Advanced Finite Element Analysis
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Lecture – 4

In the last class, we were looking at the elastoplastic behavior and we were determining the tangent stiffness matrix.

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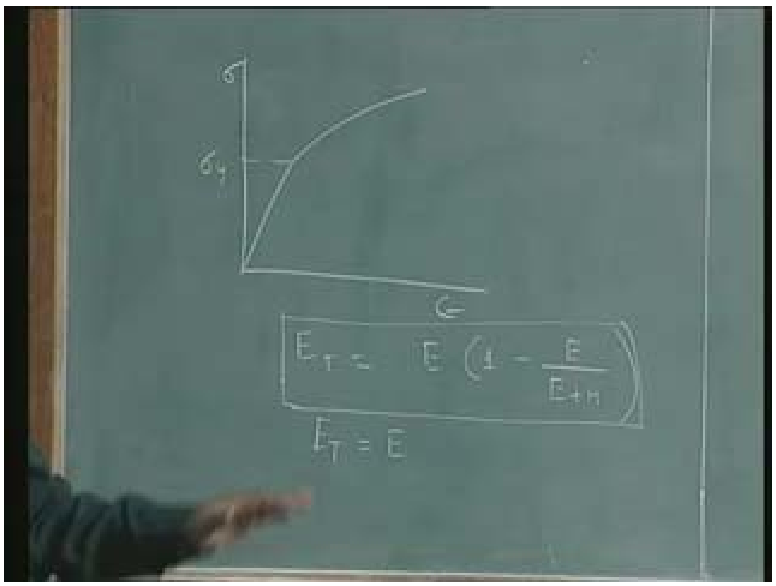

So, if there is any question let us discuss, there is any issues here let us discuss them. Yes.

Student: suppose that the material is under both in the elastic and the plastic region. Just we are applying the load. It is in the region where the curve changes.

Yes, in other words this equation is valid in the elastoplastic region. Note that this equation is valid in the elastoplastic region. Obviously, since we are going to go through an incremental procedure, we will pass through the elastic region and then go to the plastic region. So, in the elastic region this equation is not valid and we will use only E_T is equal to E; we will use only E_T is equal to E. You will see this may be in the next class or after that. After that we will develop a procedure and you will see that we will constantly be monitoring whether yielding has taken place or in other words whether we are in the elastic region or in the plastic region.

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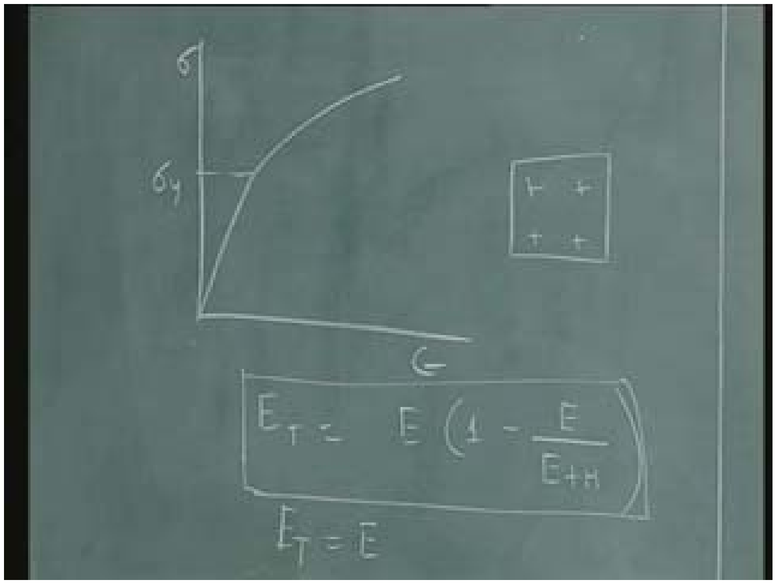

To preempt some of the things which we are going to do later, you know that there are what are called as Gauss points in the elements; there are what are called as Gauss points. Yes, that is used for numerical integration. Numerical, these Gauss points which are used for numerical integration have certain internal variables or history dependent variables as a part of the program. So, if you look at say for example, if I use 4, then I will have 4 different arrays per element of history dependent variables stored.

One of them can be called as yield switch, one of them will store the plastic strains; one of these internal variables. One of them will store the yield strength and so on. So, we will be monitoring continuously as to where the element is or in other words where the Gauss point is with respect to the stress strain curve. So once we know that for this Gauss point the material is in the elastic region, then we will not use this equation, but we will use E, go back to the stiffness matrix for the elastic case. If this say, for example it is possible, it is quite possible that this may be in the elastic region and this Gauss point may be in the plastic region in which case, when I calculate the stiffness matrix and when I come to this Gauss point, I will use E_T and when I come to this Gauss point, I will use E. Is that clear? Or in other words, there is going to be a jump even within the element. So, this causes some of the mathematical issues, we are not going to discuss that. But, this is very important implementation issue. That is why in all of the non-linear course, we have what are called as history dependent variables.

We will usually distinguish between history dependent, history in the sense that the path that it has taken has a bearing on these variables. That is what we call as history dependent variables. People distinguish between this history dependent variable and history independent variable that does not change as we go on in the solution of the problem; it does not change. It remains a constant. That is called as the history independent variables. We will see more of it as we go along in this course, but it is very important to realize that I will use appropriate things at appropriate places.

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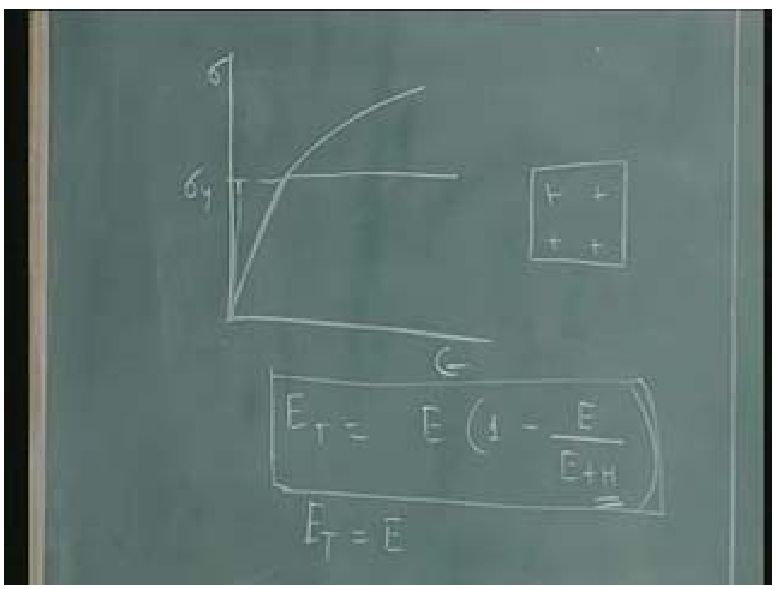

Yeah, the next question is that what happens when H goes to zero? H goes to zero, what happens? The stress strain curve, in fact you can see that when H goes to zero this guy becomes 1, so, E_T becomes zero or in other words stress strain curve takes a turn like that. This is what we call as elastic, perfectly plastic. You know that this curve is the result of what is called as strain hardening and that this is, this curve neglects strain hardening characteristics. So, we call this as perfectly plastic. In many problems, especially when we deal with forging problems, we may not be interested in the elastic part. Please note that I have exaggerated the slope. Actually the slope will not be as high as that. I told you in the last class itself the slope will be very, very close to the Y-axis and hence in many problems it may be possible to treat the elastic part to be absent, to make an assumption that the elastic part is absent and treat the problem itself as rigid plastic, rigid in the sense that there is no elastic part. In the elastic part it is rigid or the strains in the elastic part is so small when compared to the total strains we can consider that as rigid. So rigid plastic, rigid perfectly plastic combination of this and rigid; rigid means it just goes like that and so on. These are the assumptions that we make in certain problems and this, of course, this judgment is very important and comes from our engineering knowledge. It of course depends upon the problem.

Now, having said this let us see how we derive the stiffness matrix for a simple bar. We said that we should start with ……. for sigma.

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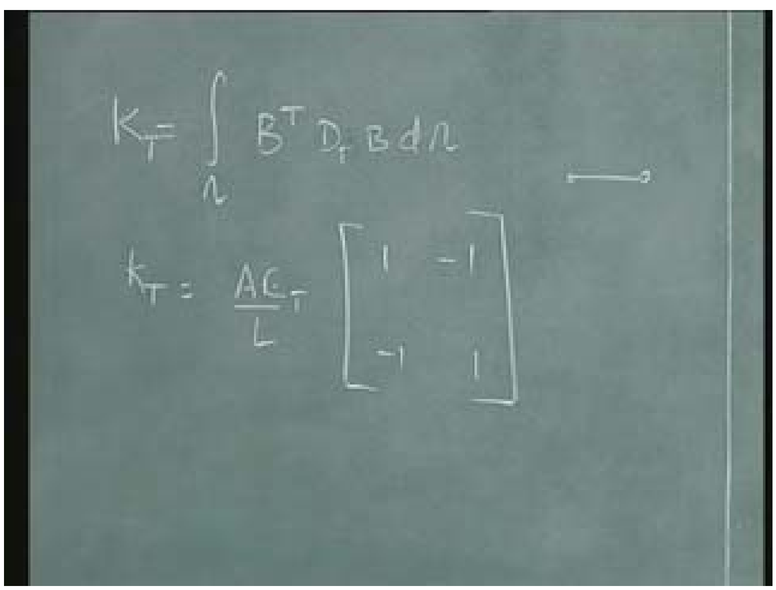

We have already seen that this whole stiffness matrix K can be written as B transpose D_T B d omega. What would happen for a one-dimensional case? Even looking at it you can say that; there is no big problem. Looking at it, you can say that the tangent stiffness matrix for a one dimensional case now can be written as, what was it before? It was A E by L; in the previous case it was A E by L 1 minus 1 minus 1 1. Note that we are just considering a bar element. In the previous case it was like that and now since that D has been replaced by D_T, E is replaced by its corresponding material parameter, E_T, so A is equal to E_T by L into 1 minus 1 minus 1 and 1. Is that clear?

Now having said this and having seen how we look at other issues, let us do a problem. But before that, I just want to bring out one more point which we will elaborate in the next class, that please note that the points, the stress strain should be such that the points which depict the stress strain should lie on the graph.

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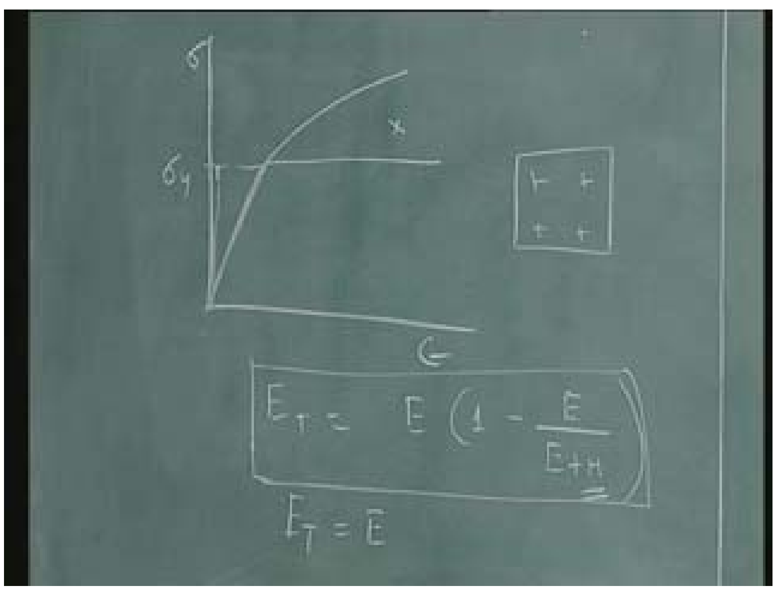

I cannot get a point here, a combination of stress and strain which lies outside or not in this graph, outside the graph; it is not possible. So, the point should lie in this stress strain curve, the graph in the sense that curve rather, the stress strain curve. We have to have some special procedures. We will talk about that in the next class, but let us now do a problem to understand all the steps that are involved in the non-linear procedures.

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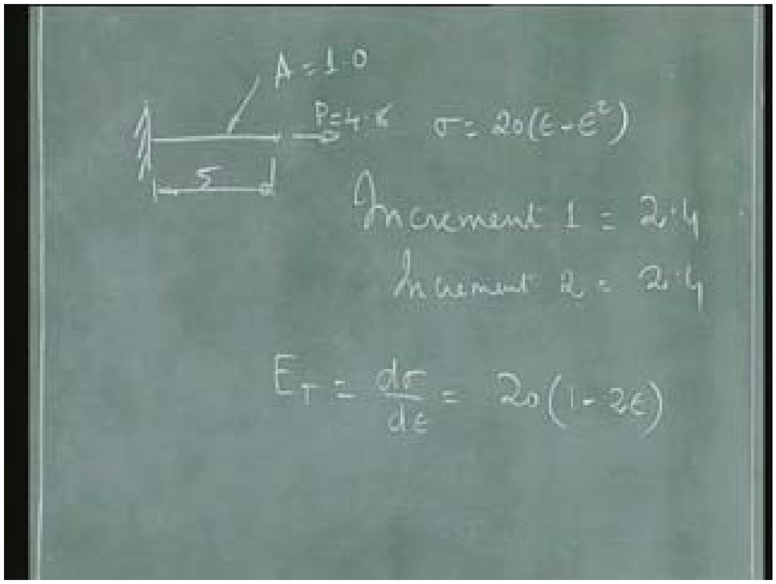

Now, take a look at this problem. This is just a bar, just a bar and the length of the bar is 5, the area is 1, the load that is applied is 4.8 and look at the stress strain behavior; 20 into epsilon minus epsilon squared. That means that the behavior is non-linear and apply the load 4.8 in two increments. In other words, in increment 1, apply a load of 2.4 and in increment 2, apply a load of 2.4 further to the first 2.4; that means that the total load is equal to 4.8. We are assuming two increments, we are assuming. So, the question is how many increments do we assume? We are assuming two increments, 2.4 and 2.4. Yeah, in other words, I have just given it as 2.4. You can, you can, have less as well; 1.2 to be applied in the first increment and so on, it does not matter. Just because I do not want to take much time, every increment is going to take time, because I have to iterate in every increment, so, I have just given 2.4 and 2.4.

The first thing that we do in this problem is to calculate E_T. So, what is E_T? Yeah, correct; so, E_T is equal to d sigma by d epsilon so that is equal to 20 into 1 minus 2 epsilon. So, obviously now, E_T is a function of epsilon. You can see that E_T is a function of epsilon, so, E_T has to vary as we go along in this curve. You have all your calculators with you and so, we will start the problem. You already know what is the stiffness matrix? The only advantage here is that, if I choose one element, if I choose just one element, then I have from the stiffness matrix one row and one column eliminated. So, I will have just one value, in order that I should solve the problem; I mean in order that I can simplify the problem, so the solution becomes easier. I can choose two elements, it does not matter; but, only thing is it is going to take lot more time. So, let us choose only one element and eliminate one of the u's.

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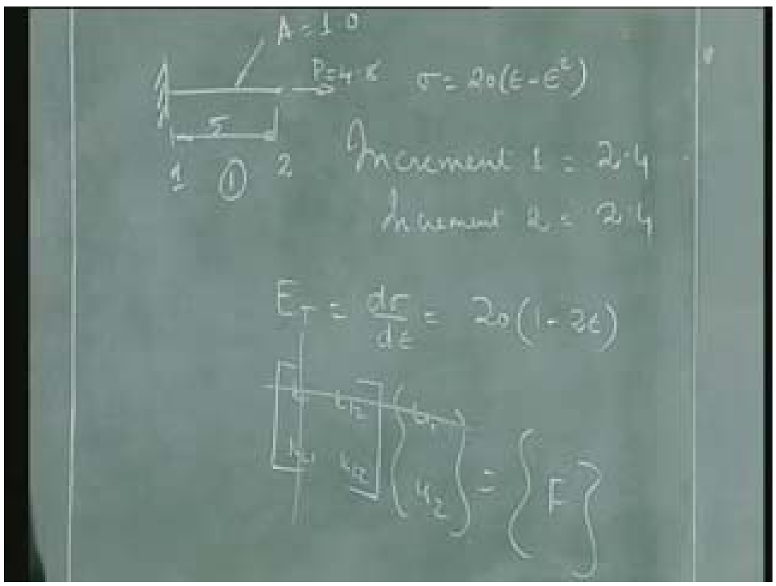

Suppose I call this as element 1 and element 2, element 1 and element 2, I can write down my stiffness matrix to be, sorry node 1 and node 2, sorry not element, not element 1 and element 2, node 1 and node 2 for this element 1, for this element 1, then, I can eliminate one of them because, u_1 is going to be fixed or u_1 is equal to zero. So, those two are eliminated. So, I have a simple equation of the form K_{22} u_2 is equal to f_2; of course, this is equal to F. So, just only one value has to be determined as we proceed in this problem.

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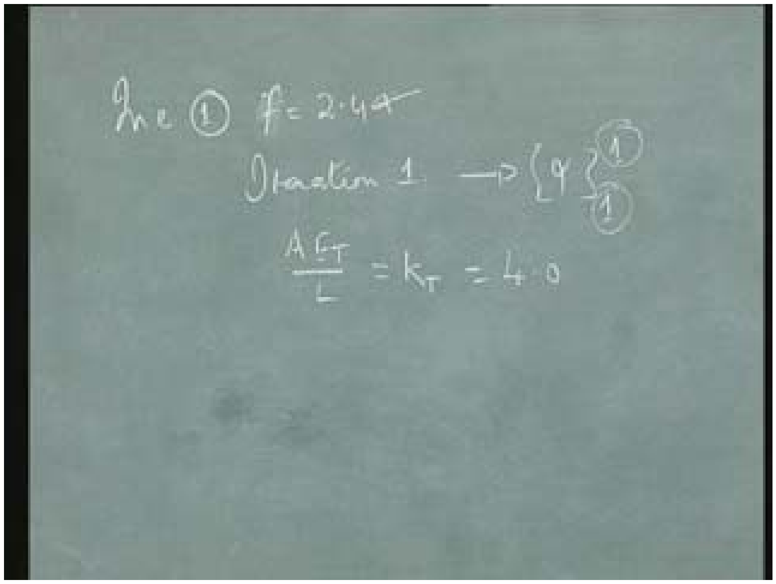

We start now with the first increment, increment 1 with load of course equal to 2.4. Let me go back to my initial symbol of f, because I think to be consistent with what we did, so that there is no confusion. Though load is usually given in all problems as P, let me go back to my previous symbol f, f is equal to 2.4 and we start the iterations now; so, iteration number 1. You calculate first of course A E_T by L and that is going to be my stiffness K_{22} for the first iteration. Please calculate that and that multiplied by u_2 should be equal to, for the first iteration should be equal to, the force, basically because I have to calculate psi. Since f is equal to 2.4, the first increment I have the error psi to be the load itself, because there is no internal force that is acting. So, I start that with the first iteration and increment. This, I will call this as the iteration and that to be the increment.

So, A E, AE_T by L is what I have to calculate. That would give me K_T. What is that value? K_T, because AE_T by L into K_{22}; K_{22} is equal to 1, so, this is going to be 1. This will straight away give me one value of K_T that is equal to, yeah, 4.0. Is that clear? So, that is the first step, calculate K_T.

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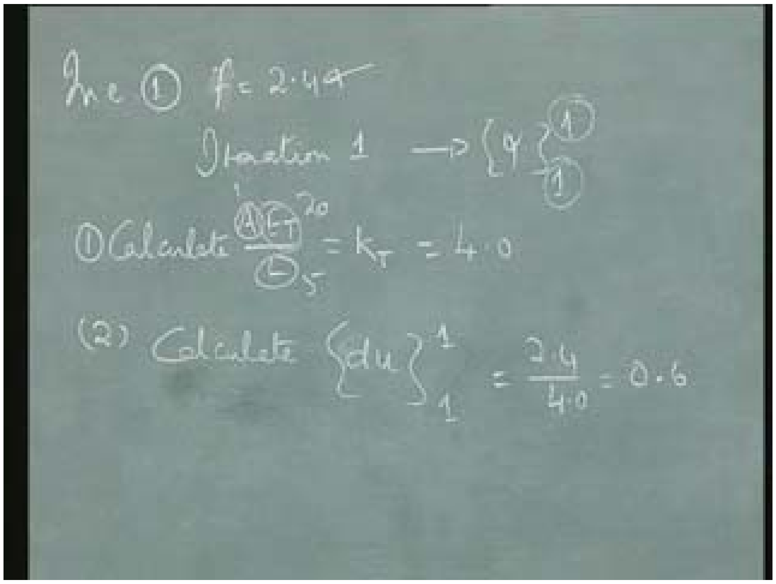

The second step is to calculate delta u. Let me call this as du. I have reason for it, let me call this as du. Yes; I just substituted A E_T epsilon is equal to zero, so this is equal to 20. Please note from here that E_T is equal to 20 into 1 minus 2 epsilon. Epsilon is equal to zero, so I got 20 here. So, 20 goes into this place. This is 20, A is equal to 1, L is equal to 5, so 20 divided by 5, so 4. Is that clear? Calculate du_1 that is equal to my psi. Psi is equal to 2.4; 2.4 divided by 4 that is equal to 0.6.

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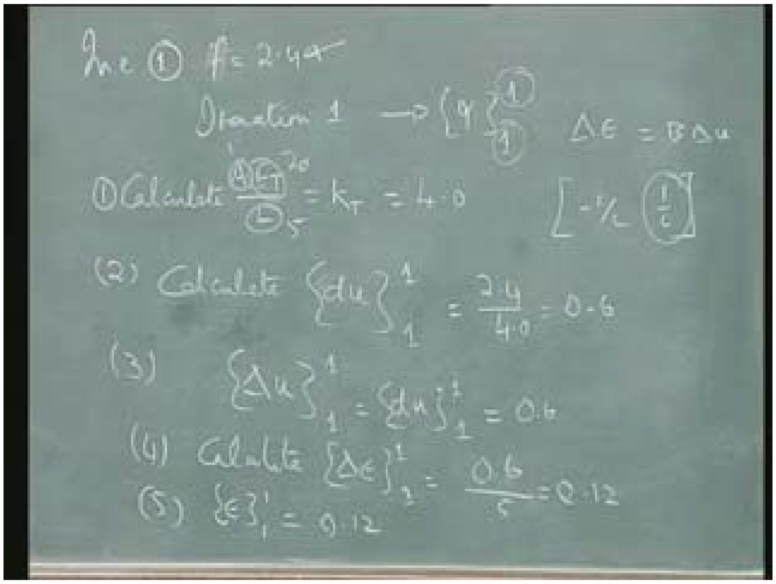

Now, the third step is to calculate the total strain from the total displacement. In this case, the total displacement happens to be that is total displacements in the sense that the total displacement from the last increment which we called as delta u, not u delta u. So, in this case, because it happens to be the first increment and as well as the first iteration, we have that to be equal to du 1 1 or else I have to sum them and put a sigma here. We will do that in the next iteration. So, we have to calculate delta u and this happens to be in this case, 0.6 and from this, I can calculate delta epsilon and what is delta epsilon?

Yes, we get this from B matrix. For this case, the B matrix happens to be minus 1 by L 1 by L. That is the B matrix for this bar element and hence that happens to be 1 by L, so 0.6 divided by L is 5; so, 0.12. So, this happens to be 0.6 divided by 5 that is equal to 0.12. Yeah, so this one I obtained delta epsilon obviously delta epsilon I just write it here separately B delta u with appropriate R and N. So, now this B happens to be the strain displacement relationship and that happens to be minus 1 by L 1 by L and so we have 1 by L there. First one is for u_1. u_1 is equal to zero. So, u_2 is determined from that quantity and hence I have 0.6 divided by 5, 0.12. So, that is the next step.

Having done that what is my next step, fifth step? Fifth step; no, I have not. It is not that we determine K_T. I have to now still determine stress. I have to look at the error and then only I have to decide whether I have to go to the next step. Usually it is a good practice at this point to calculate the total epsilon; now, total in the sense not only for this increment, but total till this increment. I am, I am going to use this total in two sense. Total epsilon means total epsilon for this increment. So, I will call this as total epsilon for this increment, which means that it is a sum of du's and then B is applied to it. That is what I mean by total epsilon or delta epsilon. Then total epsilon for the whole of the analysis till now that is what we call as epsilon. So epsilon, in this case 1 1, that happens to be for this case the same as what it is, so 0.12.

The next step is to calculate the stress due to this, so let me do it here. So, how do I calculate stress?

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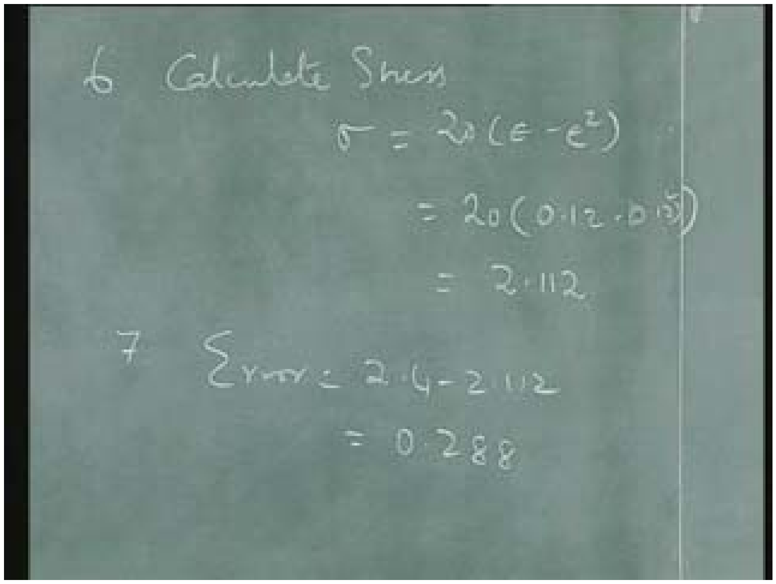

Sigma is equal to, yeah, 20 into epsilon minus epsilon squared and that happens to be how much? 2.112. So that happens to be 20 into epsilon, you put that; 0.12 minus 0.12 squared and that happens to be 2.112. Clear? Have a look at it. See, this part is usually called, this calculation of stress part is usually called, as stress update. This may be a very elaborate procedure, when we come to elastoplastic analysis, because you may not have a beautiful relationship like this and I have to satisfy what I called just now, as a consistency condition; that is the point should lie on the curve. So, this is usually called as the step, this step is usually called as the stress update algorithm. This is the step which is very important and lot of research has gone in to develop various algorithms for the elastoplastic case. Is that clear?

Now of course, the next step is to see internal forces and the error. The internal force in this problem happens to be very simple, because we are considering only one element problem. So, it just equal sigma into A. A happens to be 1, so internal force happens to be 2.112 and hence error is equal to 2.4 minus 2.112 and hence error is equal to 2.4 minus 2.112. What is that? 0.288; we will stick to three decimal places just, so that we do not keep writing it. We are going to do hand calculation, so we will stick to 3 decimal places or else, so, 3 decimal places and then we will see that the error, we will monitor the error and see that if the error comes to 1 into 10 to the power of minus 3, we stop the further iterations.

Now have a look at this 0.288. Obviously, it is better than my 2.4 from which I started. I am not still happy with it, so I have to go over to the the next iteration. So, I start with iteration 2.

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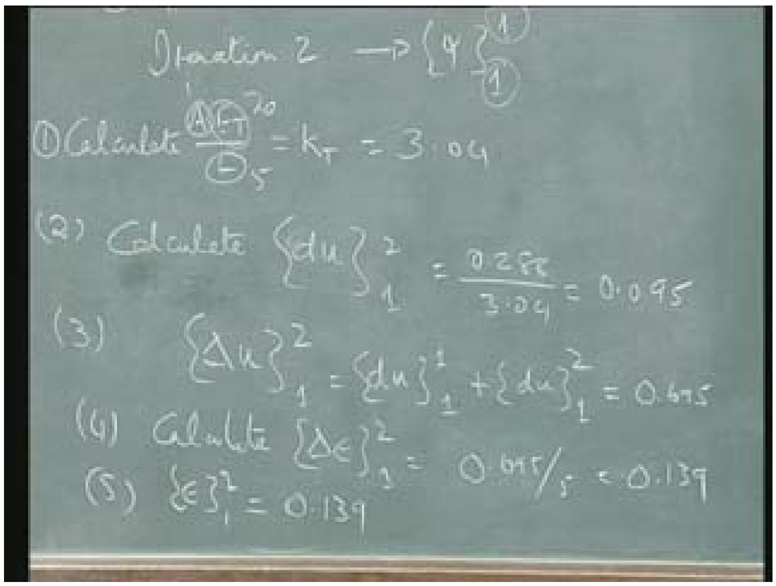

My increment is 1, but my iteration is equal to 2. Is that clear? I have to redo all these steps. Tell me what is now the K_T value? I will just remove this; see what the steps are and see what the K_T value is. In order to determine K_T first you have to find out E_T. Now E_T is, from this type of figure, let me write E_T here. E_T is what is that? 20 into 1 minus 2 epsilon. From this please calculate K_T substituting E_T into my A E_T by L and that happens to be 3.04. Is that correct? Yeah, so that happens to be 3.04. The next step of course, is to calculate same thing. The only thing is that, now I am going to shift this right here to 2. 1 becomes 2 now and I have to replace both these. So, psi happens to be now 0.288 divided by 3.04 and what is that? 0.095; clear, fine. So, now I calculate delta u 2 by adding 1 plus du 2 of 1; 0.695. It is always a good practice to calculate delta epsilon from delta u.

Note this carefully. There is a small subtle point here that we calculate always delta epsilon from delta u. Though for this problem it will not make a difference, but you will notice in the next series of problems that we are going to do and discuss elastoplasticity, that is going to make a difference. So, let us stick to the same procedure. So, delta epsilon is now calculated from 0.695 itself. Here it can add it; it does not matter, but you will see that this procedure is not valid in the elastoplastic case. We will discuss that when we come to elastoplastic case. We will stick to the procedure, so that what I do is to get to E_T in six point, sorry B into 0.695 that is my 0.695 divided by 5; 0.139. Is it clear? So, now epsilon happens to be, my epsilon happens to be, yeah of course this I have to change to 2 and here I will change to 2, so this case the epsilon happens to be the same 0.139.

Now, calculate sigma, calculate sigma.

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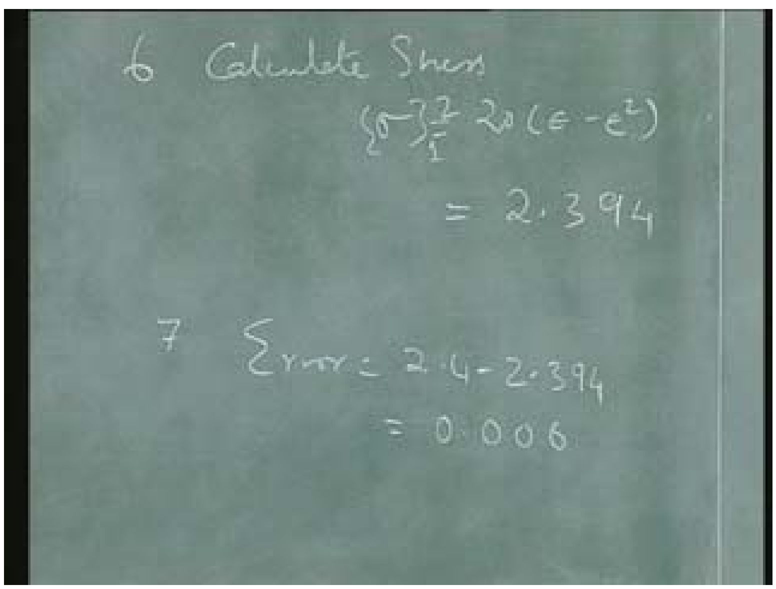

What you do is, instead of 0.12 here substitute for, so this sigma which was for 1 and 1, now is for 2 and that happens to be, calculate that; pardon. 2.394. So, what is the error now? This will become now 2.394. So, the error is 0.006; 6 into 10 to the power of minus 3. I am still not happy with this, because I said I will keep repeating these steps until I get 1 into 10 to the power of minus 3. In other words, the 6 reduces to 1, less than equal to 1.

So, I go back and do the third iteration. Increment number 1, still we are in increment number 1, but we do the third iteration.

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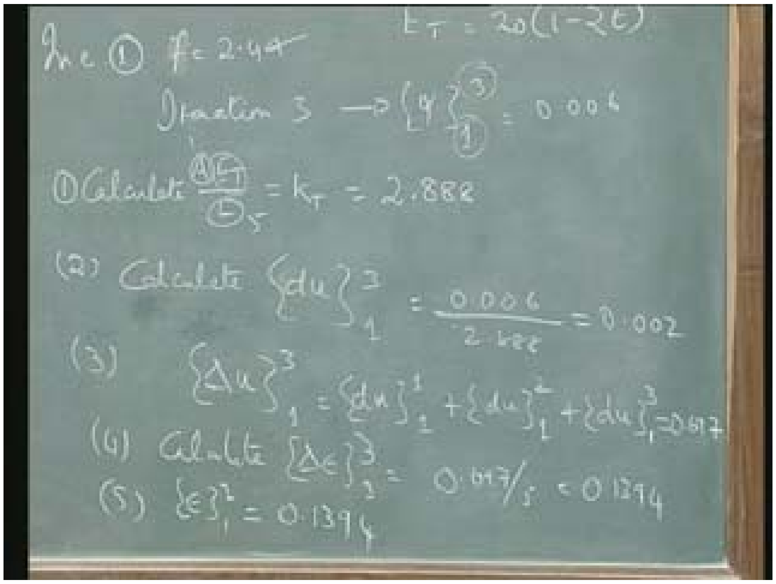

This is now in the third iteration. What is the error which we calculated? That is 0.006. So, that becomes 0.006. I hope the procedure is clear. Any question? Now calculate E_T and calculate K_T. The new K_T we have to update. We are doing full Newton-Raphson procedure and hence calculate K_T ….. Yes; K_T, K_T is 2.888, very good; 2.888. Now, I calculate du 2. So, I change all these things. I have 0.006 divided by 2.888 and what is that value? Note that I have to change this means we are now in third iteration, still in first increment. 0.002, which means that I add here du 3 1 and the total delta u happens to be what? 697; 0.697. So, delta epsilon we calculate. 5 becomes now 7 and the corresponding value here of course for the third, sorry, here again I have to go to 3, here again I have to go to 3; 1394, 0.139. Calculate now the stress.

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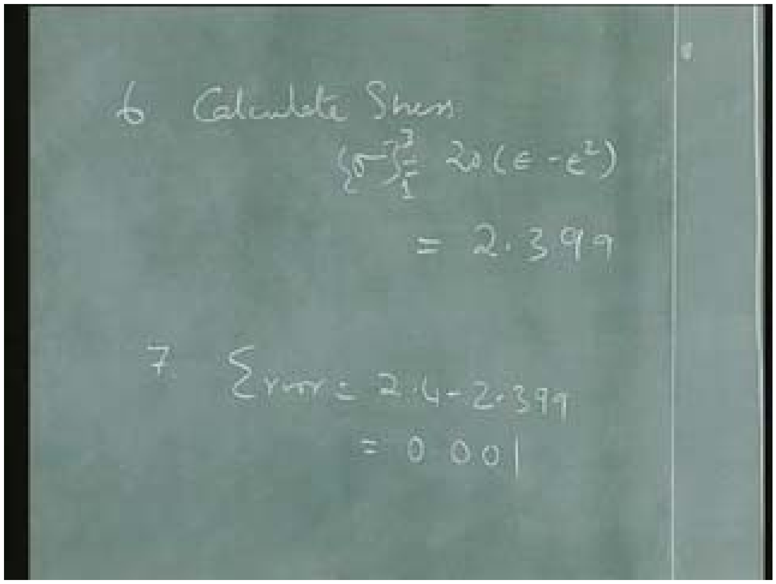

Calculate the stress; 2.399. I will give you a minute, please have a look at it; see how we are doing it, hope you are not lost in the calculations. See what we are doing, how we are doing it. Note that we are in 3, so that should be 3. Now, what is the error? The error now becomes 399 and hence the error is 001. So, I am happy with this error and I am not going to proceed further for this increment, I say that I have achieved convergence. Is that clear? ………. When to stop or in other words the question is when do we stop? What is this error? It is not based on engineering judgment, it is based on the numerical error and the corresponding accuracy of the results. In other words, what it means is that if I, I can make this error 10 to the power of minus 8.

Newton-Raphson procedure has what is called as quadratic convergence rate. If everything goes on well, if you have formulated the problem correctly, then you will hit 10 to power of minus 8 may be in the next iteration itself; next increment or next iteration itself you will hit it. In other words, the number of iterations between 10 to the power of minus 3 and minus 8 may not be very different for a well-posed good problem. Is that clear? But, for some of the problems where you loose quadratic convergence, it would take a long time for the problem to converge or the error to go from 001 to say 10 to the power of minus 6. In many of these problems, especially in the commercial world, there is a tendency to reduce the convergence to 001 or even 01 and run the problem. It is possible that you may get some convergence immediately. But in most problems, if you tamper with the error, the problem will not forgive you. In the sense that in the next step, you will not even get or next step or after two steps, you will not even get to 001 or in other words you will not even get a convergence for the error to be 001. In other words, the solution would start diverging. What does it mean? You know, you know in a very physical sense, in a very physical sense, I will just plot it here.

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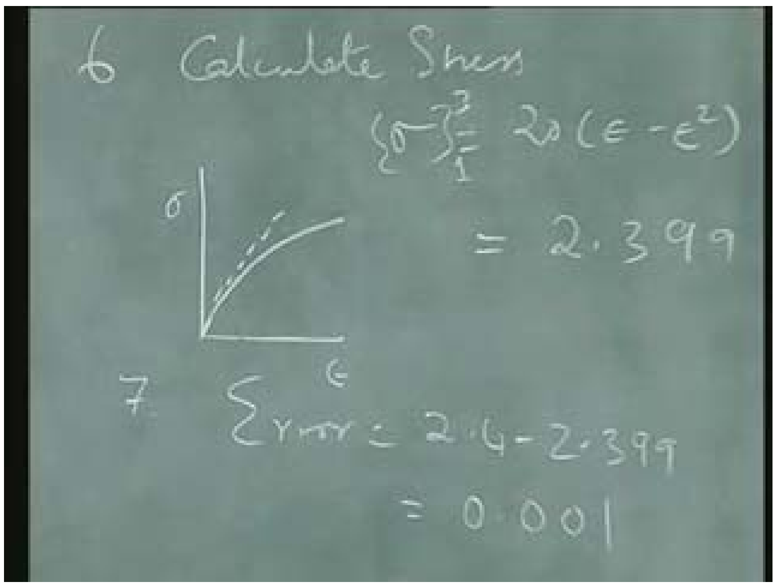

Suppose this is the graph, the stress strain curve, that you should get and if you now allow more and more error in the sense that increase this value, the solution would start diverging and instead of following this path, you would follow some other path and the error would start accumulating. It is not a good practice to run a problem by reducing the convergence requirements. It is usually 10 to the power of minus 6; some research problems we use even 10 to the power of minus 8. Is that clear? So, please do not reduce it and try to get the result. That will be, your happiness will be short-lived; you will not get it in the next 4 or 5 steps. Is that clear?

How do you decide this number? That is what I said that it is usually 10 to the power of minus 6; usually 10 to the power of minus 6 is a good figure. If you want you can reduce it to 10 to the power of minus 5, but you have to check whether it keeps converging in the rest of the steps. This 10 to the power of minus 6, yes, it is based on the experience of so many problems. Is that clear? That is why I said that many people use even 10 to the power minus 8 to check how the convergence rate is for their algorithms. Yes, that is a good question. Does it depend upon the machine? It depends upon two things. One is how good your K_T is. In many of the non-linear problems, it is possible that K_T becomes what is called as ill conditioned. We will discuss ill conditioning later in the course. But K_T does not behave very nicely and the lambda max by lambda minimum which is called as the condition number, it is quite large and this interferes with the number of digits; accuracy of the results or interferes with the number of meaningful digits. So, there is actually a relationship between the condition number and the accuracy of the machine.

Many times we use, we use 32 bit machines, you will get one type of accuracy and as you increase it you get another type and so on. It is a combination of the condition number of K and the accuracy of the machine. For some problems, it may so happen that the first 5 digits, the first 5 digits after the decimal that may only be meaningful and the rest of it may be be junk. Sometimes it may be 6 7 and so on. This depends upon the condition number. We will, we will talk more about it once we finish the first part of the lecture on non-linearities. Is that clear? Yes.

So, we will continue now with our second iteration, second increment starting again from the first iteration. Look at this carefully now, that f is not 2.4. No, it is not 4.8. Look at that. What is the, what is the, sorry f is 4.8.

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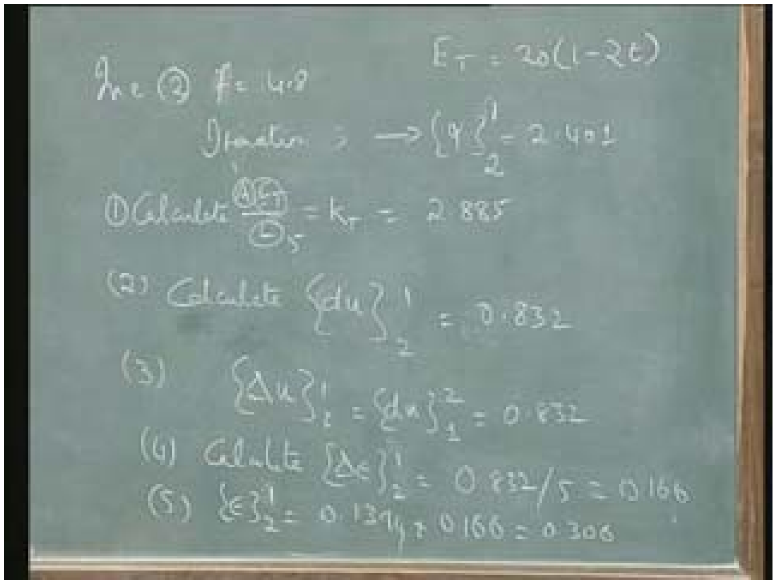

The error, the error is not 2.4. We started with 2.4 last time, f is 4.8 of course and the first thing is notice that it is 4.8 and the error, correct; so, the error is now 2.401. So, you have no escape. If you keep accumulating the error, anyway, next step you have to handle that guy. So, you have to add that to your error in this step or the second step or the second increment we start with an error of 2.401 and now what is the tangent stiffness? Yes, so the strain you have to substitute for the previous case. Yes, you are absolutely right. The strain of the previous case you have to substitute, substitute it here and then tell me what would be the value of K_T?

Yeah, have a look at it. Last time it was 2.88. This time it happens to be 2.885, so 2.885. So, now calculate du. Now it is 2 and 1 and so this, all these things would change and what would be the value of du 1 happens to be 0.832. Check that out, 0.832. Now, delta u, note this delta u is 2 and 1. That is only 2 and 1, so, I remove all these guys here, so that happens to be 0.832. But, there is a change in the epsilon calculation, epsilon calculation here. Of course that remains to be the same; sorry I have 2 1, 2 1 and 2 1 happens to be and epsilon calculation here involves two quantities, this as well as my previous increment quantity as well. So, what was my previous quantity? Delta epsilon, this is divided by 5 and that happens to be 0. 167; 166.

Yes, yes; that is a very good question. What happens for the length change? Please note that we are looking still at a small deformation case. In other words, change in length is not very high; we are not taking it into account. That is the difference between an infinitesimal case and a finite deformation case, so, 0.166. No, this is not a finite deformation case. It is an infinitesimal deformation case, because we are not considering the length change. We are considering only material non-linearity. You are absolutely right and we are not considering the geometric non-linearity as it is called. So, we are considering only the sigma as equal to 20 into epsilon minus epsilon squared case. So, calculate delta epsilon 1 and so this happens to be, epsilon 1 2 happens to be, yeah, 0.1394 plus 0.166 and that happens to be 0.306. What happens to my sigma?

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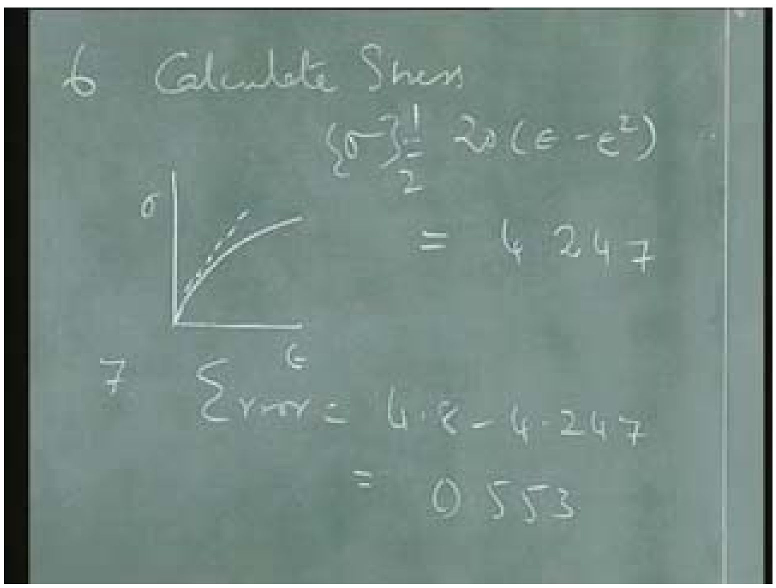

So, that is the first one second and so what happens to sigma? Yes, substitute that epsilon here and so that happens to be 4.247. So, I still have error, because it is 4.8 versus 4.247. So, I have to go to the next iteration in this increment. So, the error at the end of the first iteration in my second increment happens to be 4.8 minus 4.247; 0.553. So, we will stop here and we will continue with the second iteration for the second increment in the next class. Let us finish this problem, so that you will understand completely how we are doing. Any question, we will take it up in the next class.