Advanced Finite Element Analysis

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Lecture - 29

Yeah we had, in the last class we had looked at what is called as total Lagrangian formulation and we had, we had said that two types of formulation that we can follow. One is an updated Lagrangian and another is total Lagrangian. We had had a discussion about this.

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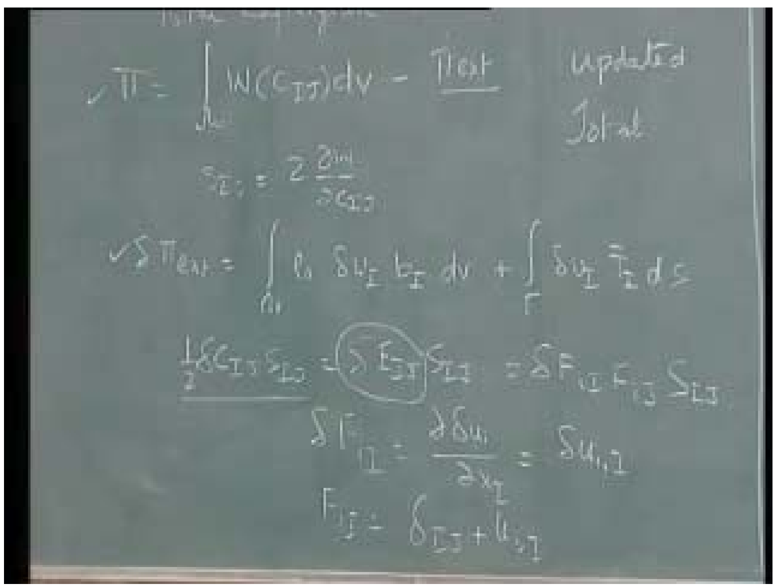

We had looked at the total Lagrangian formulation; we were half way through and we realized that there is lot of algebra that is involved in the total Lagrangian formulation. So, let us just quickly look at what these are before we go further and look at the finite element formulation for the total Lagrangian. We write down the functional. All of you are familiar with it. The functional in terms of the original configuration or reference configuration and of course we have two terms. You, you know this. Essentially what we did was to now find out the first variation of the terms that are involved in this W.

In order to do that for example, for this it was straight forward. That is the variation of the external loads or the potential loss due to external loads. Here, we found that if we have to do this, we have to have a small, you know jugglery or algebraic manipulations that have to be done or in other words, this term now has dow W by dow C_{IJ}. See, of course this can also be expressed in terms of E_{IJ}; we will see that in a minute. In order to do that, we may have to have some formula in place for this manipulation. So, say for example, if I have half delta C_{IJ}, say, sorry, delta C_{IJ} S_{IJ} which can be written as delta E_{IJ} S_{IJ}, so we can, we need to have what or we need to know what this is.

In order to do that, we will substitute or we will first find out what is the first variation of F.

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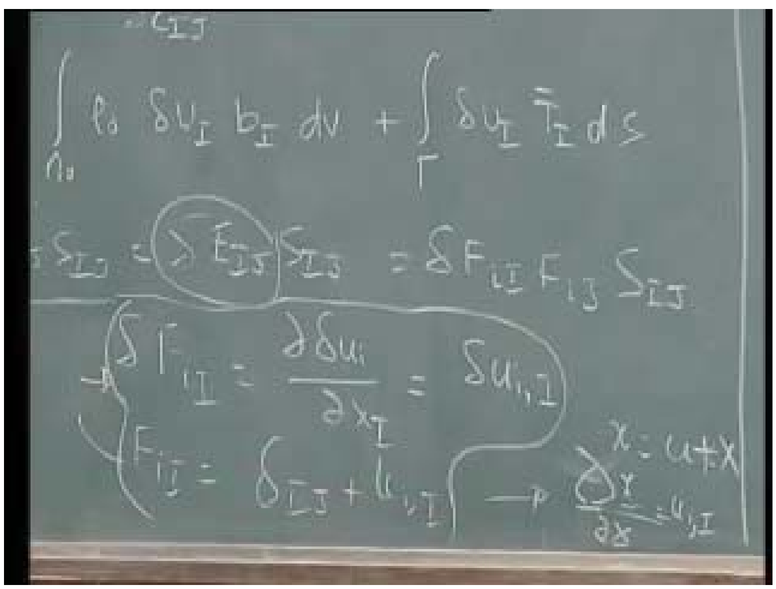

This first variation of F comes from the fact that F can be written in terms of the displacement u or in other words, if you want further going down this you can write x is equal to u or x minus X is equal to u, sorry, yeah x is equal to u plus X, sorry. From here you can write down dow u or dow x by dow capital X; you can write that down in terms of say, from here you can write dow x by dow capital X, because this is a tensor, is equal to dow u that is here it comes u i comma capital I and so on plus I. That is the term which comes here for F, so from this you get here and you use and you see that the variation of this is equal to zero. So, you will get delta u, delta F_i is equal to delta u i comma capital I. Of course you know that small i is in the current and the capital I is in the reference configuration.

With this, now with this, what we do is to look at the first variation.

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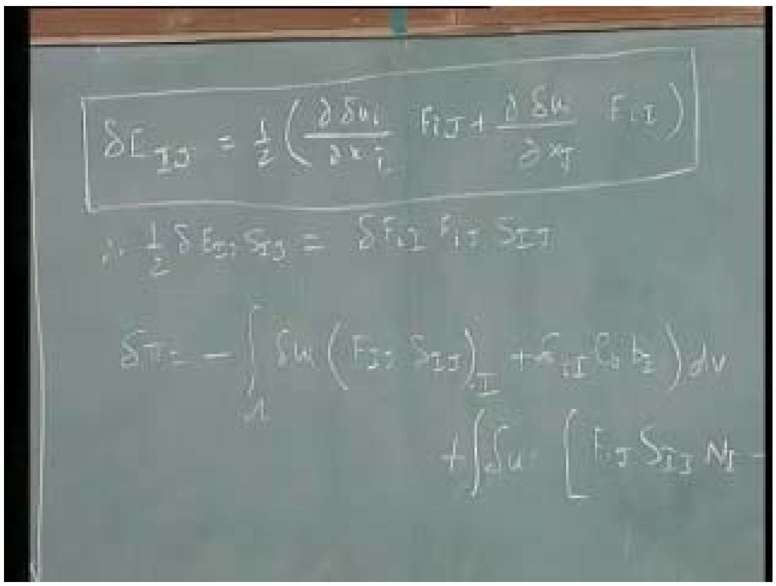

This is an important thing - first variation of E which is the, our Green strain. Then you take each of these terms. Of course we know that E_{IJ} is equal to half of F transpose F minus I. So, you apply the variation for F transpose and F. So, you will get two terms and of course the variation of I goes to zero. These two terms are now taken or this is taken along with S_{IJ} and then, or in other words, this is multiplied by S_{IJ}. So, you will get, using the symmetry of S_{IJ} you will get a term like this. Then, now using the derivation which we have done now on the first variation of F along with this and then using the Green's theorem, ultimately you get delta pi to be written in this fashion, from which we said that, what is that we arrived at? We said that, yes, exactly; so, we can get the equilibrium equation.

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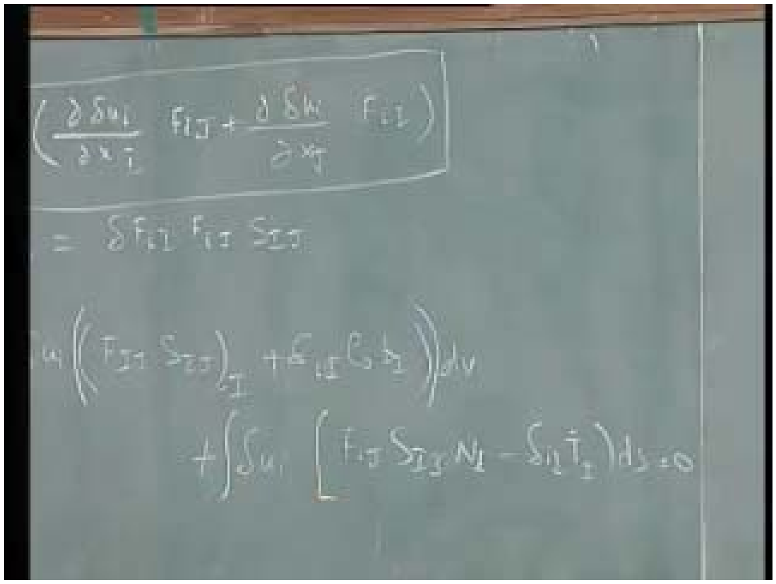

Of course, this is the whole thing. We will get the equilibrium equation, this and we will get the boundary conditions from this. This is what we saw in the last class and we also said that delta E for example, if it is, if it is to be written in terms of 11 22 33 12 and so on, you can write down delta E to be like this.

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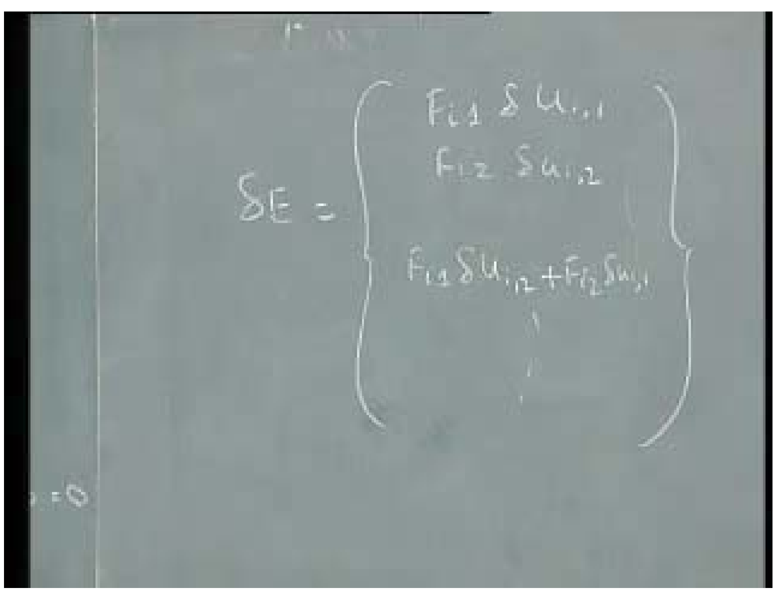

So, this will be the starting point of our finite element formulation. Is this clear? Now, if you closely look at it, if we look at the concepts, forget about the terms that are involved. They are no different from what you did in your earlier class; they are the same. The only difference now is that the type of strain measures and the stress measures that you are using, the stress and strain measures that you are using, are different. So, that involves lot more algebraic jugglery. The concept is exactly the same, there is no difference with it; but because of these large number of terms, the expressions becomes extremely complex. Is it clear?

We said that we can go and do the finite element formulation straight away from this.

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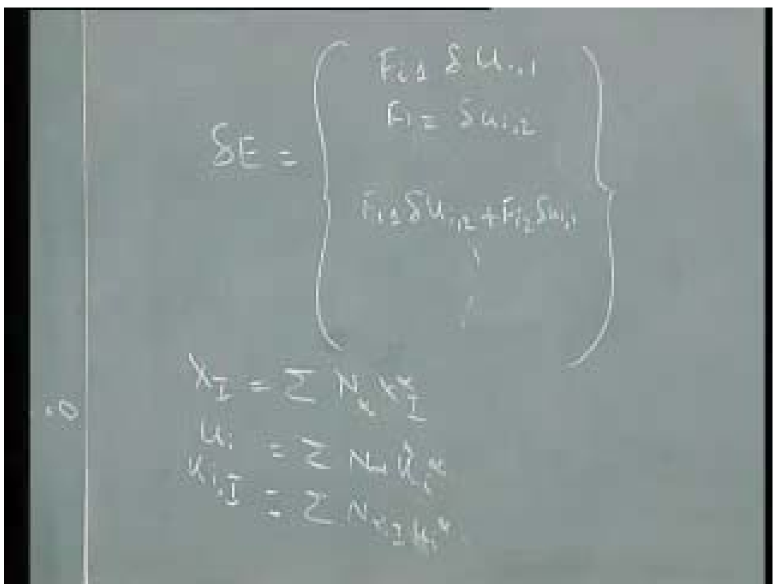

The finite element formulation starts with writing down X in terms of say, the X_I in terms of the sigma, what is it? Shape function times the X nodal, very good, in terms of say, I am putting alpha here, where alpha is summed up to the total number of nodes and so this is alpha I and so on. This is our regular formulation for the iso-parametrical formulation which we had used again in our earlier classes, so that now u is again written in the same fashion. If you, if you look at u_i, then it is again written in the same fashion N alpha u i alpha and so on, where again alpha indicates the nodal quantities. Is there any question?

Essentially what we are doing now is, note this carefully; forget about for the time being the algebraic manipulation. We are exactly following again the same procedure. Only thing is now we are going to get all the terms that are involved here from the discretized quantities. All the terms there we are going to obtain from the discretized quantities. For example, if I, if I have to have u i comma of course I, what do you do? The same thing what you did in your earlier classes and that is what resulted in, for example the B matrix. Now, B matrix is slightly going to be different, but the concept of B matrix is very similar, so that this can be written as again N alpha comma I u i alpha. Is it clear? Okay. Now, what we do is to substitute this into this expression, in order to get a B matrix. Actually, let us see, let us just do that. Let us see what we get? Is there any question so far, clear?

Let us write down for each of the node, let us take a three dimensional problem, because this is a three dimensional element is what we are looking at. Delta E we have written, written in a column, as a column vector with six entries. So, what we are looking at is a three dimensional formulation.

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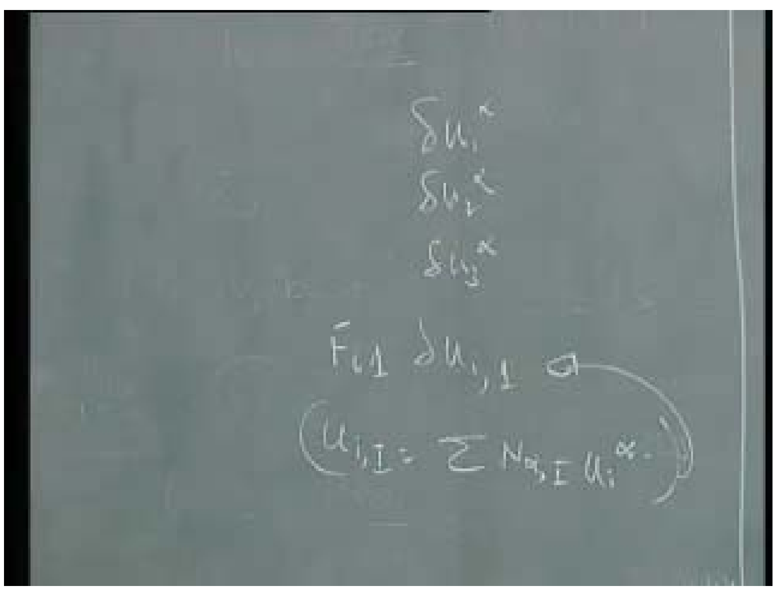

For the three dimensional formulation let us say that we have, each node has of course three degrees of freedom. Let us say that, let us call that as say, delta u 1 alpha, delta u 2 alpha, delta u 3 alpha, where alpha is the nodal degree of freedom. Let us see what we get. Let us say, look at the first expression here. What I do is look at the first expression. This is where some doubts were there. From here, from here I substitute it into this expression. Note that, from here I substitute it into that expression. Note that I is repeated, so that you will get actually there, what is that you will get? Substitute that say, F. Let me just take that first term what we had written F_{i1} delta u i comma 1.

How do I now get the first term in my B matrix, what we had written in the last class? I am giving you a clue, u i comma I we had written this as sigma N alpha comma I u i alpha. So, substituting it, substituting this into this expression, we get actually a big expression for sigma. For sigma we will get a big expression. Let us, let us just substitute that, so this here. What are the terms you have? You will have F.

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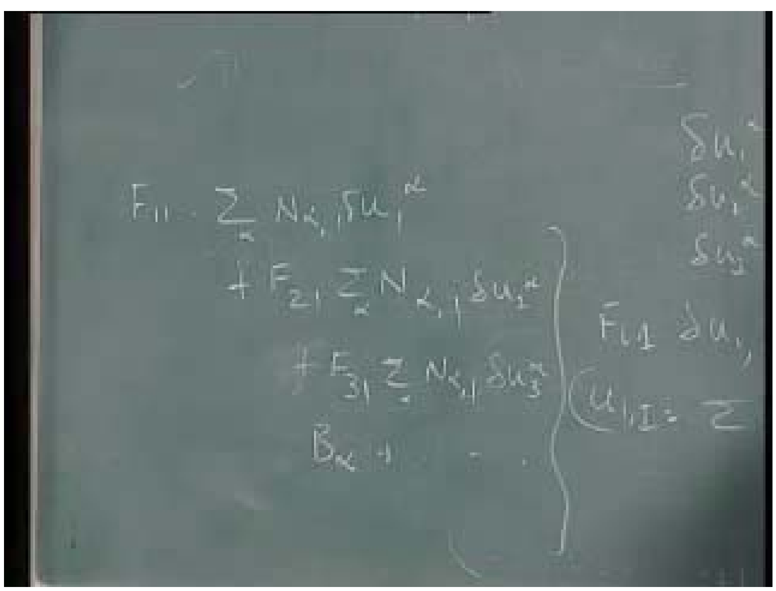

It is going to be really big; F_{11} into delta u_{11} delta u_{11}. So, that I have to obtain from this. So, delta u_{11}, which will involve alpha 1 u 1 alpha; sigma of N1 u1 alpha. Since it is delta here, I will get delta u_1 alpha. Is this clear, plus what is it that I will get? Second term, plus F_{21} sigma N alpha comma 2 delta u 2 alpha plus F_{31} sigma N alpha comma 3 delta u 3 alpha. No, sorry alpha comma 1, 2 1 yes alpha sorry 1; 1 1, because that 1 is constant here. So, this sigma were summed up by from alpha. So, this will be like N 1 comma 1 delta u 1 plus N 2 comma 1 and so on. It will, it will be a big expression.

In order to, in order to not to write the whole expression we split this B into B alphas; we split this into B alphas, that say for example, this we split it into a number of alphas properly aligned where each of them belong to one node. Then comes my first expression or first delta E term.

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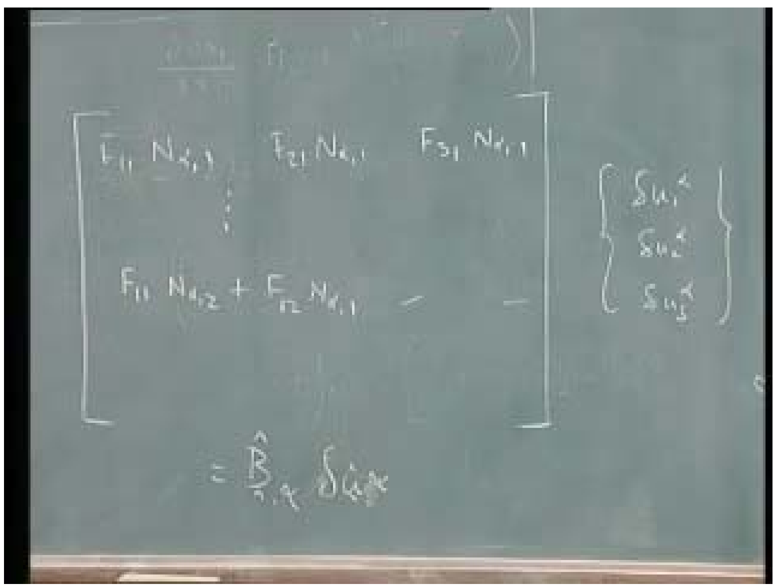

So, that will be, that is first that is B delta E in terms of B alpha, of course sum of them, can be written as, from here first term F_{11} N alpha comma 1, F_{11} N alpha comma 1; let me multiply that by delta u_1 alpha delta u_2 alpha delta u_3 alpha and of course, sigma of alpha, I have just taken it outside. Then, the second term will be F_{21} N alpha comma 1 and the third term will be F_{31} N alpha comma 1. That is the first entry here. Is this clear? That is the first entry. So, you can, you can imagine how big it is. Now, the fourth entry which is there is going to be still more bigger; I would like you to complete it, this fourth entry here, okay. Can I, can I give you two minutes? Can you write down and tell me what it is.

Though these are not important from concept wise, but I just want you to understand how it is written. So, F u i comma 2, so F_{11} first term will be F_{11} delta u i comma 2, you have to put N alpha comma 2 that will be the first term plus, plus the second term plus i 2 F_{i2}; note this here, note this here. So, that will be plus F_{12} delta u, now again alpha; alpha comma 1, very good, alpha comma 1, sorry, delta N, not delta u, N alpha comma 1. This whole thing will be multiplied by delta u 1 alpha.

This will be the first term. Then, there will be a second term, there will be a third term. The second and the third term will be terms which will be multiplied by the term delta u 2 alpha delta u 3 alpha. This whole thing can be, forget about this; you, you can fill this up, it is not a problem or I mean you can get this matrix also from very, all standard text books give this. Look at Zinkevichs for this. But, ultimately what we get is that this can be written as say B alpha, this alone, one term alone, into delta u alpha. Is this clear?

Now, if you, if you look at this, actually this can be split again. I am not going to do that, split again into two terms. Actually this has L terms and NL terms or B matrix can be split into a linear part plus non-linear part. Linear part is the same as what you did in the previous course and there will be a non-linear part. My next step is exactly the same as what we did in our earlier classes. What is that we did? We substituted this back into my expression for the variation, the first variation in order to determine K. So, I substitute it back into my first expression what we wrote and then let us see what happens for delta Pi. Substitute that into that expression and take delta u alpha out; remember this was, this was very similar.

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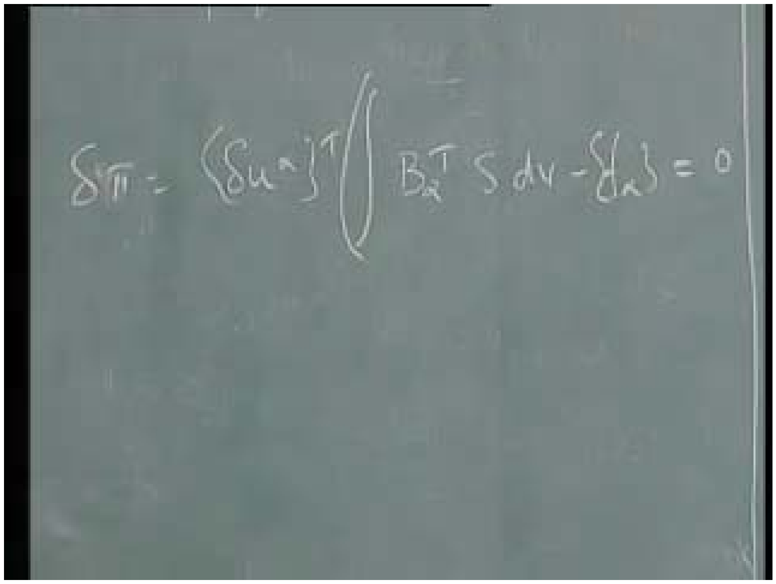

Now, only thing is this alpha I am writing there just to state that, since it is big I cannot write one B; next step I am just going to write that down, but just stating there, because sum up, you have to sum up with respect to each of the alphas. S dv minus f_{alpha} is equal to zero. This was exactly the way we had written before and this gives rise, where f_{alpha} is of course, if you want a matrix and f_{alpha} is of course the ones which consists of the body forces and the surface forces. In other words, this is my starting point for what? For the Newton-Raphson scheme; remember that this is the starting point for my Newton-Raphson scheme. If you remember these terms, where B transpose sigma, delta u was outside, B transpose sigma minus f is equal to zero. If you remember this is how we started Newton-Raphson. Let me write it in the same way as we did in the last, you know, some time back, where we had B transpose sigma; remember that we had put this as B transpose dow sigma by dow epsilon and so on.

But right now, I cannot write that next step in the same simple fashion as we did before, because this B, in fact B if you had noticed in my last session, when I wrote the Newton-Raphson scheme, B was well behaved, nice guy, it was independent of f's, there was no f, independent of u. So, I was, I was able to write this as how did we write? B transpose dow sigma by dow epsilon into dow epsilon by dow u into delta u, so that we had K_T. If you remember we had a lovely K_T term without much problems which was B transpose D_T B. This is for small strain non-linear problem. This is the K_T we had. But now, this B has or is much more complex than what it was in my small strain problem.

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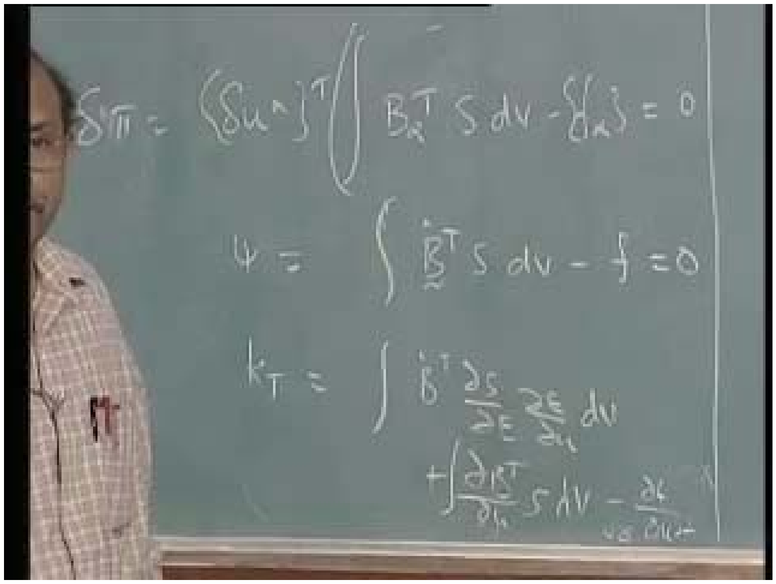

So, let me, let me write down, now that you know what this B's contains, B contains you know, B alphas we can write this down as say for example, B transpose S all the sums of it; I hope you will do the correct this one dV minus f is equal to zero. This is my starting point for my Newton-Raphson scheme. Now, how do I write this down, what are the terms that are involved? Let us see what, what these terms are? In other words, what do you think would be the difficulty now or what are the other terms, other terms which I have to add than what I had done in my previous course? Let us see whether you are, whether you are able to say that. Since B involves u and f may also involve u, actually when I want to calculate K_T, I will have three terms. One term involving, first term is very similar to say for example, similar to what we did in the last class that is B transpose dow S by dow u. That can be written as B transpose dow S by dow E dow E by dow u, so transpose dow S by dow E dow E by dow u dV. Of course, delta u is there but I am writing only K_T plus what is the other fellow? Dow B transpose.

Now this, this has u in it. B has u in it. This is how we had written down, because it has deformation in it, F in it. So, that term also has to be differentiated with respect to u, so that term will be dow B transpose by dow u S term dV. That is the next term which I will have and f is not going to be or f may or may not depend upon u. f may depend upon u, I will come to that in a minute, what it is. So, that term also will be there. We will investigate each one of them and what they mean. Is it clear, what is the difference between this and small strain? Yeah, we will come to that in a minute. How does this term come into picture? We will name each one of them.

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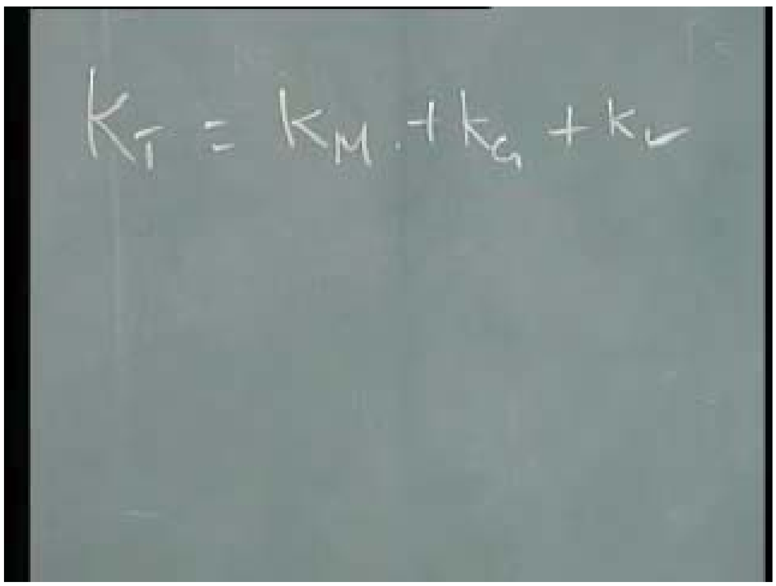

This guy is due to the material contribution and is called as the, if I put K_T, it is called as the K_M matrix, the first guy here. That is the material contribution. The second guy here is due to geometric stiffness. That is the beauty of this whole formulation. You can, you can understand each one of these terms. First one is there is a material contribution to stiffness. The second term, note this, this is due to B which has f in it. This is what is called as the geometric contributor for stiffness and the last term is due to the load and that is called as the load contributor of the stiffness. So, you have, because we have split this P_L plus P_{NL} or in other words, there are non-linear terms associated with B that means that the stiffness matrix has contribution from geometry or geometrical changes as well. These are what are called as the stiffness terms due to load.

Some times people call this as follower forces, follower forces. How does it change or why is there, it is a very good question; why does it contribute suddenly to the stiffness. Let us see that.

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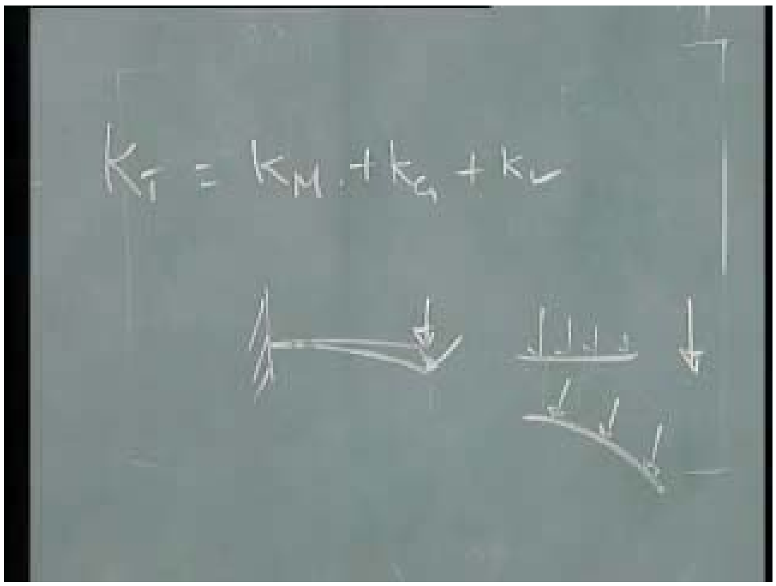

Let us just understand this term first and let us come back to this question in a minute. Let us first understand this term, what it means. Suppose I have a cantilever beam and I apply a load. If it is a very small deformation, you would not have even bothered to ask a question whether the load is going to act in the same vertical direction or is it that this load will act say, normal to it. If the, if the deformations are small you would not even bother to do that say, pressure is acting on to a surface. Now, if the deformations are very small, you would not bother much to look at the pressure, if you had defined for example, pressure in terms of x direction, you would not define or you would not bother if this x direction there is a change. But, on the other hand when the deformations are large, the pressure which say for example, was acting like this, where say for example, this is x direction or this load which was acting in the downward direction, if this deformations are large here in this cantilever or on this pressure, then there may be a large variation in the way the force acts. In other words, there was no horizontal component to this when it was like this, but when the deformations are large, there may be a horizontal component to it and so on. In other words, the load follows the deformation which means that f terms varies with u. So, that is the contribution for the stiffness matrix and that is what we call as follower forces. Yeah, any questions?

Student: f term is outside that integral or it is inside that integral?

Of course, it is outside. B transpose sigma minus f is the, is the side term. But, when I take dow u dow psi by dow u into delta u for my Taylor series expansion, then dow f by dow u into delta u will come here. In previous case it was not there.

Student: When we are doing this one, then this term will be there.

Yeah, that is why it is, this is not inside the integral. It is outside dow f by dow u term. That is what is the contribution for my K_L term.

Student: Here but, the force always reduces the stiffness of the ….. Is there any reason?

Force reduces I am not, I did not comment, yes. Yeah, yeah, but I do not know how dow f by dow u is going to vary, whether that is going to be negative or positive. So, you cannot make generalized statements whether the deformation reduces the stiffness or not; very, very interesting example. Yes, but, but the, but you cannot make very general statement whether deformations, you cannot make a general statement that deformations reduces stiffness; it may not.

I will, I will, yeah, I will tell you a physical example; that will make things clear. In other words, why you cannot make a statement, with deformation, will a system stiffness decrease? Suppose I have a, I have a sheet of paper. Let us say that we take this cloth. The cloth is say I am just holding it just like this. No load, no loads are applied. You apply the load on top of it like this. There is going to be some deformation. Suppose, now I deform this cloth like this. I apply load in this direction and even in the biaxial way, both directions, I apply a load. Now, this cloth is now deformed. If you apply the load perpendicular to the cloth what will happen to this cloth? The deformations are going to be less. In other words, this has been stiffened by stress. It is called as stress stiffening. The deformations in the other directions had actually in fact enhanced the stiffness of this cloth in a perpendicular direction. So you cannot, that is why you cannot make a general statement that deformations reduce stiffness. Is this clear? These are the three terms that are involved and they are called as K_M, K_G and K_L terms.

Now, let us go to the next step of defining it for updated Lagrangian. How do you define this whole problem of updated Lagrangian? No, different, very straight forward. The only thing what I am going to do is transform all the quantities which I had written to the current co-ordinates. What is the current co-ordinates? Current co-ordinates, yeah, are small i, but current co-ordinates in this case, is actually not Eulerian, but the converged state at say t_n, at say t_n, time step n. This is the, actually what is called as the current configuration. In other words, you update, you update all the co-ordinates of the body to the current or t_n by adding to the coordinates the displacements, the total displacements. Let us see how we are going to transfer this. How are we going to transfer all the terms? Not these terms, I mean you have to go back to my original statement where we had say, delta E_{IJ} S_{IJ}. Let us see how we are going to transfer.

Let us take that term. So, what we do? Term by term I am going to transfer. I am going to carefully transfer this to the current co-ordinate system. So, let us take, because of that let us take delta E_{IJ}, the first variation of E_{IJ}. Let us see how you can write that. Of course, what is it that is going to be useful to us when we transfer it? Deformation, f; very good. Where are, where are these kinds of things that we did? Note, we did this when we did Nanson's formula. We did the similar thing when we calculated dv to capital DV ratios; d small v to d capital V ratios, where we did, you know, similar pull back operations and push forward operations. What essentially we did was to play with the current co-ordinates and the deformed co-ordinates. Let us see, we will do that. Let us see how we write this for upgraded Lagrangian. Unfortunately, I do not have them on the board. So, the old ones, so, you may have to write it down from your notes. Let us see how you are going to write delta E_{IJ}.

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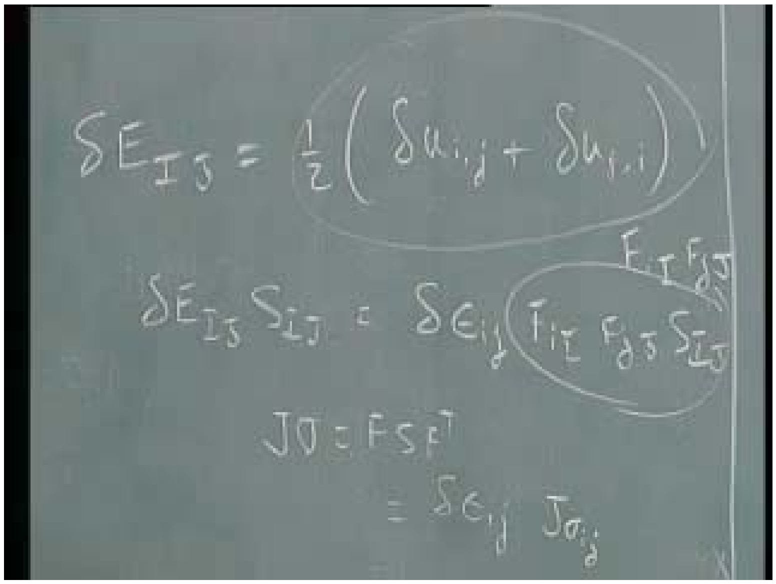

Let us do a small exercise. You just try it out, so, delta E_{IJ}, we had written this some time back. How did I write? Half of say, half of delta u i comma j into or I will, I will add this and multiplication I will do it ultimately; delta u j comma i into F_{iI} F_{jJ}. Look at this term what is inside, what is inside? That is nothing but delta $epsilon_{ij}$. So delta $epsilon_{ij}$ F_{iI} F_{jJ} look how neatly whole thing falls in place when we move from the reference to the current configuration.

Now, let me E_{IJ} S_{IJ}, what happens to this term? Let us see how many of you are able to recognize and say something; S_{IJ}, the second Piola-Kirchhoff stress, S. So, what is the relationship between the second Piola-Kirchhoff stress and Cauchy stress? Sigma is equal to FS F transpose 1 by j or J inverse or in other words, F SF transpose j, F SF transpose is equal to J sigma. Let us see if you are able to recognize any of them here. Look at what these three terms in indicial notations means? Fantastic, so that is it. So, this is equal to dow $epsilon_{ij}$ into J $sigma_{ij}$. Is that clear? Now, what is that you do? You go back and substitute for my delta E_{IJ} S_{IJ} in my previous expression for delta pi. Let us see what you get.

Why? How come, how is that, how is that possible? See, what we have essentially, essentially what we have done is to, absolutely, absolutely; please note, yes, very interesting question. Why, why did I get J? If I look at the total energy term, yes, but when I look at the total energy term delta E_{IJ} S_{IJ} d capital V is what I have there, when I integrate it. I will have now small dv. If I now substitute it back into my expression, so that is why I said you have to be careful and they are not blind transformations. Go back and put this. This is very interesting thing. So, you put this back into my first expression there, delta E_{IJ} S_{IJ} term.

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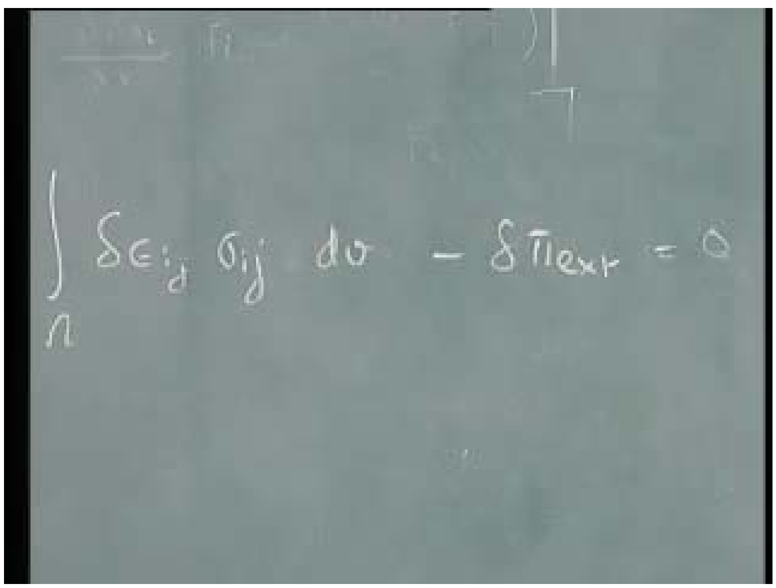

So, you will get delta epsilon_{ij} sigma_{ij} J d capital V. So, that J d capital V will go now to small dv. What was omega_0 now, when I now convert this into small dv will go to the current configuration. Of course you will have, this is my delta pi you will have minus delta pi external. This is my first variation. The first variation should be equal to zero. Is that clear? So, you have to be careful in converting every term.

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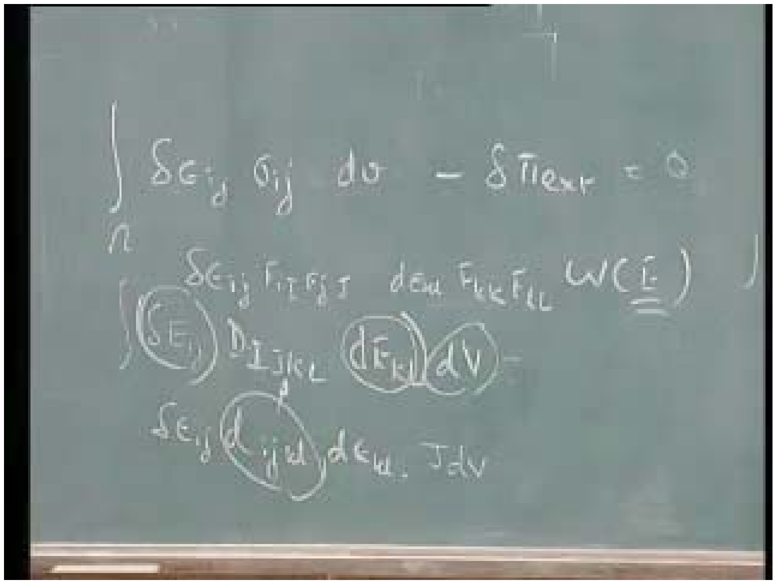

In the same fashion, if I write it in terms of say, dt delta E_{IJ}, so, integral delta E_{IJ} when I, when I differentiate it and write it as say, D_T into dE, in other words, it is, D_T is the relationship between delta S and delta E and write this as IJKL into dE_{kl} dV. Let us see, if you want, this is my first term for my K expression. Say, let us see how you write this. Substitute this, one by one term and see how you write this. This has to be d delta $epsilon_{ij}$ d_{IJKL}. This is, this is a small catch in what we are doing into delta $epsilon_{kl}$. But note that, we always define d capital IJKL or D_T to be delta S by delta E. That is how, that is the theory. So, I have to calculate an equivalent d, equivalent d in the current configuration.

In other words, in other words, note this carefully. In other words, we know how to transform S, stress, we know how to transform strains; we just now saw it. How do you now transform the D that is capital D? If I, if I can call it as tangent modulus, tangent elastic modulus, how do you now transform this, because usually you are given only D capital. If you look at most of the strain energy forms, all the strain energy forms that you write say, W is a function of E only. Obviously you cannot write that straight away here. You have to transform it to the current configuration. Please do that, let us see.

You need not do push back pull forward, you know, right now. Actually that is what it means, but let us see how you arrive at this. Just substitute it or in other words, how am I going to write delta $epsilon_{ij}$, if I write say, with respect to small d as the one which belongs to the, as the elastic modulus for the current configuration ijkl d $epsilon_{kl}$, then what is this relationship with respect to this? That is the question. Not very difficult; this is delta $epsilon_{ij}$ F_{iI} F_{jJ} and this guy here is d $epsilon_{kl}$ F_{kK} F_{lL}, substitute that. So, I define, note this carefully, I define d_{ijkl} to be F_{iI} F_{jJ} F_{kK} F_{lL} into D_{IJKL}. The only other thing that is there is dV is equal to J dV and usually that is also included here, so that j, sorry, not here in the next one here, so that J into J into d_{ijkl} is equal to F_{iI} F_{jJ} F_{kK} F_{lL} into D_{IJKL}. That is how you find out; this is very, very important expression. Let me write that. That is how we find out the elastic modulus in the current configuration.

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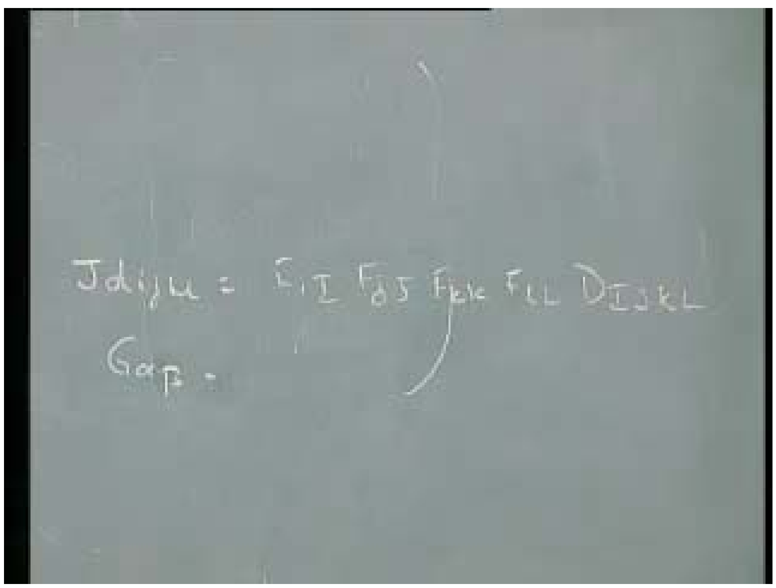

Jdijkl is equal to F_{iI} F_{jJ} F_{kK} F_{lL} D_{IJKL}. No, but we are in the indicial notation. No, we are in the indicial notation; we are not in matrix notation. Matrix notation, of course, will have an effect. These are indicial notations that we are doing. To make things clear, so that you know where the summing up and so on happens, we are doing the indicial notations. Now, of course, we will have a look at the KG terms slightly more closely in the next class. So, that also has to be transformed. That term is usually called as $G_{alpha\ beta}$ term.

We will, we will look at that thing in the next class, but before that what I want to comment is, very important comment, that by this time you would have realized, by this term you would have realized that both the updated and the total Lagrangian formulations are equivalent, there is no difference between them. They are the same, basically because mathematically in a very correct fashion we are trying to get from the reference to the current, the current to the reference and so on. So, no way we miss out anything in this. The results should be the same. So it is, it is almost meaningless to talk about whether updated Lagrangian is better or total Lagrangian is better from an accuracy point of view, from an accuracy point of view. But, at the same time, but at the same time please note that these guys here reduced to such a nice extent and so close, so close to the linear formulation, not only linear formulation even non-linear small deformation formulations that the extension of the code from or implementation of the code from linear to non-linear is very simple with updated Lagrangian.

Though we know that it is not just that if you put delta epsilon$_{ij}$ I will get the current configurations small deformation code, because there is one more term which we have to look at that. I told you that we will have a look at it in the next class, but the whole point is that the only difference or only major advantage for this formulations are that they are easy to implement. Is there any question? The second term we will expand it and we will have a closer look in the next class and we will finish the finite element formulation. But before we close, there is one more comment just coming up in the next class that this is only a displacement based formulation, but unfortunately, very unfortunately, many of these formulations may not be valid, may not be valid if you do large strain large deformation problems.

Very good; there are problems where the problem can be geometrically non-linear, but the strains can be small, so that we can still use an elastic type of constitutive equation, but add the geometric part of the stiffness matrix. In other words, this formulation is also valid for linear elastic case when the strains are small, when the displacements and hence the geometrical variations are large. So, that kind of problems, there are quite a few problems that exist. For that kind of problems we can still use this formulation. But, when the strains are large and the displacements are also large, then we have certain problems which we have to take care of. We will see that in the next class. We will stop here and continue in the next class.