**Advanced Finite Element Analysis**

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**Lecture - 28**

Yeah, we were looking at yesterday the total Lagrangian formulation.

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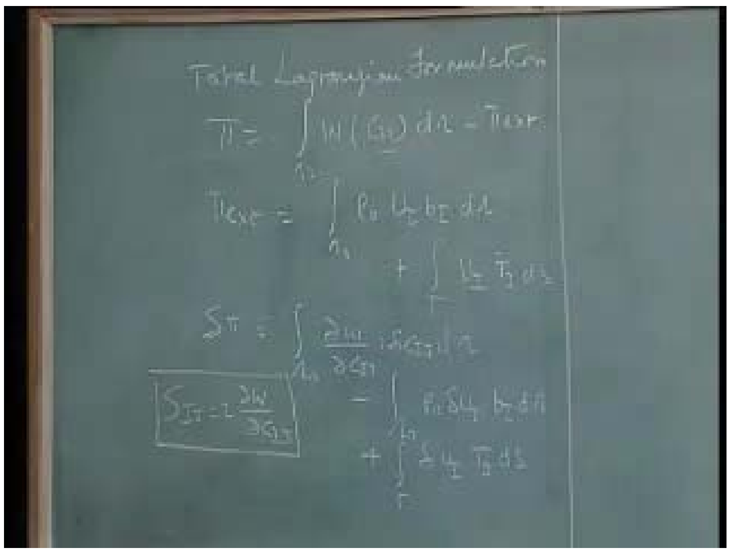


We will just take a look at the first few steps which we had done, before we continue with this. What we said is that, we defined a functional. Functional is now in terms of strain energy. We said that psi will replace it by W, which is common in most of the papers. This is the strain energy term, this is the work done by the external forces or the potential loss due to external loads, where you know that pi external consists of two terms - one is the, one is the internal force term and the other is the external force term, where T is the traction vector that is acting on the surface and so, we take the first variation of this delta pi, the first variation of delta pi, first variation which is delta pi is given by dow W by dow $C_{IJ}$ into delta $C_{IJ}$ d omega that is for this, because we had written W in terms $C_{IJ}$ and the second variation of a, sorry, variation of the second term is given by $rho_0$ dow $U_I$ and there is a delta $U_I$ delta $U_I$ $T_I$ and so on. Yes, so, this is, this is where we stopped and we of course know that $S_{IJ}$ can be written as 2 into dow W by dow $C_{IJ}$.

Let us now take this term. This means that I can of course, I can do a small substitution. What is the substitution I can make?

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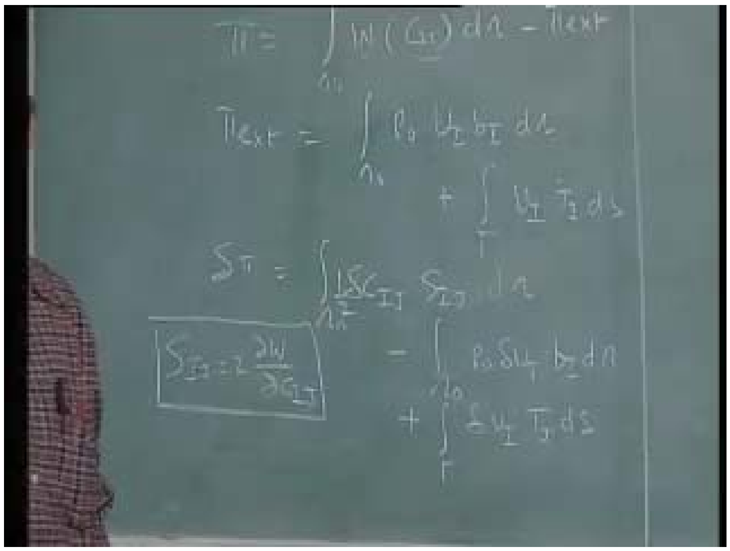

I can just remove this and I can substitute this into that expression, so that I can write that as dow $C_{IJ}$ into, half of it the 2 is there, so that has to come here, $S_{IJ}$. Now, let us see what these terms are. Ultimately, I have to write the whole thing in a form, a variational form, such that it will, one, we can derive the equilibrium equation that is incidental. We know that, we have done this before and next is to make the way clear for implementation of the finite element formulation.

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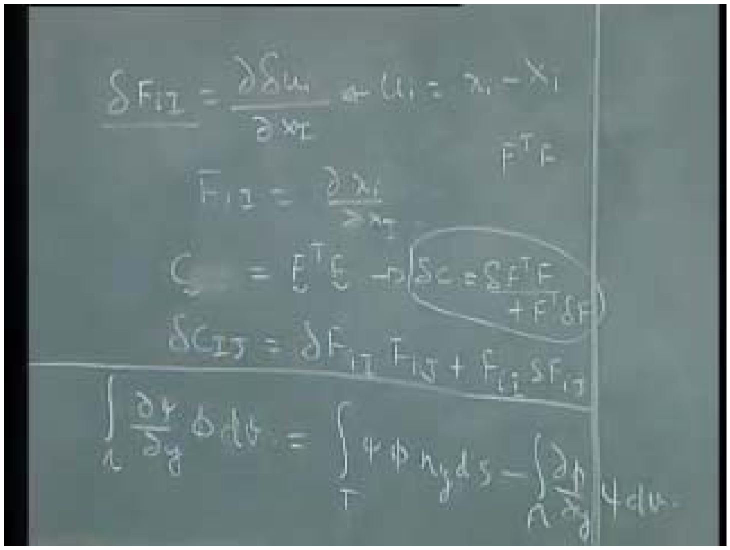

Of course, all of you know that dow $F_{iI}$ can be written as, how do you write this? This is equal to dow of this we did say, we had seen this before, $X_I$. This comes from the fact that $u_i$ can be written as $x_i$ minus capital $X_I$, so delta $u_i$ and dow of dow $X_I$ from here, we will get that dow of del $u_i$ is equal to, note that if you do not remember, this is equal to dow $x_i$ by dow capital $X_I$. Yeah, these are the things which we saw in the last class. Let us now try to replace this del $C_{IJ}$ or dow $C_{IJ}$. What is $C_{IJ}$? We had seen this before is equal to half what is C half, or sorry, C is F transpose F and that is what we substitute in E to get half F transpose F minus I. So, this is equal to, so C is equal to F transpose F. Right, this is in terms of the tensor notation.

Suppose I want IJ, what is that I get? Actually I can say from here delta C is equal to F transpose F plus F transpose delta F. So, write that in terms of, let us see, indicial notation here. How do you write that? Note that $F_{IJ}$ is a, so here you have to write in terms of indicial notation, how will you write that term? Delta F transpose delta F transpose F, how do you write this in indicial notation? F transpose F, how do you write this, how? So, delta F transpose F can be written as delta F, so will it be IK KJ or or say, iI say for example, $F_{iJ}$. This is what is F transpose F, right, plus F again same way I write it iI iJ, right.

Now, substituting this into my previous expression and invoking the, I am, I am going to step, I mean skip a few steps and invoking the Greens theorem, I can write down this expression here. Look at this expression, I can write down this expression as, and by the way what is, what do we mean by invoking Greens theorem or the integration by parts in say, 2 and 3 dimensions? Let me write down that. That is quite simple. Suppose I have omega, I have some quantity dv is equal to uv minus integral v du, so this will be phi $n_y$ dS minus integral v du that means that dow phi by dow y psi dv. So, invoking that I can write down this expression here. Let us, let us see how you write this term. Substitute for delta $C_{IJ}$ and then you have these terms here. Write that down and what is that you will get? Look at these terms; look at this term, look at this term, look at this. Let us see, in fact, the first step is that delta $C_{IJ}$, so, you can combine this.

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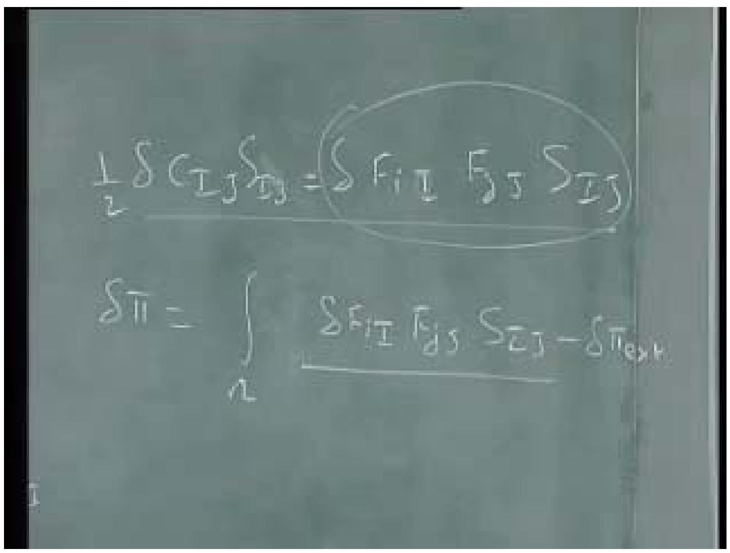

You can say that delta $C_{IJ}$ can be written as delta $F_{iI}$ $F_{jJ}$ $S_{IJ}$, if I want to include. You can verify this that you can write down delta $C_{IJ}$ $S_{IJ}$ as delta $F_{iI}$ $F_{jJ}$ $S_{IJ}$, because F transpose F is, what is F transpose F? F transpose F is symmetric term, $S_{IJ}$ is symmetric. You can verify that, you can write down this, this fashion. Now, let me substitute this into my expression in delta pi, my expression into delta pi, so that the delta pi expression can be written as, I am doing it for you; so, delta pi expression can be written as what is the first term? Integral half of I $F_{jJ}$ $S_{IJ}$ minus delta pi external terms.

Now, I think, just a second; this is half, because that is symmetry, so it should be half here, half of that will be, sorry, this has to be half here, so that half will go off.

Because of the symmetry conditions, you will get only one term. It is quite straight forward, you can verify that. In other words, you multiply this, this term here into $S_{IJ}$, you will get what I have written there; just multiply that. Now, let me invoke my integration by parts in three dimensions with respect to that.

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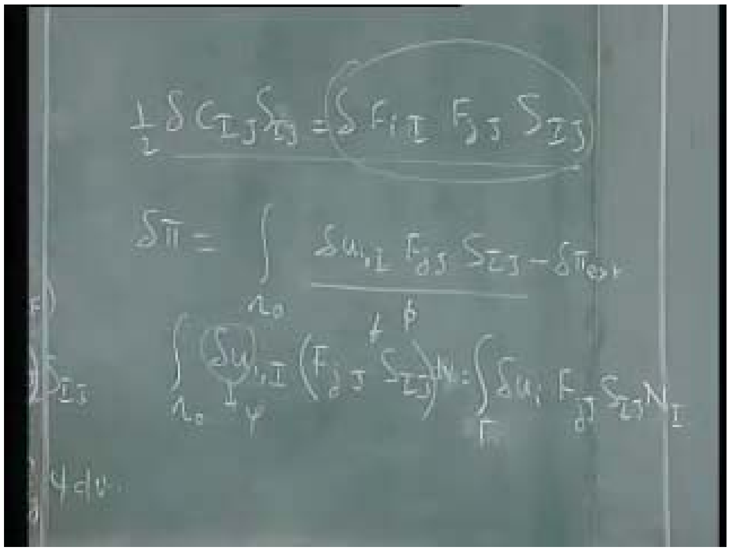

One of the things I do is, before I do that is, to substitute what I know as dow of dow $u_i$ dow $X_I$, this place as delta of u i comma what is that I should get, I right? What I have done is essentially delta $F_{iI}$ I have substituted it from this expression here into that expression. So, I have written this as delta u i comma I $F_{jJ}$ $S_{IJ}$ minus delta pi external terms. Let me now at this stage invoke my integration by parts theorem, so that this term quite simple, but only thing is it involves, you have to be very careful with the notations, indicial notations; that is all. That is nothing more than that, very simple terms. So, that is the left hand side. That will consist of two terms, so the first term is, sorry, we have to put dv and of course, all these integrations are with respect to the reference configuration, so, is equal to first term is surface term, so this gamma term. What is that we have written here? What do we get? From this how do you write that?

We can, I will give you a clue. We can treat the phi to be what, in this case? That is our phi, if you have a doubt this can be treated as our phi and this delta u or delta $u_i$ can be treated as psi. Now apply the theorem. What will you get there? You will get delta $u_i$, phi will be there, $F_{jJ}$ S capital I capital J and then instead of that $n_y$ there you will get $N_I$, $N_I$ term at this place instead of $n_y$ here, $n_y$ here. If you need time I would just wait for a minute, you can write it down, so that there is no confusion. Let me, let me repeat just what we did, so that there is no confusion. What essentially we did was to write down the potential energy in terms of C and S. Then, what we are trying to find out is the variation of C that is delta C from the fact that C is equal to F transpose F and we can write down delta $C_{IJ}$ after this multiplied by $S_{IJ}$, we find that because it is a symmetry, can do a small jugglery and find that half of delta $C_{IJ}$ $S_{IJ}$ is equal to this quantity.

What we are going to do is to substitute this back into our potential energy expression. That is what we are doing, so that is the first step. After this integration by parts that is the next step that we are doing.

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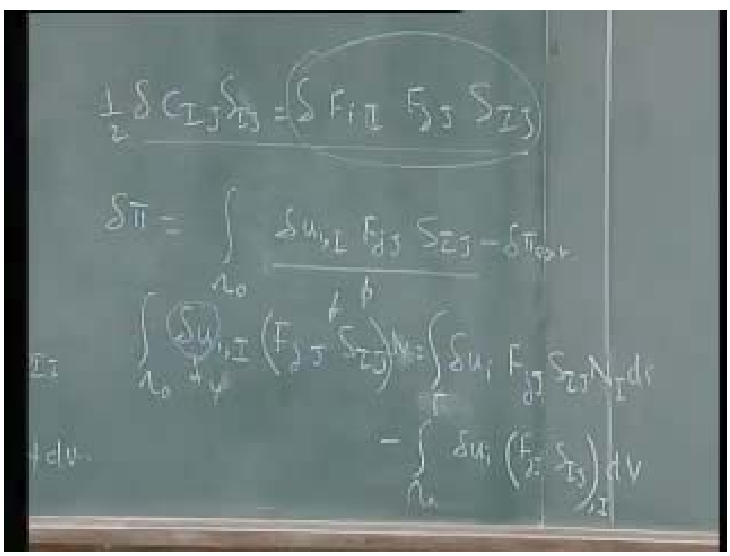

So, in integration by parts, I find that the first term for this can be written like this and I have identified the delta $u_i$ to be phi $F_{jJ}$ $S_{IJ}$, sorry, psi and $F_{jJ}$ $S_{IJ}$ to be phi, so that the second term here minus integral at $\text{omega}_0$, how do I write now? Delta $u_i$, very good, then the second term $F_{jJ}$ that is the second term F capital J, then $S_{IJ}$ the whole thing comma, shifting this I comma, I dV. Is that clear? Yeah, psi and phi, this is psi, this is phi. So, the first term consists of phi, psi, N. That is the first term. The second term consists of, since psi has a derivative we are shifting the derivative. So, psi dow phi by dow y is what we had in the formula. Now this y is now this capital I, so you will get dow phi by dow $x_i$ or $X_I$? that is what you will get, which means that this whole thing is derivative of I. Is that clear? I will give you a minute, just see it. If there is any question, I will answer. Pardon?

$N_I$, yeah, $N_I$ term will come here. The second term here, here it will not come, obviously. Look at this, please look at this formula again. Please look at this again, you will see that this is our psi and phi psi comma i phi dv psi. So, these are the two terms. First you have $n_y$. You do not have a $n_y$ term here, so this is exactly what we did here as well. Yeah, normal; yeah, yeah, normal in the reference configuration $n_x$, $n_y$ and ….., so that is the $N_I$ term. Of course, all of you know that the summation is valid here. We have assumed that the summation, whenever I write that is automatic, I need not even tell you.

Now, this is my first term in my delta pi. So, substituting it into my delta pi expression, it consists of two terms.

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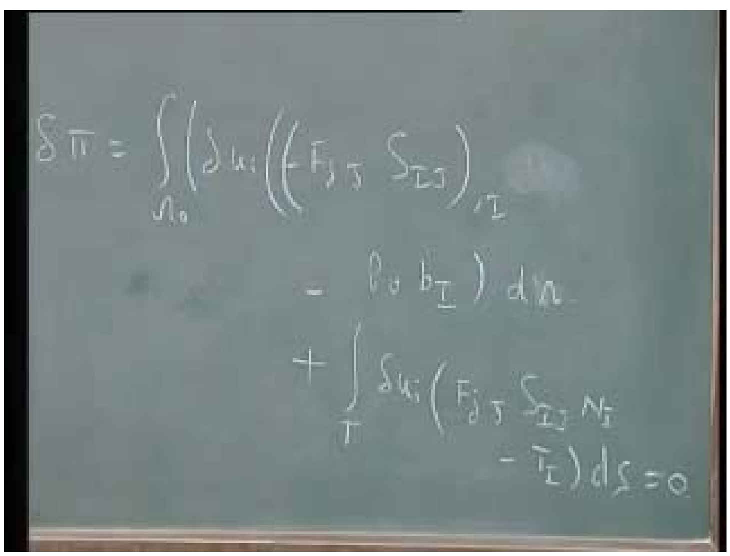

How do I write delta pi? Is equal to, there are two terms there in the delta pi external; one term is due to the, what? Internal, what we call as the body forces, so, both of them would come under the complete integration, so that you would write that as delta $u_i$ into $F_{jJ}$ $S_{IJ}$ comma I dV or let me put the dV a little later. Then, what is the second term? Minus, get back to the first one delta $U_I$ $rho_0$ $b_I$ so minus $rho_0$ $b_I$ dV or yeah or d omega plus the second term here delta $u_i$ into $F_{jJ}$ $S_{IJ}$ $N_I$ plus or rather minus $T_I$ term dS that is equal to zero. Yeah, yeah. $F_{jJ}$, so this, that is correct, which gives us the equilibrium equation. The first part is the equilibrium equation that you get and the second whatever is inside the bracket, both of them are equal to zero; this is equal to zero and that is equal to zero. That gives us the since delta $u_i$ is a variation that you are going to impose delta u small i can be assumed to be delta U capital I. So, this is going to be our equilibrium equation and this is going to be our traction …. condition. This is exactly what we would have got in the case of the small strain as well.

The approach that we have followed here is exactly the same as that of the small strain case. Yeah, I will write down the equilibrium conditions now separately if you want. Let us get that. What is inside the bracket, the first one is the equilibrium equation for the reference configuration which can be written as I, one second, I think it should be i I, sorry, iJ. Please, please note this thing. We had, I think we wrote that in the previous one. This is the problem. There is so much of i's and j's that we have to be very careful. Please note this.

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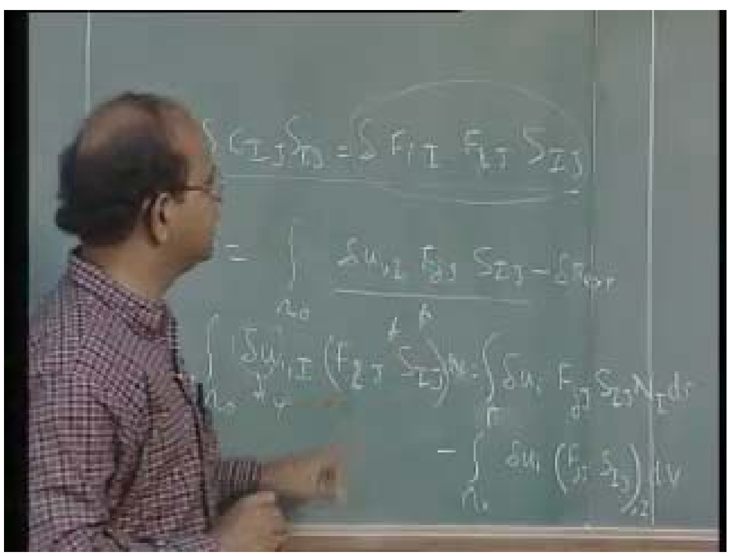

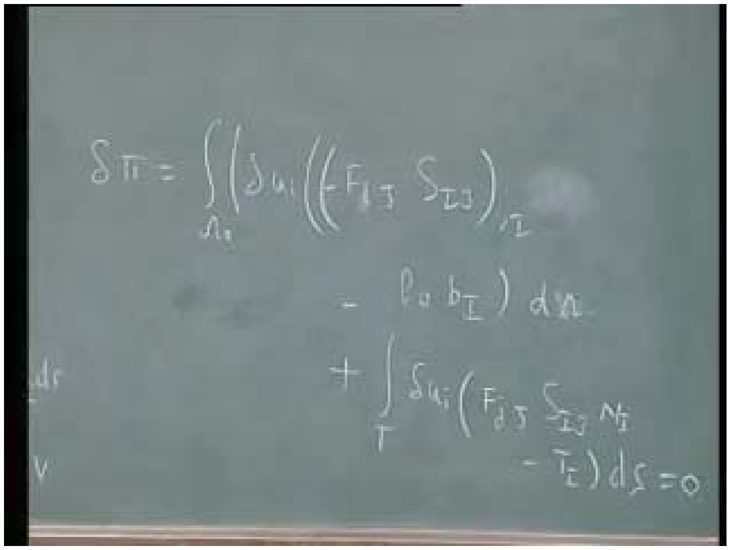

This is i, this is i, this is i, so that the first equilibrium equation can be written as iJ $S_{IJ}$ comma I plus rho$_0$ $b_I$ equal to zero.

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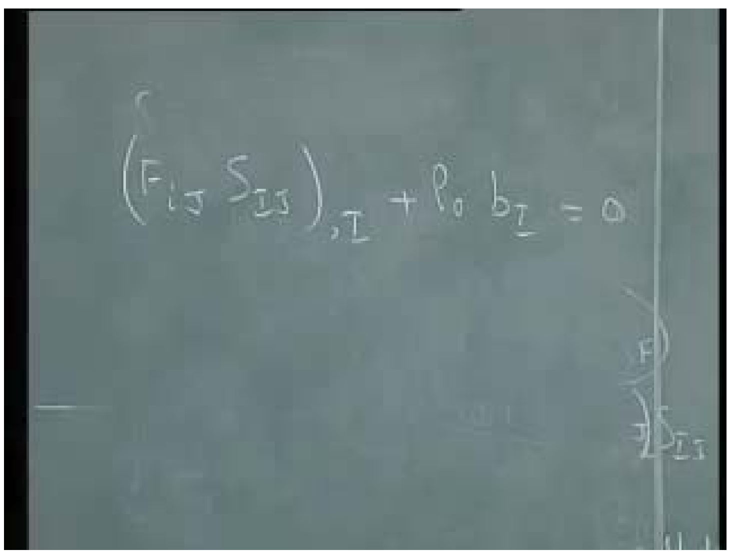

Now, it is, it is usually that since this had a delta $U_I$ term here, it is usually there is a small I would say, jugglery that is done.

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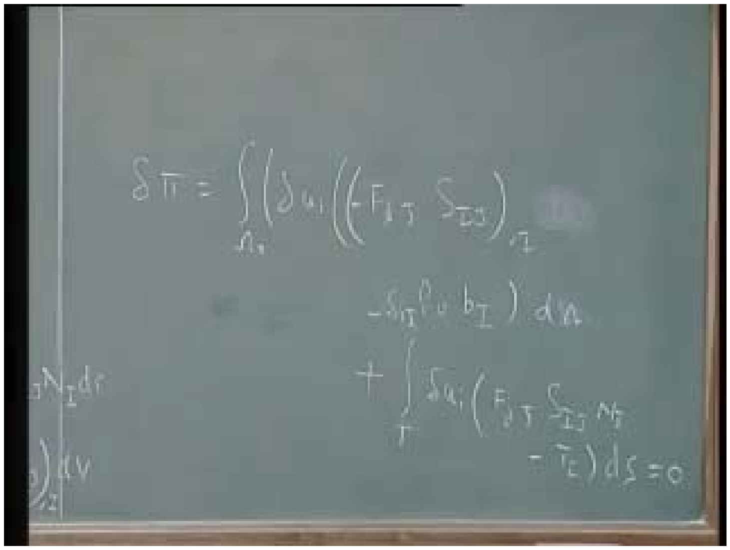

Instead of making delta $u_i$ the same as delta $U_I$, you usually say that this term has iI, so that both of them are the same, I want to show, so that this would actually have plus delta$_{iI}$ rho$_0$ $b_I$ equal to zero and the second term is the, there is a second term what you have here, is the surface traction term and that is written as $F_{jJ}$ $S_{IJ}$.

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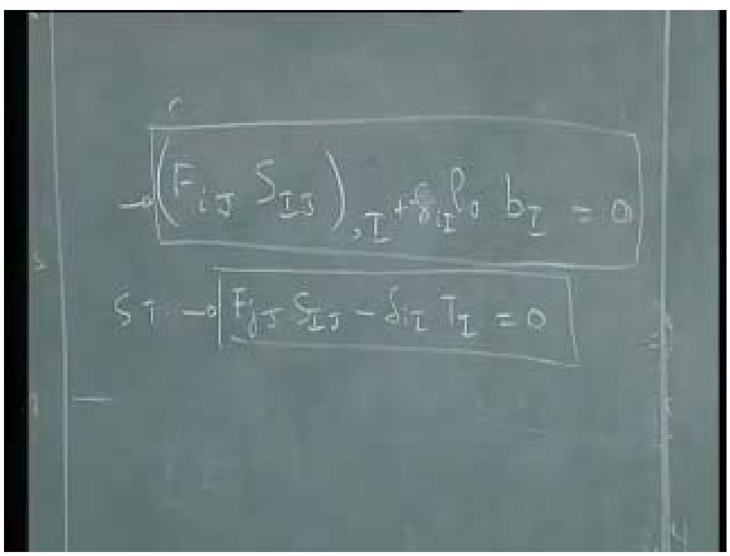

The same way, put a delta I delta$_{iI}$ T bar, is that I? Yeah I is equal to zero. This is the surface traction term and this is the equilibrium equation term. Check; I hope I have all the indicial notations are correct. Yeah, obviously because delta $u_i$ and both of them should be the same, so that that is why we had delta u capital I delta u small i, we had. So, the connection is that, it should be delta$_{iI}$. Delta$_i$ should be equal to I, that is why we had put this. When both of them are the same then this term will exist, because you remember that delta u capital I was the one which we, with which we multiplied the second one, because all of them are summations. Hence we had put that. You can sum that up and you will see that that exists. I know it is very tedious with too many indicial notations, but nevertheless the concepts are very simple and having crossed a major hurdle, now things are much more easier to work out, once we go into the finite element form.

Student: That will be $F_{iJ}$ or $F_{jJ}$.

$F_{iJ}$.

Student: The second one, second one.

$S_{IJ}$.

Student: no, no, below that, traction.
Traction, look at that.

Student: That will also be $F_{iJ}$; that will be iJ.

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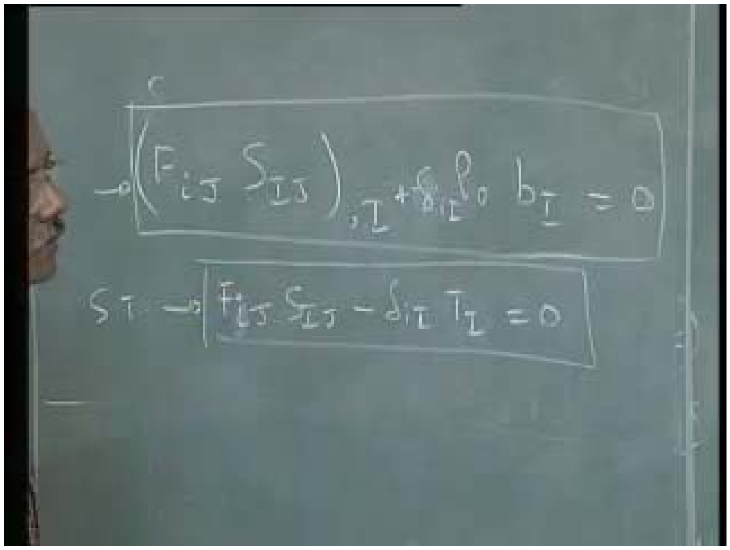

$F_{iJ}$; correct.

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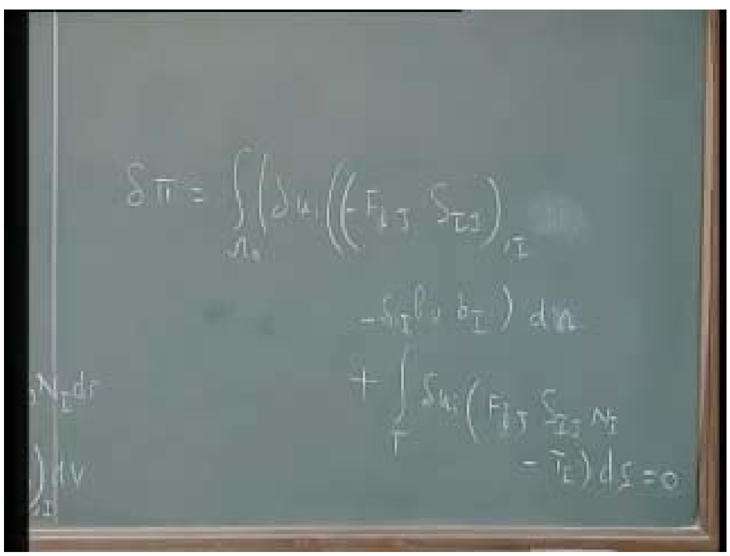

Then, there also, there also where is the, yeah, so that means here also $F_{iJ}$. Yes, $F_{iJ}$ $S_{IJ}$; see when you just logically follow, it would be very simple. $F_{iJ}$ correct $S_{IJ}$ minus delta$_{iI}$ $T_I$.

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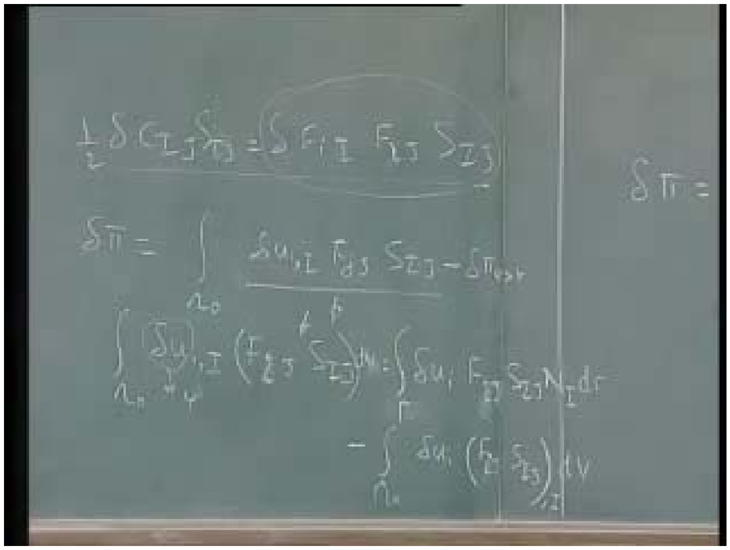

I hope there is no other, yeah, sorry, here also because that will be term right; down also, right. I will just give you one minute. Just look at this. I hope my i's and j's are right now. Just look at this and that, everything is fine. Though we have derived this, this is not of much importance for us, because we will go back to the first step. Remember that the first step is a potential energy theorem and the second step what where we said delta pi is equal to zero term, the delta pi is equal to zero is actually our virtual work type of principle. So, we will start from there and the procedure is exactly the same as we did before, but only thing is that our B matrix, there are additional terms like F's and hence our B matrix would all be different.

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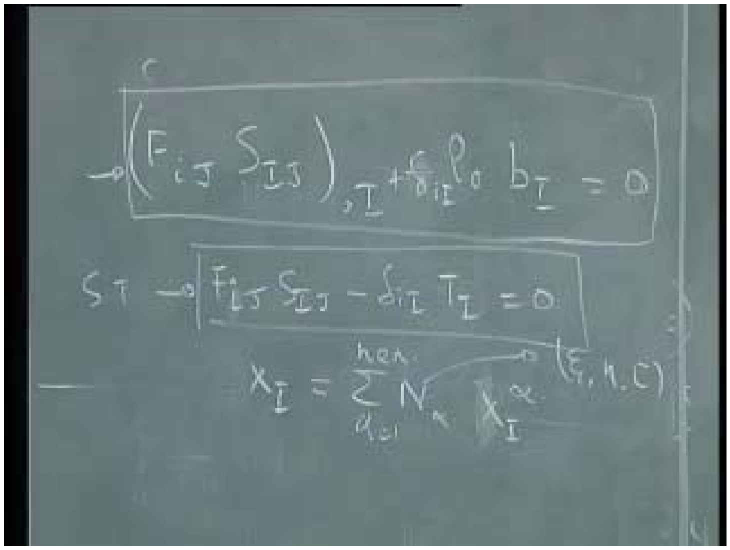

So, as the first step, let me write down the, for iso-parametric formulation $X_I$ is equal to sigma of, I will just explain what this alpha is …., X I alpha where alpha is equal to 1 to number of nodes per element. This is our regular, where of course this N is the shape function written in terms of, if it is 2D, psi eta or psi eta tau, if it is 3D and alpha is equal to 1 to nen. So, if it is an 8 noded element, then this will be 1 to 8. It is going to be quite cumbersome to write these expressions now. So, I am going to write only a slightly different form than what we had written before. So, that is my first expression for finite element. Just please watch the indicial notation. I hope I do not make any mistake on that indicial notation.

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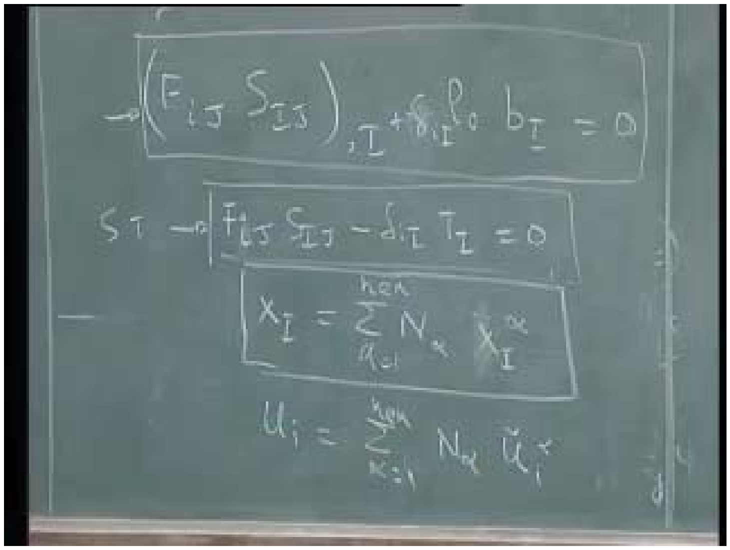


The second one is say u, sorry, $u_{iI}$. How do I write that down? $u_{iI}$, because $u_i$ will also be again written in terms of the same thing. So, let me write down that say, $u_i$ is equal to sigma alpha is equal to 1 to nen N alpha u i alpha, where this u i alpha are the nodal displacements from which now I can write down u i comma I. This is what I want, because delta $u_i$ is what I want.

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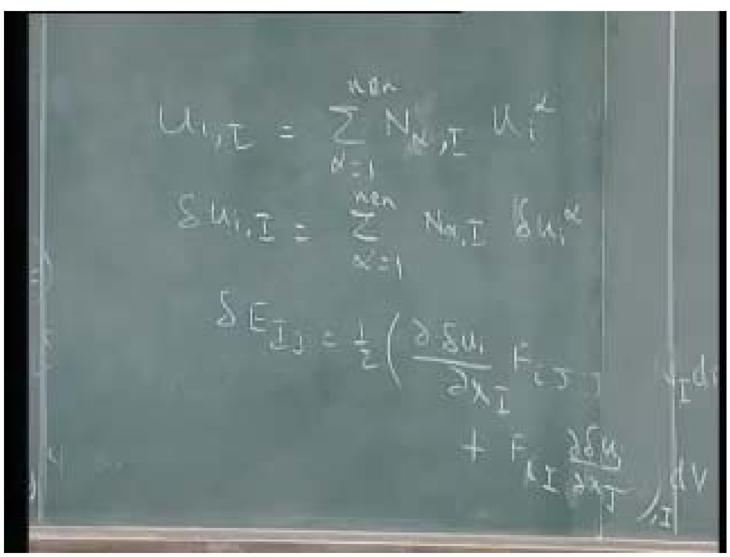


u i comma I is equal to sigma N alpha i comma I u i alpha. I think alpha was here or where did I put alpha? Yeah, alpha comma I, sorry we will put that as not i; alpha comma I u i alpha; alpha is 1 to nen. Yeah, so, that is u i comma I. So, if I want to write this as delta u i comma I, that will be 1 to nen N alpha I delta u i alpha and suppose I want to write down delta, delta E, how do I write down delta E? Say for example, delta $E_{Ij}$ is equal to half of delta u, dow of delta u i dow; this consists of two terms. How do I get this? This I get from E is equal to half of F transpose F, so, minus I, so delta $E_{IJ}$ is equal to, the same thing what we did for C F transpose F, so delta u i delta $X_I$ $F_{iJ}$. I hope I am correct.

Yeah, that is the first term F transpose F, the second term plus the second term will be, see this is F transpose F. So, dow of F transpose that becomes this F plus F transpose, so F transpose how do I write? F so F J, no, no iI, right into dow of delta $u_i$ J. Just check the indicial notation. Is that right? $F_{iI}$ dow of delta $u_i$ dow $X_J$; right. So, this will be delta $E_{IJ}$. Now the reason why we write delta $E_{IJ}$ is because, ultimately we are interested or we are interested to write down in finite element formulations in terms of strains, Green strains that is the reason why we have written this down in terms of delta $E_{IJ}$.

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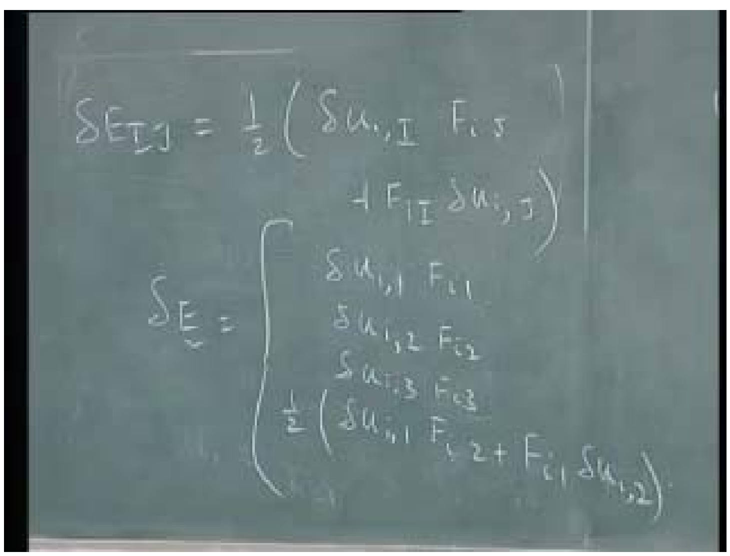


In other words, that delta $E_{IJ}$ can now be written as, in a more, in simple terms as delta $E_{IJ}$, I have a reason to write it like this, half of delta $u_{i,I}$ $F_{i\,J}$. It is going to be huge expressions plus $F_{iI}$ delta u i comma J. This is the delta $E_{IJ}$ expression. It is very useful to write this in slightly different form in the matrix form, because that is how we are going to work. So, delta $E_{11}$ would be, of course, there is summation in terms of i. So, in terms of, 1 1 can be written as delta u i comma 1, sorry, 1 $F_{i\,1}$. Right, look at that. In other words, delta $E_{11}$ is 1 and 1 I substitute, summation about i. So, if I want to write down 2 2, then it is delta u i comma 2, $F_{i2}$ delta u i comma 3 $F_{i3}$. So, if I want to write down in terms of the, I mean, $F_1$ suppose I want to write down 1 2, how do I write this term? Half of delta 1 2, yeah, delta u i comma 1 plus, sorry, F i comma 2 plus second term, second term here $F_{i1}$ delta u i comma 2, where summation of I is intended. That is in other words, this whole thing is, the whole thing consists of three terms here, three terms here; huge terms, delta u 1 comma 1 F 1 comma 1 comma 2 plus delta u 2 comma 1 F 2 comma 2 plus delta u 3 comma 1 F 3 comma 2 plus $F_{11}$ delta u 1 comma 2 delta $F_{21}$ delta u 2 comma 2 $F_{31}$ delta u 3 comma 2. So, those are terms that are involved. It is a, there is huge terms. Delta?

Student: i 2, $F_{i\,2}$ the second term one positive)

i comma 2

Student: i - there is no comma.

Oh, sorry, not, yeah i 2 there is no comma, sorry, I am sorry, $F_{i\,2}$. Now, it is usual practice to write this in terms of 2 gamma. If you remember that is what we had written down. This delta E epsilon terms were not written as in terms of epsilons, but in terms of gammas.

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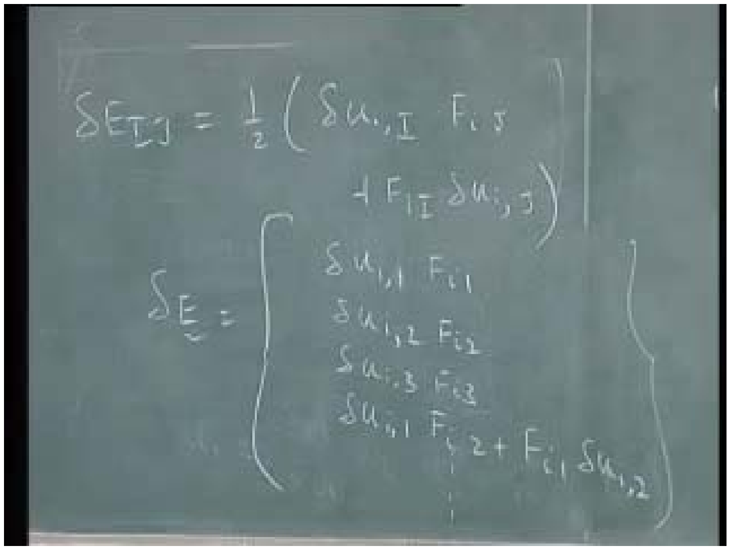

So, this half is usually taken out and that is how you write delta E. This is what we did in the first course as well. So, you can continue and fill this up. Please fill the rest of it. Now, we have another huge expressions coming up, because unfortunately we have to now substitute what all we have here, what all we have here we have to substitute it into this expression. Then, let us see how delta E would become. Let us see. Please try to substitute it. It is just an exercise in algebra. So, let us see just substitute the discretized quantities here into this expression.

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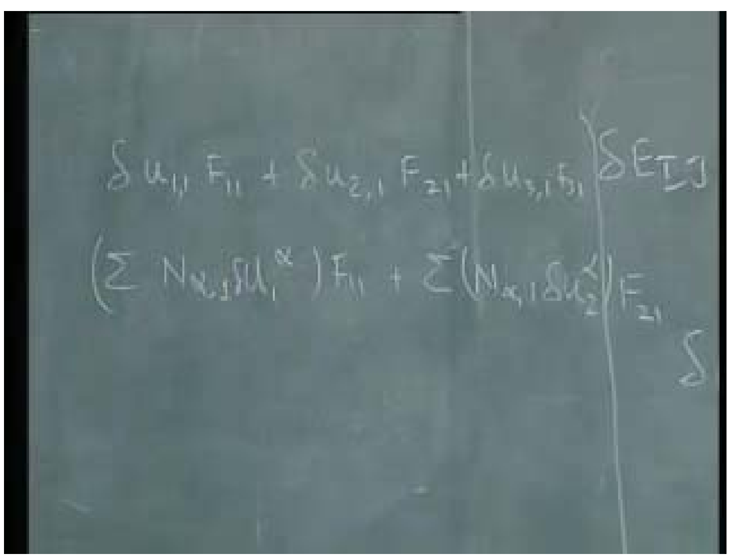

First term I will, I will only write, I will only write the two terms. Then, you have to fill this up. Yeah, so, let us, let us just write this down, first only the first term. Delta $u_{11}$ 1 comma 1 $F_{11}$ plus delta u 2 comma 1 $F_{21}$ plus delta u 3 comma 1 $F_{31}$; that is the first term; we have it like that. I will just remove all this. We are just looking at only the first term. So, I am going to substitute now delta u 1 comma 1 from my discretized quantity. I have remember, u i comma I in terms of N alpha and so on. So, what will, what will happen to that? So, this will be sigma N alpha comma 1. Then, the second term will be u 1 alpha. Since delta is involved there will a delta u 1 alpha sigma of this multiplied by $F_{11}$ plus sigma N delta u 2. What will I have?

Alpha comma 2 comma 2; so, delta alpha 2, no, 1 1, into u delta u delta $u_2$ alpha into $F_{21}$ and so on. You know, it will, it will be a huge expression again. So, now I am going to divide this into two categories. In other words, sigma of N alpha is divided into, N alpha will become B alpha, so that let me write down the first one like this. This expression can go on, you can verify that.

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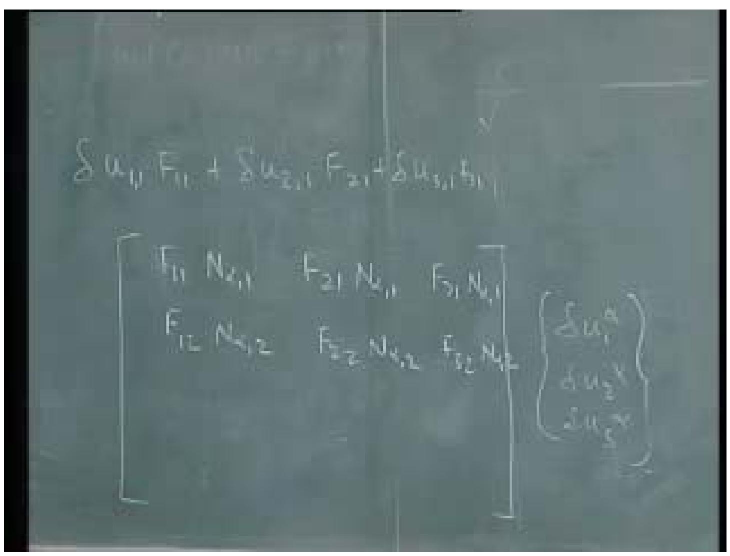

The first term, so what I am going to do is I am going to write that down as in terms of, note this, in terms of alpha, so that this will be in terms of delta 1 alpha delta u 2 alpha delta u 3 alpha. So, that will be $F_{11}$ N alpha comma 1, then $F_{21}$ $F_{21}$ N alpha comma 1 and $F_{31}$ N alpha comma 1 multiplied by this. This will be the first term of the sigma terms. This is what I am going to call this as B alpha. This will be a huge matrix again. So, you can work with this, the first term. You understand, so this will actually be, it will just, this will go on. So, this will be B alpha term; there will be another. For alpha is equal to 1, this will be like this. So, there will be another B for alpha is equal to 2, another B alpha is equal to 3. Complete this, I am not going to write this down; may be, second term, second term we can write down.

$F_{12}$ N alpha comma 2 $F_{22}$ N alpha comma 2 and $F_{32}$ N alpha comma 2, that will be the second term. That will be multiplied by this. In other words, you can, you can please fill this up.

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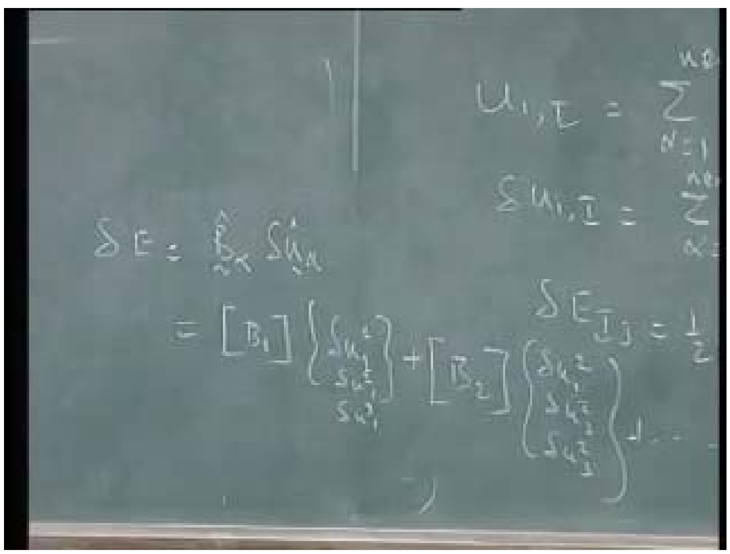

In other words, delta E can be written as B alpha delta u alpha. This is just to indicate that they are nodal quantities, where summation of alpha is intended. There will be say, if there are 8 nodes there will be 8 B's like this. For each of the node, you will have delta u say, we are talking about fourth node. In other words, this will be written as, there will be a $B_1$ having instead of, where alpha is equal to 1 in all these places $B_1$ into delta $u_1$. This 1 indicates this is a vector. In other words, this will become 1, 2 1, 3 1. This will be the three entries for the three degrees of freedom of node 1.

Like that you will have, look at that and this is the problem with the total Lagrangian; huge matrices you are going to get and programming all this very carefully even when we write i and j you have to be careful. This is again delta u 1 2 delta u 2 2 delta u 2 3 and so on; you will keep getting this for. Yeah, 1 2 3 are the degrees of freedom for node 1 in the three directions. Yeah, because alpha, I had written it like this, if you want alpha, keep it at the top.

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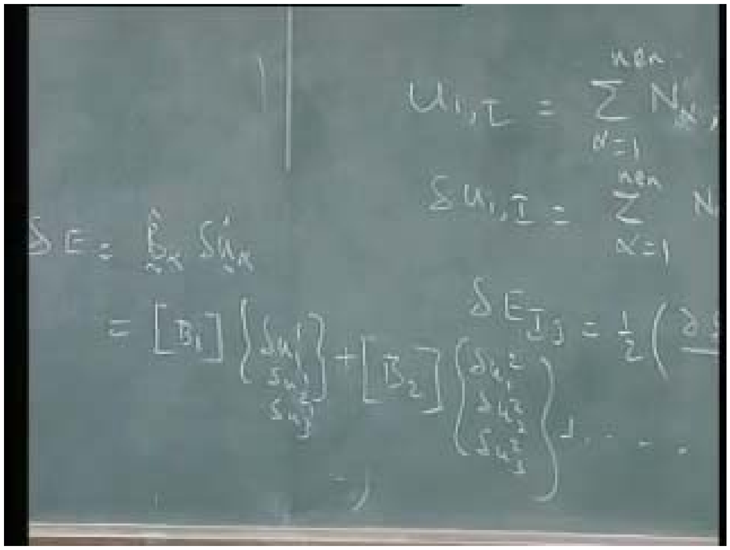

So, if you want, you write it like this. This is what you are saying because alpha is here I will put it here, so you can write it like that. So, 2, yeah, this is right 2 2 3 like that you will keep adding B alpha terms. So, there will be a, this will be a big chunk but, instead of writing all this, instead of writing all these things we will stick to just B alpha delta u alpha term.

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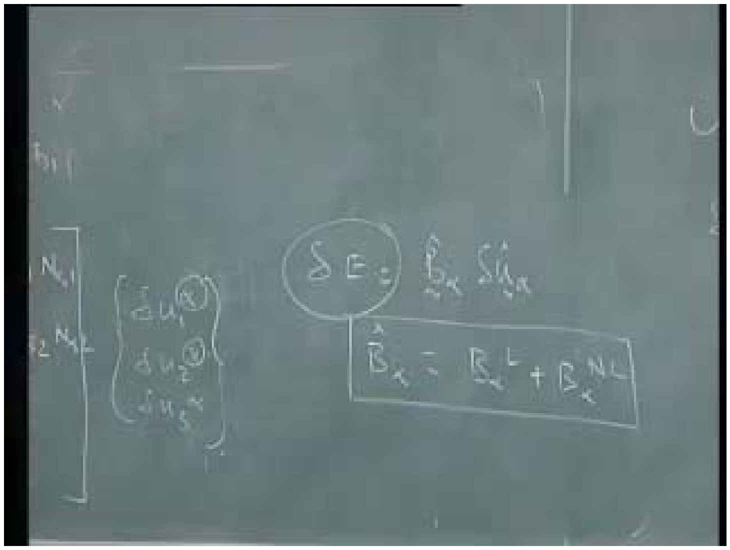

In fact, this B alpha can be viewed as which I am not going to do again, because they involve huge matrices. I refer to Zinkevichs volume II, all these expressions are available. May be, tomorrow I will just summarize and write down all the expressions in one shot, because deriving this is again going to take lot of time. Actually, you can see that this can be split up into two terms - a linear B alpha term and a non-linear term. So, this B NL, very famous term B NL, is the addition to BL. BL is what? Our old B matrix; is actually the one which converts delta epsilon IJ which we had seen to delta $E_{IJ}$. This is what we are going to use.

We will continue in the next class. I will summarize all the matrices that are there and then finish this expression. I am not again going to come back to total Lagrangian. We are not going to derive it in so detail fashion, but we will finish this derivation putting all the matrices in place just to give you a flavour of, feel of how terms are involved in the total Lagrangian formulation for non-linear finite elements. We will stop here and continue in the next class.