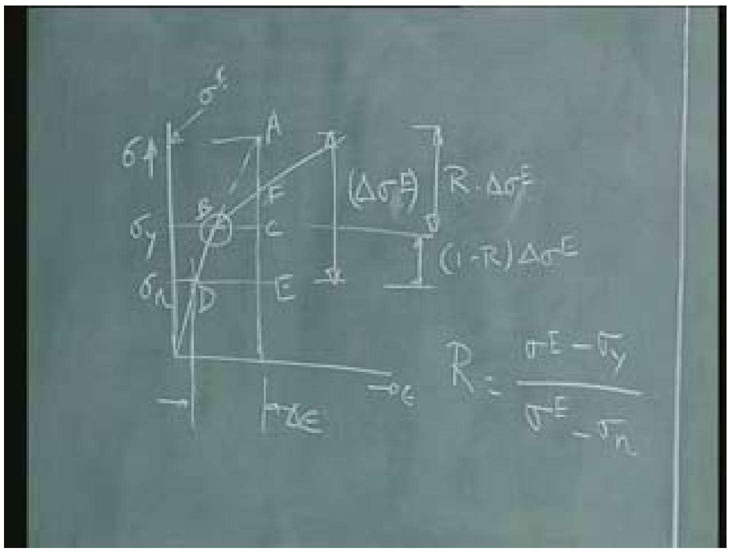

Let us quickly recapitulate what we did. We saw that there are number of conditions. There are four conditions which may be possible as we move from one step in the iteration to the next and said, we looked at a general category where we were moving from the elastic case to an elastoplastic situation. This is where we stopped in the last class. One of the most important things that you see here is that I calculate where I hit that yield point. In other words, I recognize that I have crossed that yield point and I calculate where I hit the yield point. So, this is going to be a major step when I extend or expand this to the multiaxial case, in which case I will have a yield surface and I will also look at where I cross the yield surface. So, that part of the stress update or in other words that part of the correction to the elastic predictor, delta sigma E which is the elastic predictor, was written as 1 minus R and that part which has to be corrected is written as R into delta sigma E.

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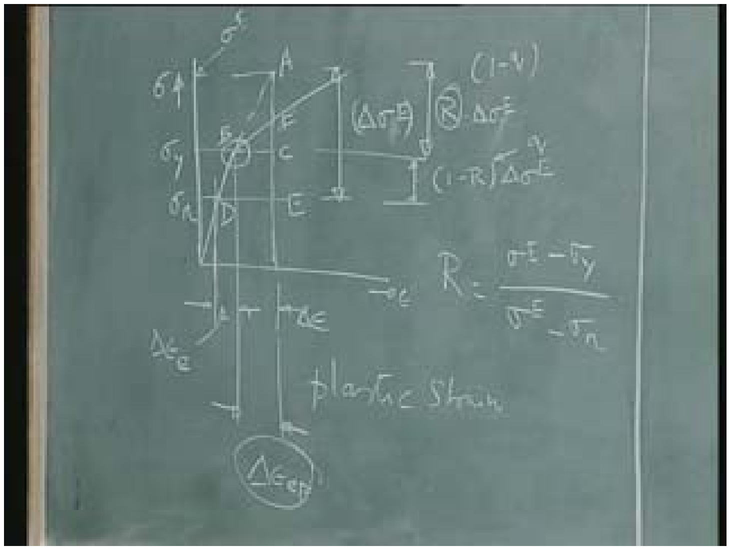

Various algorithms call this as 1 minus R or 1 minus Q and this as Q and so on. It does not matter, we may also shift this 1 minus R to Q and R to 1 minus Q and so on, but it does not matter. But, as long as you understand that of the total predicted stress, I have to correct a part of it. Now, having understood this and having understood how you write down the updated stress, there is a very important quantity which requires attention. What is that quantity? That quantity is the plastic strains. What I have essentially done is to look at the total strain and from where? I said we can also separate out the elastoplastic strain. May be, we call this as delta epsilon_{ep}, ep subscript. So, this is the elastic part; of course, you know that. That is the elastic part, delta epsilon_{e} and that is elastoplastic.

We are interested to now see what is the plastic contribution to this elastoplastic? Elastoplastic in the sense that it has both the elastic as well as plastic strains and what is the plastic contribution to this? Why do we need that? Why, why do we need to calculate plastic strains, because I have to update the surface or in other words, in this case the yield strength of the material itself. So that makes it important and let us see how we calculate delta epsilon or delta epsilon_{ep} and that is my next step. Let us see how we do that.

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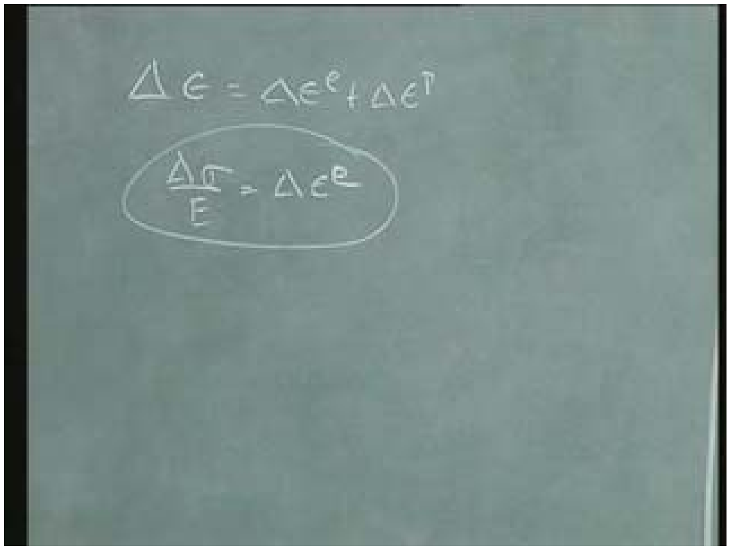

Let us say that delta epsilon can be written as delta epsilon e plus delta epsilon P, which we did right in the beginning and then write delta epsilon e in terms of delta sigma and Young's modulus. Note this carefully. This is confusion for many people that even in the elastic region, sorry even in the plastic region, my elastic strains are written in terms of e. This is very obvious, very obvious when you consider that when I unload from any point in the elastoplastic region, I unload it in the elastic mode and hence the elastic strains, even in the elastic region is given by that. Is that clear and that brings us, in fact let me comment a bit, I think may be you know it, but it is very important to comment, because people usually get confused that there are elastic strains, increase in elastic strain in the plastic region also. Please understand that elastic strains do not stop growing, do not stop growing when you reach e, but it keeps on increasing even in the elastoplastic region.

Fantastic question; so, what does this physically mean? Because, the physical mechanism, the phenomenological mechanism, behind it are different for the elastic strain and for the plastic strain. What happens or what is the reason for plastic strains or what mechanism is responsible for the plastic strains? What is the mechanism? Slip, yes slip; dislocation motion. So, there are what are called as defects in the material and the defects that are responsible for the plastic deformation is called as dislocation. There are of course other defects which help us, which help us; note that defect does not mean all the time it is not good, which help us in various other technological pursuits like for example ……, which is very important for diffusion and so on. But, here is a case where there is a defect called dislocation and you know from your earlier material science courses that dislocation is responsible for or the movement of dislocation is responsible for the plastic strains.

Now, what is responsible for elastic strains? What is responsible for elastic strains? What is that? Strength or the, this is called lattice distortions. Actually when you subject a body to strains which are less than the plastic strains, in other words, where noticeable movement of dislocations are absent, note that word carefully, noticeable movement of dislocations are absent. That is why you always have a ……. stress 0.2% to define yield; then you see that you have only lattice distortions. In other words there is a tug of war between two atoms. So, one hand you are trying to pull them apart, on the other hand they would like to come to an equilibrium position. So, you distort the lattice by applying the load and when you release the load the atoms come back and stay in their designated position, the lattice positions or in other words, the lattice distortions go to zero. So, on one hand you have elastic strains which are due to lattice distortions, on the other hand you have plastic strains due to slip. They are two different phenomena and they are not linked.

So, inspite of the fact that when I reach the yield point, my slip starts and starts growing or in other words the dislocation starts moving. My lattice distortions do not cease. They do not cease; they also take place along with my slip. So, when both phenomena are active in this elastoplastic region I have both elastic strains and plastic strains. Does it answer your question?

Student: when there is, when there is no, when there is slip there in the material, so plastic strain, the elastic strain recover. I mean those lattice distortions they will not be there. I mean those….

No; what we are trying to say is that when there is slip, when there is dislocation motion, lattice distortions too take place. Lattice distortions are independent of the dislocation motion.

Student: But, that is due to the forces that we have. We are splitting each other.

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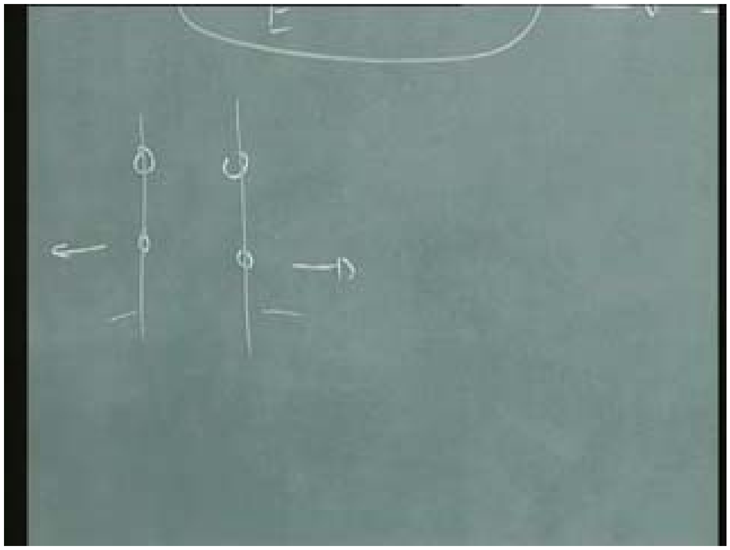

Yes; forces that exist between the atoms that is responsible for lattice distortions so that also take place so there is even though two atoms sitting like this, these are the atoms. When you, when you apply forces so, these equilibrium positions are disturbed and they go out. That takes place apart from total rearrangement of bonding in certain planes which are slip planes, so that one pair of that extra pair of that atom goes to the next position. That is a different thing, different story altogether. So, this lattice distortion keeps happening and they happen to all the atoms. That is the reason why we have elastic strains. Clear, fine.

Let us get back and see how we calculate epsilon P and why do we need epsilon P? I hope things are very clear now. We need epsilon P, so that we can, we can predict the strain hardening parameters. Delta sigma can also be written in terms of what we called as plastic modulus H.

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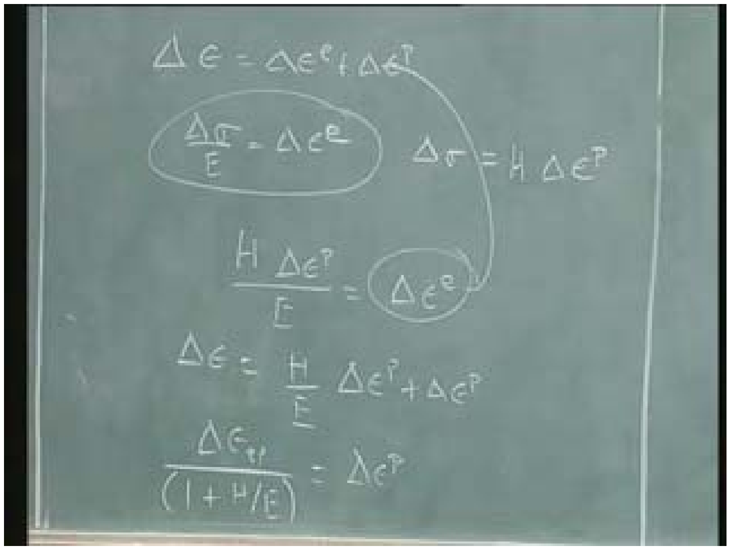

So, that can be also written as H into delta epsilon P. Is that clear? So, I have a very simple relationship, looking at these two equations; I see that H into delta epsilon P divided by E is equal to delta epsilon e, so that when I substitute this delta epsilon here in order to solve for delta epsilon P, I get delta epsilon is equal to H by E delta epsilon P plus delta epsilon P and from which I can write delta epsilon divided by 1 plus H by E is equal to delta epsilon P. But, note one thing carefully. What is delta epsilon? No, that is exactly the mistake most of you make. This is not total strain, elastoplastic strain.

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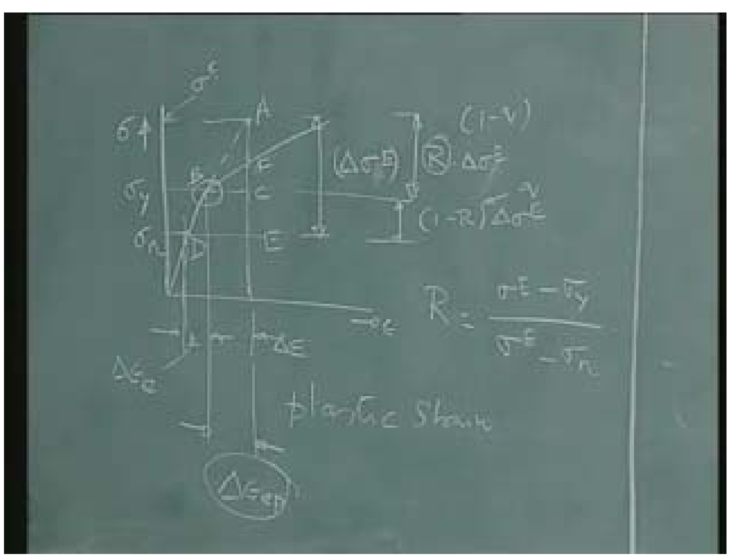

So, note that this is the total strain and this whole thing from here to here, this is the part and of this total, I have to make these adjustments. I have to bring in the concepts of elastoplastic strains only for this part which I call delta epsilon_{ep}. So, what is to be here is only the elastoplastic part of the strain, ep. Is that clear?

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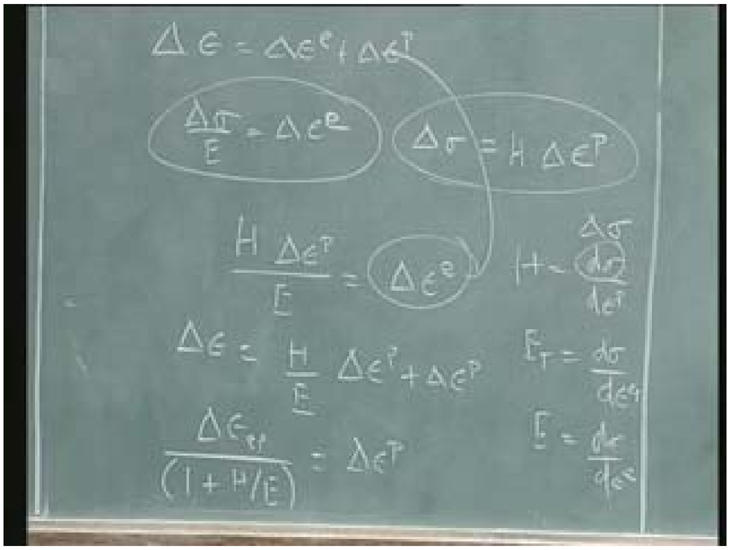

Yes, so these are the two definitions that come about from the elastic part of the strain and the plastic part of the strain, because we had defined the plastic modulus H to be d sigma by d epsilon P from which we get that equation, so together we get delta epsilon to be, yeah, so d sigma is replaced by delta sigma. So, delta sigma by delta epsilon because they are incremental and hence we get, yeah please note, these are the things that we have. These are the relationships we have. E_T is equal to d sigma by d epsilon_{ep} and we have E is equal to sigma by or d sigma by d epsilon e. So, these are the relationships we have, from which we are writing this. These are valid, there is no problems about it. It is quite simple.

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Now, the elastic strain or plastic strain is the sum of plastic strain in the previous case, converged value. If it is naught, in this case it is, there is nothing there and so we have the total epsilon ep.

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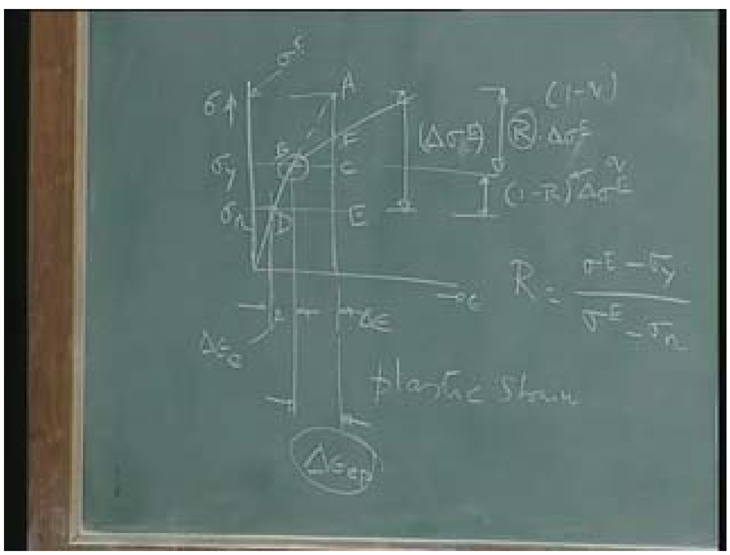

If we had started from an elastoplastic position, we would have done exactly the same thing, but only thing is that my R now should be equal to 1, because all the stress that I had predicted, I have to now correct it. So, R should be equal to 1; that should be …..

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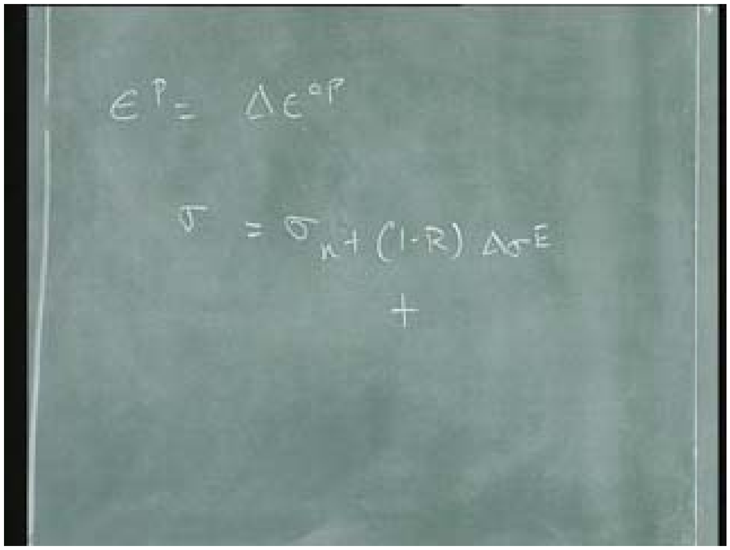

Please note that again you know these are very important points, because these are the basis for research as I told you in the 80's and which would actually give rise to various algorithms and which will be helpful to us also to look at an algorithm and say, yeah this algorithm is good, they take into account these issues and so on. Note these things carefully that when I predicted ultimately or rather corrected from the predicted stress ultimately in the last class, I had put this, the elastic sigma E, sorry sigma_n plus I had whatever I had not corrected; delta sigma E plus what did I do, the last step, last one? Yeah; that is whatever is corrected, R delta sigma E that is whatever is corrected, what is that? That will be equal to E_T into E_T into R into delta epsilon.

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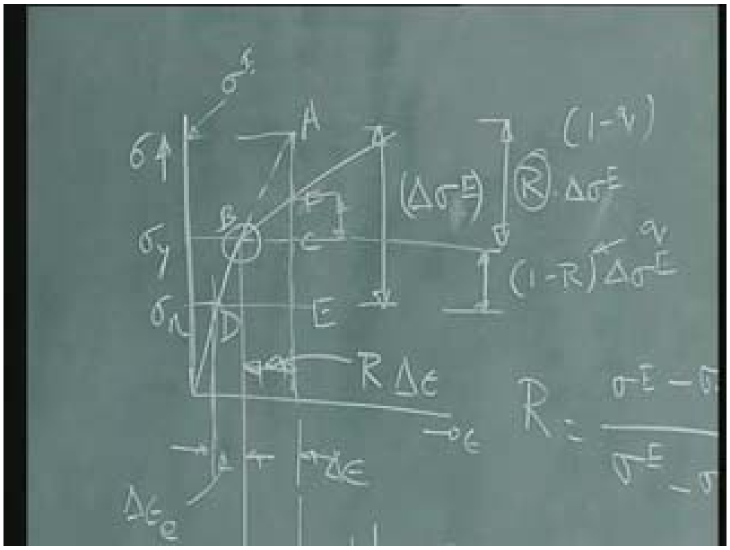

Please note that we had calculated ultimately this part. What was that? Delta epsilon$_{ep}$ and that is equal to in terms of delta epsilon from similar triangles we wrote down and said that that is equal to R times delta epsilon. So, I said that we can calculate this delta sigma. What is this delta sigma now? FC, this distance; from this delta epsilon$_{ep}$ and E_T, exactly.

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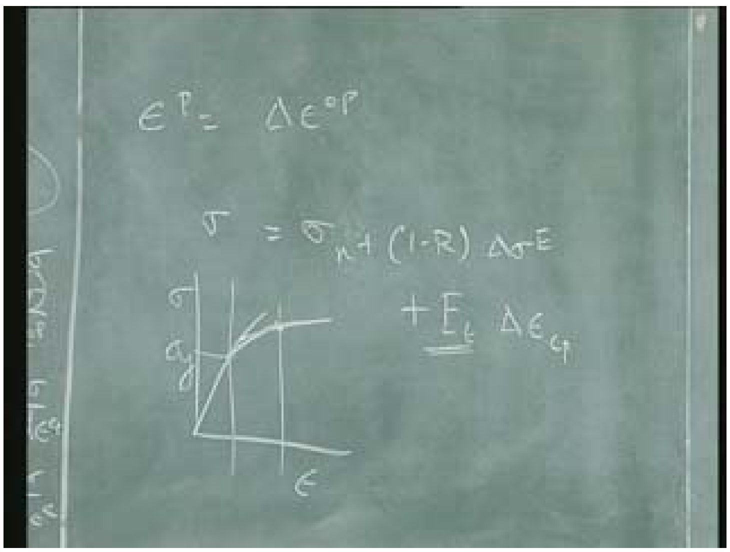

So, we said, in other words, this is equal to E_T into delta epsilon$_{ep}$. In this particular problem we did not have much issues, because we had considered a bilinear curve.

But if this curve is not bilinear, what does it mean? It means that my curve would look like that. That is $sigma_y$, that is sigma, that is epsilon and see that my E_T values are not constant in the elastoplastic region and they keep varying. It may be like this, like this. So, if I have to go to that point, I can have a series of E_T's. Then, you can ask a question what E_T do I use? So, the problem now becomes an implicit problem. You can solve it explicitly by assuming E_T to be at this position or you can solve this by taking E_T at the middle position or you can integrate it by various schemes, various finite different type of schemes, you can use that E_T. Alright; so, this has given rise to the issue of calculating E_T.

What E_T to use has given rise to a number of schemes - an explicit scheme, where I have the knowledge of E_T at the start here or an implicit scheme or a midpoint rule and so on, so many ways of calculating E_T. Is that clear? Note also that this E_T has one more role to play. What is that role? E_T also enters into my stiffness matrix. Remember that for this problem we had written the stiffness matrix to be A E_T by L. So, there is an E_T there, there is an E_T here. So, E_T appears in two places, one in the stress update and another in the stiffness matrix. So people say, from their research, first promoted by Taylor and Semo, in the 1980's, if I remember, about 86 or so people said that the E_T's that you use should be consistent between these two applications. In other words, E_T at the stress update should be consistent with the E_T that you use in the stiffness matrix calculation.

If you do not do that, if they are not consistent, then the Newton-Raphson scheme that you use in order to get convergence will not behave with quadratic convergence. In other words, the convergence rates will be small or will be high, sorry low, so that the number of iterations you require will be large and in fact you may get into even problem of convergence with larger time study.

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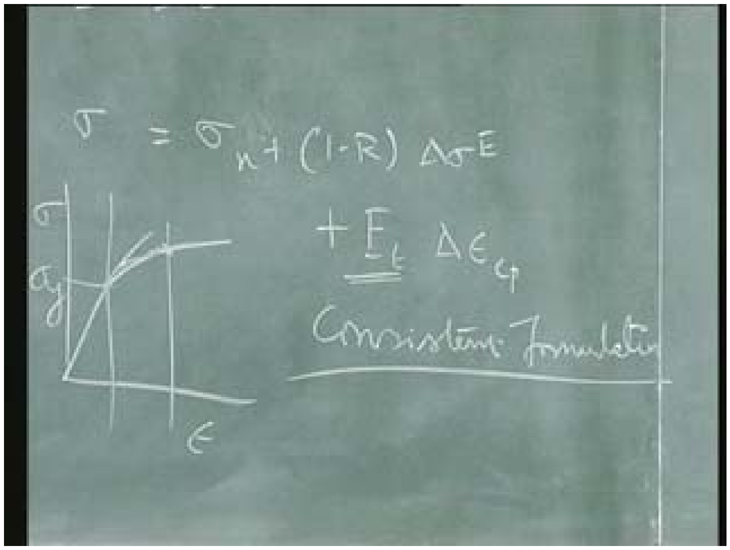

So this has, this E_T business has given rise to what is called as consistent formulation. There is usually a confusion between, you know what is this consistent formulation and what is this continuum formulation? There is usually confusion among students, but I want to state very clearly that consistent formulation means it is just consistent between the stress update and the stiffness matrix calculation. They are consistent that means I use the same E_T. Yeah, yeah, no, no, no.

What consistency means is that I use the same way of calculating E_T, whether if it is implicit same way of calculating E_T here as well as there. Not same value, so the same procedure I use. We will see, we will not, we will not go to the derivations of it, but at least we will indicate when we come to multiaxial case, what we mean by consistent formulation. As against this, there is what is called as a continuum formulation; we will see that also as we go along. Having calculated epsilon P, having updated it and having updated the stress, we go to the next iteration.

We calculate now with the new sigma that we have done. Where is that new sigma that we had done? Here; with this, we calculate the internal forces, look for equilibrium. It is a well established procedure; all of you know it, look for equilibrium, look at the error and then go to the next step and so on. Note that if I am only in the elastic case and moving from elastic to elastic my predictor is the actual value. So, I will have to update the stress only using the elastic constitutive equation and so, I can move ahead in the same fashion without calculating all the plastic part, period.

Yes, so, in other words, the question is that in plastic part what you will do when the elements move? Do you take into account the movement of the elements?

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This is exactly what we said, the difference between small strain and finite strain formulations. When I update my positions of the nodes or in other words the length differences are important for the definition of the strains and the stresses, we go into finite deformation. In other words, the change in length to original length; that original length business is now going to be changed, that there is a difference between the original length and the current length, then I will go to finite or else we will call, we will keep to small. So, whatever we are doing now is only small deformation. We have not come into finite deformation; we will see that later in the course. We will stick to only small deformation.

Having done the stress update and understood all the nuances, it is very important that, I have spent a lot of time here, because these steps are very important to understand my next foray into the multiaxial case. Let us review the theory of plasticity quickly, whatever we require in order to extend, whatever we have done in the uniaxial case to a multiaxial situation.

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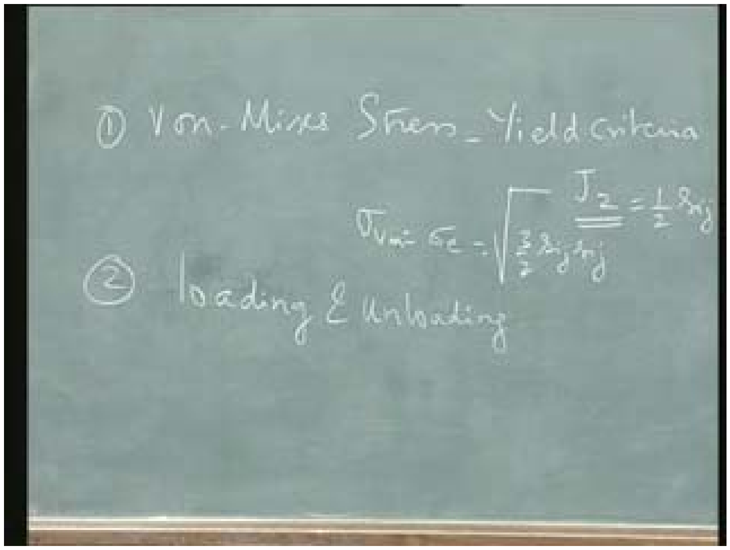

In the previous course, you had already seen what is called as Von-Mises stress. If you remember, Mises stress was the one which you used against the yield strength of the material to see whether yielding takes place or in other words, Mises stress combined with the factor of safety has to be less than the yield strength of the material is what it used.

Actually what you had done essentially when you did that was to look at what is called as an yield criteria. In other words, Mises stress came out of an yield criteria called Von-Mises yield criteria. If you remember that the yield criteria was a function of the second invariant of the deviatoric stress tensor J_2 and J_2, we had defined as half s_{ij}, half s_{ij} and the Mises stress vm, which we will call as the equivalent stress is equal to root of 3 by 2 s_{ij} s_{ij} is what we derived. We will keep it at that. We will come back to the other yield criteria later in the course, but we will say that we have accepted and we have understood Mises yield criteria and see what the other things are that are required in order that we can model elastoplasticity in more than one dimension. Is that clear? So, what are the other things that are required?

One, of course, you require the yield criteria. The second one that you require, look at the 1D case and you can easily see what the other things are that are required. Look at the 1D case. So, we require loading and unloading criteria, loading and unloading criteria. Why does this become important and is slightly different, this multiaxial situation that means in 2D or 3D, why is that? Basically because, when I go to these 3D, three dimensional case, I have six different sigma's and I cannot decide on unloading based on one value of stress, as I had done in the previous case. I can reduce $sigma_{11}$, I can increase $sigma_{22}$ or $sigma_{33}$ and so on and ultimately, I may load or unload or in other words, combination of these sigma's can either give rise to a loading case or an unloading case. So, a proper definition of loading and unloading is important.

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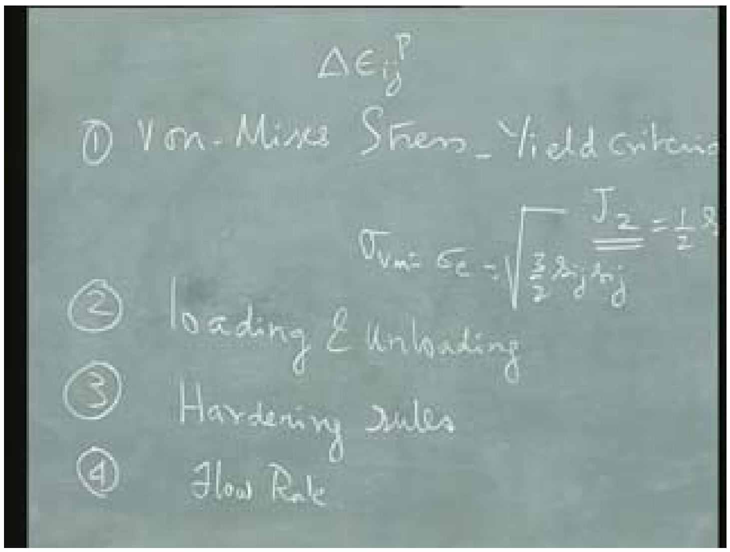

The third thing you look at, as you had done in the one dimensional case, is what is called as hardening, hardening rules. We will come back to that in a minute and look at more important things on hardening and fourth thing or fourth issue that did not figure to that extent in the one dimensional case, which is important for us to calculate the individual plastic strain components; here I had a beautiful H, so I was able to calculate delta epsilon P, but in a multiaxial case, I need a proper rule to calculate individual components of delta epsilon P. What are, what do you mean by individual components? That means that delta epsilon ij P, delta epsilon ij P that is delta epsilon 11 P, 22 P and so on. I need a rule to calculate these quantities and I have to resort to what is called as flow rule or normality flow rule.

The analysis of this whole issue of the plastic deformations can be carried out in a strain space or a stress space. There are lot of, always an argument where we are, what is that we use, whether we use a strain space or a stress space? There is one school of thought that which says that strain space is better, there is another school of thought which says that stress space is better and so on. In this course we are not going to look at the nuances of stress and strain space. We are going to look at the classical flow rule, which I will explain later, may be in the next class we will see what it is. May be, we will pass one or two comments on the importance of the flow rule, but before that let us understand what this loading and unloading is. So, we will come in this order to look at this.

We will, we will look at the phenomenological issue of the hardening and we will see how we can make some modifications to the yield criteria as well or the definition of stress as well with hardening. What is this loading and unloading criteria and why is that it becomes important? What is this yield criteria now?

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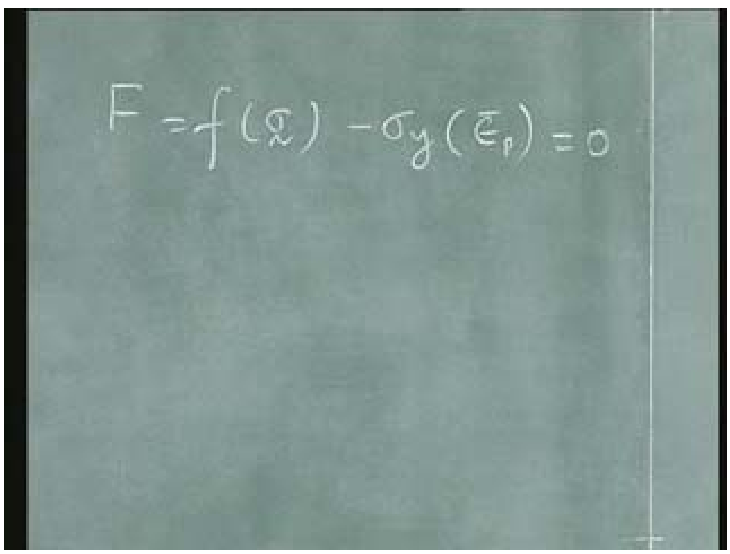

If you go back and look at the Mises yield criteria, Mises yield criteria can be written as say capital F consists of a function of sigma; when I put an under score as squiggle here which means it is a tensor and it has one term which is the stress term, related to the stress term, which is nothing but my J_2 terms there, minus sigma_y. We will see what this epsilon bar is in a minute, epsilon bar P is equal to zero; simple way of writing a Mises yield criteria. On one hand, you keep increasing the stress. That is this guy is going up. So, as you keep increasing the stress, the resistance of the material also keeps increasing on the other hand. So, both of them compensate one another, so that the result of addition due to the stress and its ensuing material resistance is equal to zero. So, this is very important. Now, what does this indicate?

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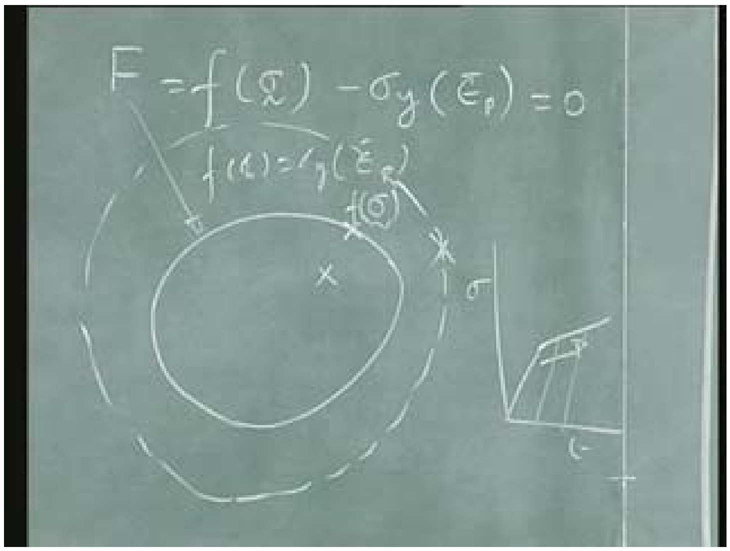

This indicates that in six dimensional space, we can imagine that there is a surface. Of course this is not six dimensions, it is just a schematic way of representing the yield surface. So, this is called as the yield surface. So, you will understand immediately that this is the yield surface designated by F and that points will lie either on the yield surface, which means that this is a combination of the function of sigma's, will be on this yield surface. This is the stress space, it is a function of sigma and this gives down or in other words, if I want to write down the equation of this, it is nothing but f sigma is equal to sigma$_y$ epsilon bar P, you can look at it like that. A point may lie on the yield surface or it will lie inside the yield surface. A point cannot lie outside the yield surface.

Why is that, because what is, what is meant by outside the yield surface? That means that you are increasing this part of the equation or you are putting in external forces to increases the stress. But, the beauty of the material is that it responds with you and so this part also should expand in the same fashion or to the same extent as the first part of the story. So, when I now increase this point to this point through an increase in this small f sigma, this would also increase such that I have now a new yield surface which is given by that particular dotted surface.

What does it mean? It means, simply means, in one dimensional case that as I increase, as I increase my stress, I will move in that line. I will follow that line and that line is what I would say that yield point keeps on increasing and that is what it means, it means that in the 3D case that there is an increase in the surface.

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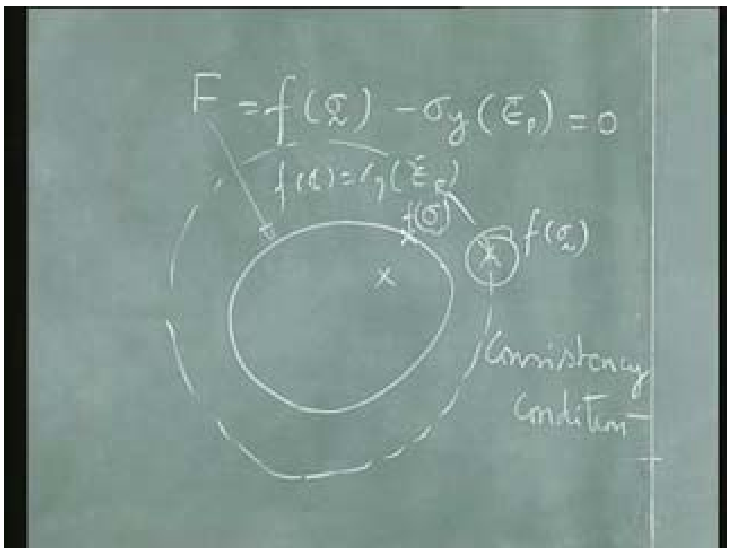

So, this point that the stress, f sigma, function of sigma should lie on the yield surface is called as the consistency condition. So, this is called as the consistency condition. How the yield surface actually deforms, we will see it in the next hardening rule, let us just wait for some time.

Yes; yeah if there is an outside, can a point lie outside the yield surface? A point cannot lie outside the yield surface at all, because it is, what we are talking about is what is called as the rate independent plasticity, which means that when I increase the loading that means that I increase or change the first part of the equation. The material will respond with me and that results in the corresponding increase here, so that any increase here is compensated by an increase in the second part so that the sum of it will always be equal to zero. So, points can lie only on the yield surface. So, this is what is called as consistency condition. On or inside, of course; we will see, when it is inside. We are only loading it now.

Now, there can be a question. This is fine for hardening. What happens when the material is perfectly plastic? In that case what happens to this point? What is perfectly plastic material? My sigma$_y$ remains the same, my sigma$_y$ cannot increase.

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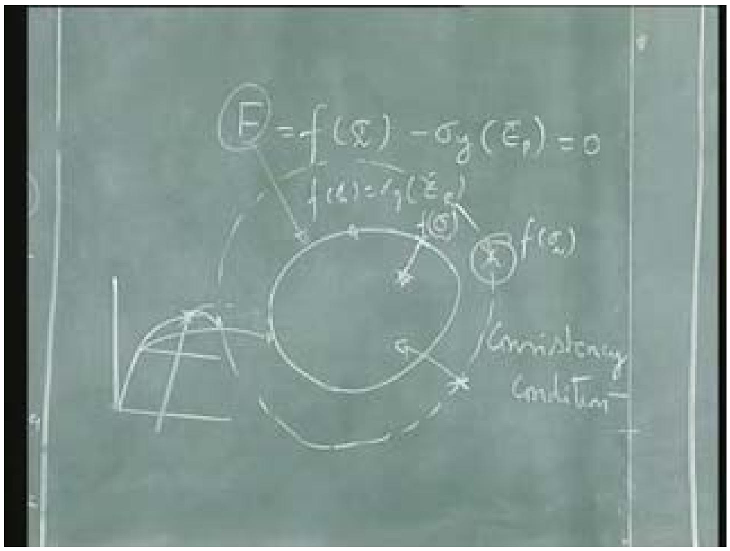

So, if I have a stress strain curve, then my stress strain curve will be like this. From the point of view of yield surface, what does this condition state? This condition states that the surface cannot expand, the surface cannot expand or in other words, this guy here to the right hand side, he cannot keep increasing to meet the challenge of this guy. So, whatever you do, whatever you do to the loading actually makes the point move from one point to the other in the original yield surface.

In other words, even though I load it, even though I load it, I cannot move outside the yield surface and the yield surface also will not dilate, but will remain there and what I will ultimately achieve is to move from one point on the stationary surface to another point of the stationery surface. This is what is called as neutral loading. Of course, the last category is that I can bring that f in such a fashion that I can bring the point inside or I can unload making this capital F to be less than zero. Obviously, when I reduce this, this f which is called as unloading, whatever has happened to the material, the response of the material does not revert and is left as it is and that is why when I have this picture and when I unload, my yield point remains there.

The analogous situation in a multiaxial case is that this depicts the yield surface, sorry, rather this point, this yield surface, this depicts that yield surface. So, after I come to say this point on the new yield surface, when I unload or reduce this f, I will go back into the yield surface, but will not take the yield surface along with me because, what has happened, has happened. It is an irreversible, dislocation motion and slip are irreversible phenomena and so I will leave the yield surface there as I had left the yield point here; I will commit. This is what is called unloading.

You see that there are three types of things that are possible. One is loading, another is neutral loading and the third is unloading. Neutral loading is especially important in the case where we have perfect plasticity. Is that clear? Now, let us develop simple equations to say when we are loading and when we are unloading. So, let us come back here. Now, what is the condition I should satisfy in order that I can have conditions for loading? What is that condition, which I have to satisfy? It is a consistency condition.

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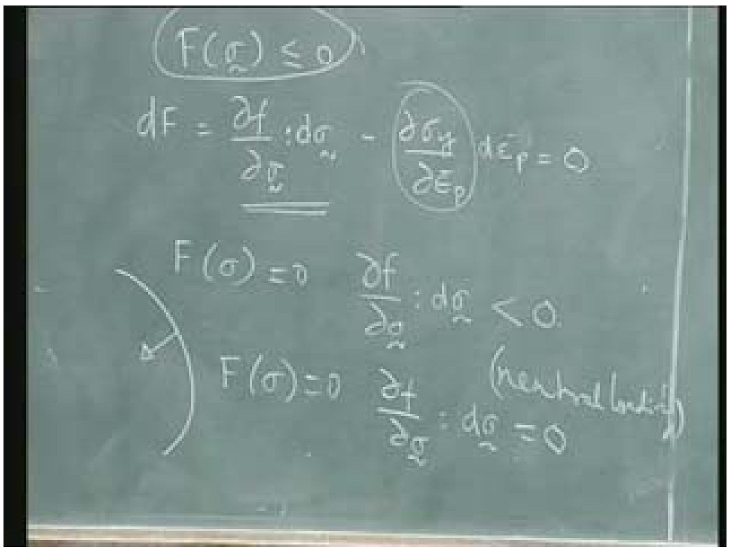

So I can state that F sigma should be less than or equal to zero. So, let me write, what dF is which is equal to zero of course is equal to, there are two terms in my F; look at this, there are two terms. So, dF can be written as dow f by dow sigma into d sigma. Actually it is a double dot product minus this means that dow f by dow $sigma_{ij}$ into d $sigma_{ij}$; this is what I mean by this double dot product; it is a contraction operation, you know that from the first course and the second part is that dow $sigma_y$; that is the second part dow $sigma_y$ by dow epsilon bar P, d epsilon bar P that is equal to zero. Epsilon bar is what is called as equivalent plastic strain and we will define it in the next class, we will do that in the next class.

Now, this d $sigma_y$ by d epsilon bar P is actually the one which gives rise to my hardening. So, when do we say I am unloading? Very simple; when I have this guy to be less than zero. So, when I am, when I have F, say, let us look at the situation F sigma is equal to zero and dow f by dow sigma d sigma is less than zero, what is my condition? What is my condition? So, with F sigma is equal to zero, I am already in the yield surface. I am in the yield surface and I am now unloading. Is that clear? Exactly; so, I will not be on the yield surface, when I do that. So, if this is the yield surface, when I have unloaded it, I would have come in leaving the yield surface. So this, I am reducing this part, the first part of the story, now, this fellow would have come in. So, that is why dow f by dow sigma is less than zero.

In other words, F, look at this carefully; F can be less than equal to zero. Now, when I unload, what would happen to F? It would have become less than zero, because this has reduced now. So, this is reduced. This will remain the same and hence I am in the unloading path from an elastoplastic case, common to whether we call this as perfect plasticity or hardening elastoplastic.

What happens in the perfect plastic case? When I have to load it, obviously my F is not going to increase. So, when F is equal to zero and dow F by dow sigma d sigma should be equal to zero, then what is the condition I have? Loading or in other words, called neutral loading, neutral loading in the perfect plastic case. So, what essentially we are doing is to pick up an analogy between the uniaxial case and the multiaxial case and trying to express conditions, because I have to calculate these conditions for my stress update also. Also for elasticity point of view it is important, there is no doubt, but from our point of view in order to develop a numerical algorithm, it is important that I understand these steps. So, what happens when I have elastoplastic case?

What is the loading situation in an elastoplastic case or unloading situation? First, let us look at the unloading situation. Unloading situation is the same, so, that condition remains the same. Now, what is it for loading?

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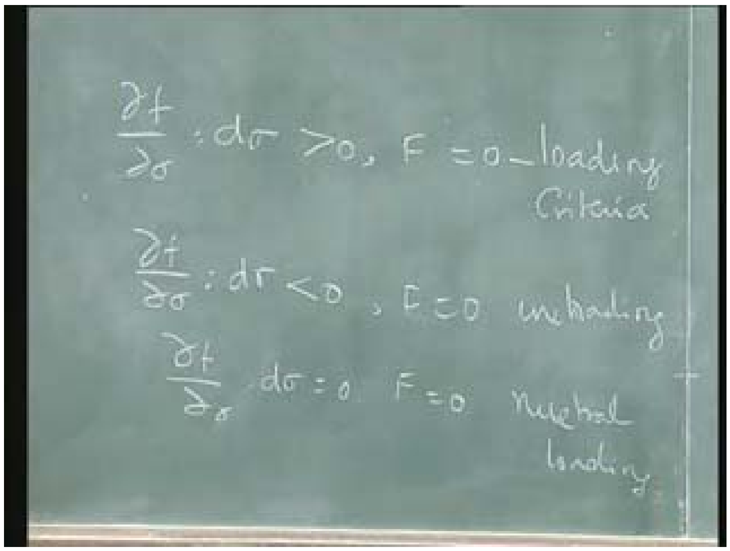

So, dow f by dow sigma d sigma double dot d sigma, when it happens to be greater than zero along with F is equal to zero, we call this as the loading criteria. But, here we can have three cases. In fact, one is loading and this is unloading and I can also have neutral loading which means I just travel on the yield surface when dow f by dow sigma d sigma is equal to zero and F is equal to zero; I have neutral loading. We will stop here and we will continue with hardening rule and pick up some niceties and whatever we have defined, in the later class.