Advanced Finite Element Analysis
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Lecture - 27

Let us complete our discussion on constitutive equations by taking some very specific examples of this strain energy function.

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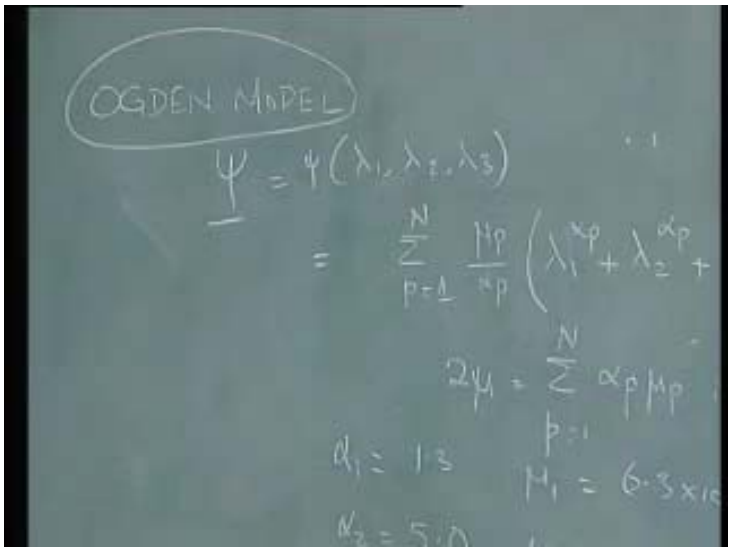

This we had already discussed, but anyway for the sake of completeness, let us do that and do a simple problem to understand how we apply, not the finite element problem, but an example problem to see how we apply these models. As I told you, one of the most I would say, important model is the Ogden model and we had already seen that most of these models can express the strain energy function in terms of the, what are they? Stretch, lambda$_1$, lambda$_2$ and lambda$_3$. So, a general model of Ogden is given by this equation.

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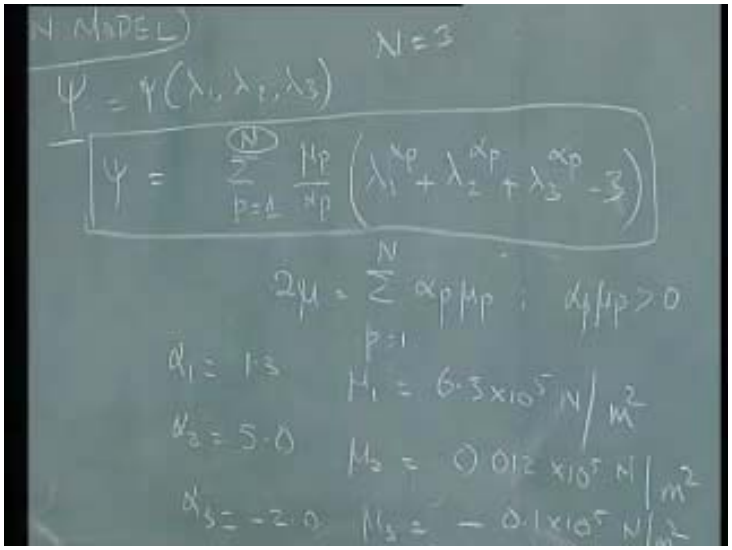

Psi is equal to, so, that is the equation that you use to define the Ogden model. There may be a small variation. In fact, if you look at Abaqus, I think there is a small variation as to how this is done, but nevertheless there are two constants mu_p $alpha_p$ and this p varies from 1 to n. In other words when I expand this, you will get mu_1 $alpha_1$, mu_1 by $alpha_1$ into this plus mu_2 $alpha_2$ into $lambda_1$ $alpha_2$ plus and so on. So, you will get a set of mu_p $alpha_p$ functions; p is equal to 1, 2, 3, 4 or 5. In other words, this N is in your hands, what is the N that you choose? Usually it has been found that when you choose N is equal to 3, when this N equal to 3, you get very good results in the sense that the strain energy functions that you get out of N is equal to 3 are able to express the or they are able to meet the values obtained from experiments. So, most people use N is equal to 3 in order to calculate mu_p and $alpha_p$. But, there is a bit of warning that I want to give here because, if you look at most of the softwares, the software asks for stress strain curves and then you will be able to give the stress strain curve and the program feeds these values.

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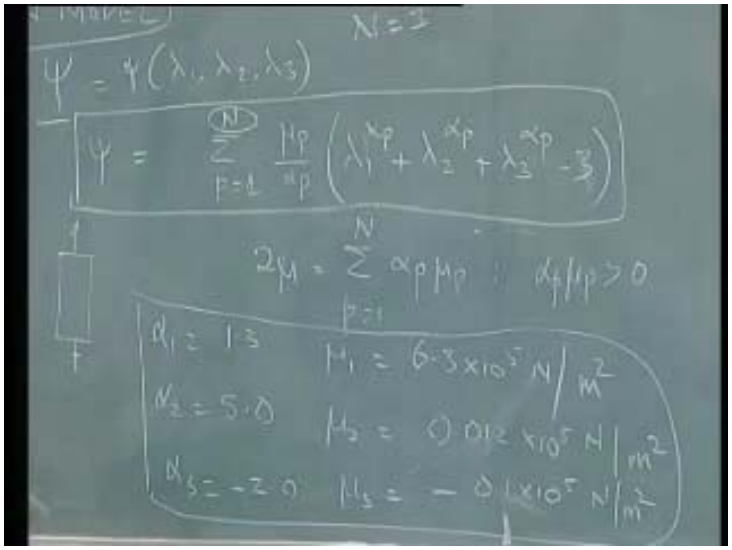

In most cases, because of lack of availability of data or whatever it is, we give only a uni axial tension case. So, this is what you do, an experiment you do with the uni axial tension case and then you give this stress strain curve and ask the program to fit mu_p, $alpha_p$ data. Usually that is what is done, but this is not adequate. Actually the mu_p and $alpha_p$ should be arrived at when say, when N is equal to 3 should be arrived at from not only a tension data, but also at least a bi axial tension and shear. These are also required in order to fit these parameters properly. In fact, there are lot of anomalies not only in finite element and finite element is a difficult task; we are going to see that also, but also the definition of the property values itself, the property values itself, for example we have found in, I mean experienced that a tension curve may not be reflected about the centre point for compression. In other words, there may even be difference between tension curve and compression curves. Then, the data that it fits when you give that together is different from when you give only the tension curve. So, when you give a bi axial curve, then it is different and so on. But, it is unfortunate that people do not spend lot of time in getting this data; they do not want to do it, so, they just take values.

Say for example, they take typical, this is a typical value; does not mean, does not mean this is the, these are the values that can be used for any rubber. It is a typical value of the coefficients. People just use this kind of value and expect to get a result. You will get a result, but may not be the correct result. The key to the success of finite

element analysis of elastomeric components is the correct definition of the mechanical behaviour through such constants like mu_p and $alpha_p$. Fortunately most softwares give or have the ability to fit these kind of curves. For example Marc has it, Abaqus has it and all that, but please note that you cannot take a default value and do a problem. As I told you in the last class, we can divide the whole problem of this kind of the hyperelastic material as compressible or incompressible hyperelastic material. Incompressible hyperelastic materials also to a certain extent consider or take into account nearly incompressible hyperelastic materials.

Let us do a small problem to understand what we mean by pressure term and how we actually calculate pressure. Just take down a problem. Consider an incompressible, consider an incompressible hyperelastic membrane that is subjected to bi axial deformation.

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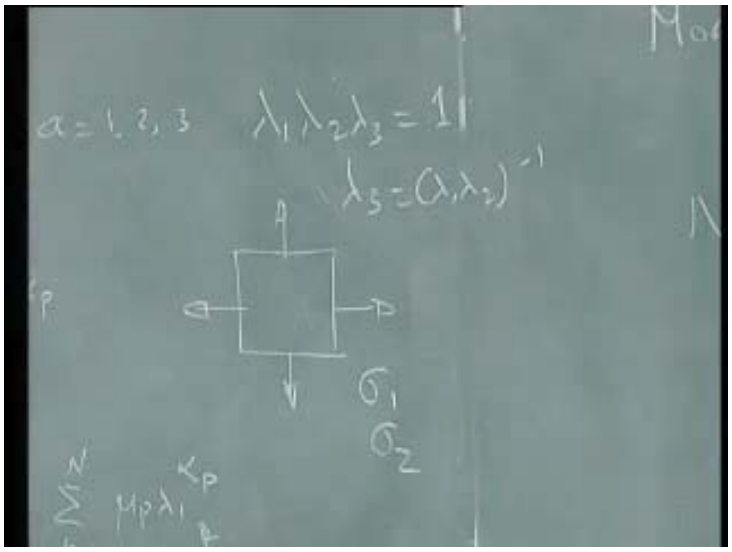

That means that say, sheet is taken and the sheet is subjected to bi axial deformation; membrane means very thin sheet, plane stress problem, subjected to a bi axial deformation. Assume of course, the plane stress state and specify the Cauchy stress or for in other words, find out the Cauchy stress $sigma_1$ and $sigma_2$ using the Ogden model. In other words, develop a relationship between $sigma_1$ and $sigma_2$ and stretch $lambda_1$ $lambda_2$ and of course $lambda_3$, which you can get as a function of $lambda_1$ and $lambda_2$. Let us see how you do this problem. One of the first things that I have

stated is that the material is incompressible hyperelastic; correct, hyperelastic incompressible material. So, that means that my $lambda_1$ $lambda_2$ $lambda_3$ should be equal to 1, so that $lambda_3$ is equal to $lambda_1$ $lambda_2$ whole power minus 1.

Let us see how we do this problem. We start from our general relationship.

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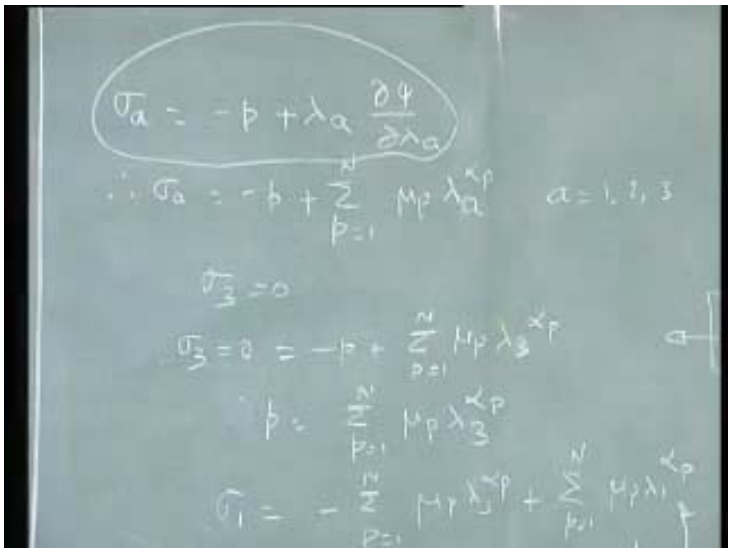

In fact, most times such general relationships are used to calculate the stresses even in finite element analysis. So, that is the general relationship. Now, in this relationship what you do is that is now defined by the Ogden model. You calculate this part, this part, from the Ogden model which means that take this, differentiate that with respect to delta $lambda_a$ and then substitute this into that expression. Is that clear? This is where various or different types of material models come into picture. The other material models are, before we proceed, we will just look at other material models.

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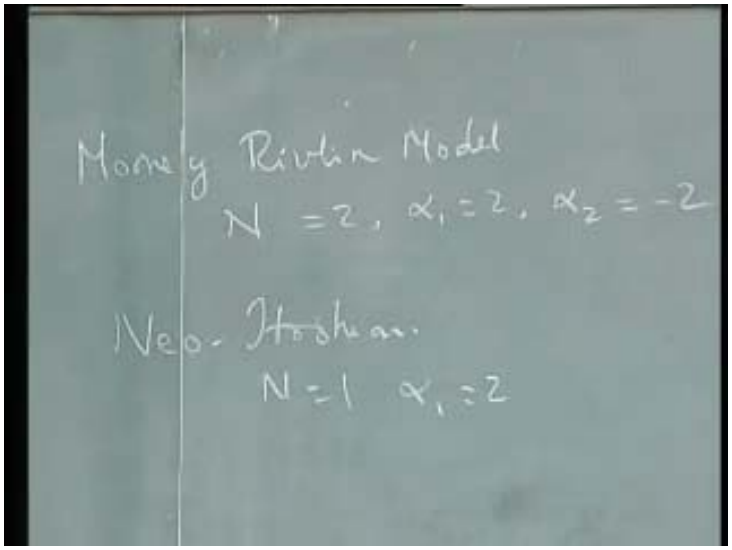

The other material models are what are called as Mooney-Rivlin model, Mooney-Rivlin model. Look at the way Mooney-Rivlin model is associated with the Ogden model. When N becomes 2, $alpha_1$ is equal to 2 and $alpha_2$ is equal to minus 2, then we reduce the Ogden model or Ogden model gets reduced to the Mooney-Rivlin model. Though there are, there is lot of theory behind it, I am not going to details of it, it is, it is not that Mooney-Rivlin is just like that obtained from here, but this is one of the relationships, but Mooney-Rivlin has its own theory behind it. As well as what is called as a Neo-Hookean model when we have N is equal to 1 and $alpha_1$ is equal to 2, we reduce this to Neo-Hookian model. Neo-Hookean model usually as you can see, the level of sophistication is quite low when compared to, if you call that as sophistication, in the sense that when N is larger, these class of Neo-Hookean rubber material models are good or elastomeric material models are good to look at not very, very large deformations, but moderate deformations you can get away with it by using Neo-Hookian models.

Now you may ask me a question. If I, if I have that kind of an Ogden model which encompasses all these models, why should I come, come down to Neo-Hookean or Mooney-Rivlin? Answer is very simple, because there you have to calculate mu_p and $alpha_p$ as you keep increasing p; 1, 2, 3, 4, 5, suppose 5, look at how many constants you have to calculate. That means that the test data that is required is also quite large. Many times it may so happen that the deformations are not very large. It is still in the

large deformation trend, but moderate, moderately large 7, 8, 10% maximum, in which case you can use, get away with using a much simpler model where your calculations are reduced to only one constant and your calculations here reduced to two constants and so on. So, you sort of make an assumption in N_1 and in fact this is, this Neo-Hookean is not just a reduction, it is also based on some concepts of statistical mechanics. That is the reason why this seems to work. You can sort of use these simplified versions of the Ogden model and then get away with it.

Many people use Mooney-Rivlin models to a great extent; that seem to work for many problems. But again, there is no one prescription to say that use always Ogden model and so on. This is one area where your knowledge of finite element is not sufficient. Knowledge of also the material behaviour of these kind of elastomers or biological specimens or whatever it is as well, is important as well. So, you have to know what models to use.

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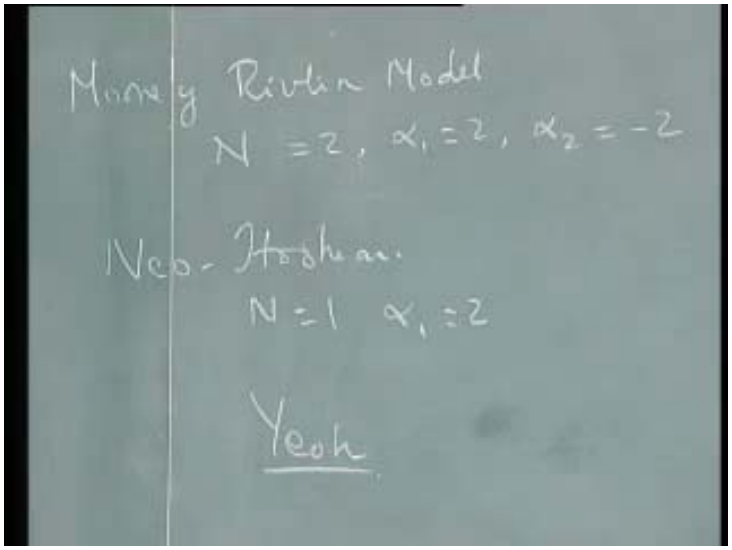

As I told you, I do not mind repeating it, that there are models like Yeoh model which has very specific application in carbon filled rubber which is used for say, tyre analysis. These are latest models. These are 50's model, this is the 90's model, but the procedure is the same. You need not worry about it. The procedure is the same which is just plug and play concept. All the concepts which we have seen is the same. Only thing is that that expressions keep changing. Of course, I have not done lot more thing

than, I know, than what is required. For example, there are certain relationship between static models and these models. In other words, with our linear elastic case there is a relationship between them.

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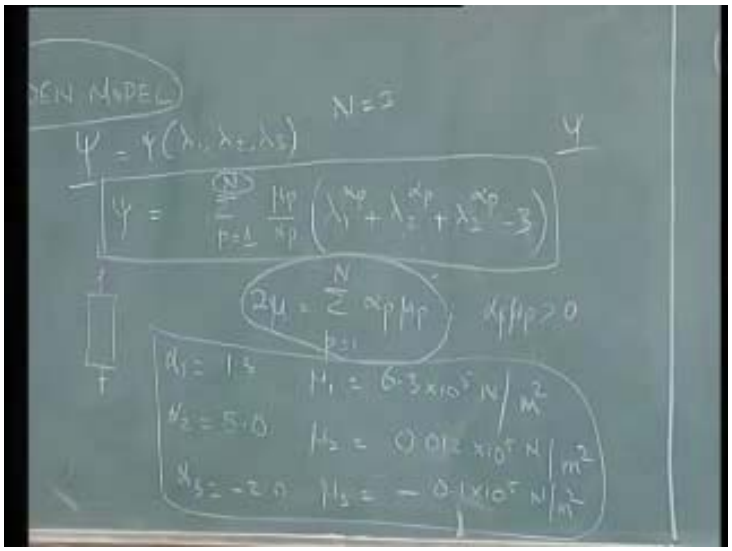

This comes about because of certain restrictions, further restrictions that you have to put on the strain energy function. I am, I am not going to the details of it. For example, for Ogden model we can see that sigma of $alpha_p$ mu_p is equal to 2 mu, where mu is the shear modulus of the static case that is in the sense that, of the reference state or for small strain case. We call this as the base state; we will call this as the base state that is just at the point where you apply a small deformation and the shear modulus what you calculate from your small strain case is equal to $alpha_p$ mu_p and so on. There is a reason behind this, why this comes about. This is because w 1 comma 1 or strain energy mu psi 1 comma 1 comma 1 should be equal to zero. Also, there are some other conditions as well. We will not go into the details of it. Just I want to state that these kinds of conditions are also required.

Now, let us come back to this problem what we had just defined for a plane stress problem where we have a bi axial tension case.

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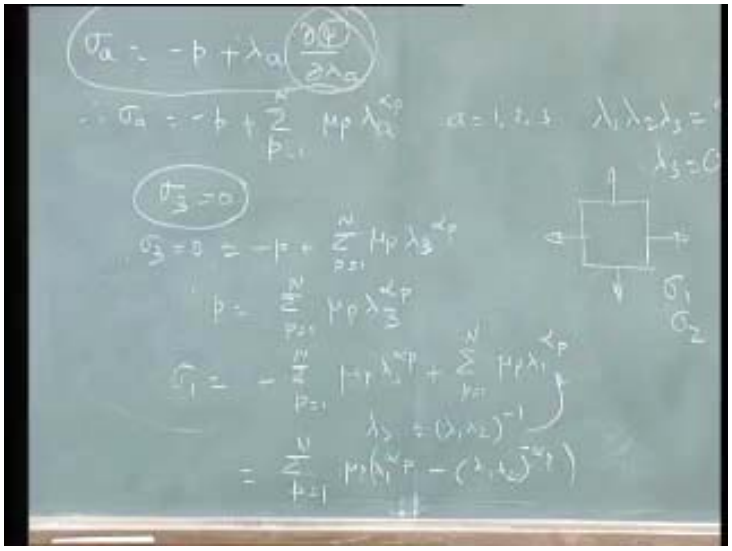

Now, since I had already defined that to be a plane stress problem, so, automatically $sigma_3$ is equal to zero. So, look at how I am calculating pressure or else pressure cannot be, is indeterminate, cannot be determined due to this kind of conditions either boundary condition or this kind of behavioural conditions, that you calculate p. So, $sigma_3$ is equal to zero in this case; kinematic conditions that comes about that is equal to zero means that minus p plus p is equal to 1 to N mu_p $lambda_3$ whole power alpha p, $lambda_3$ power alpha p, that is zero from which I can get p in this fashion.

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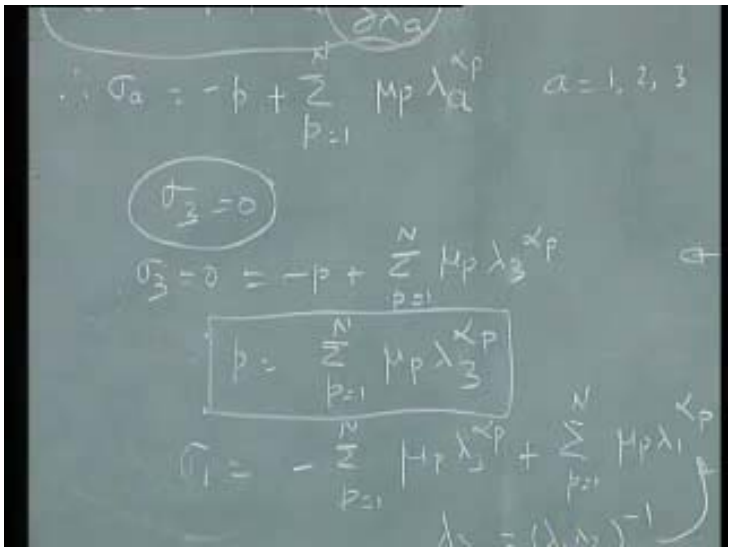

Now, substituting that into the expression when a is equal to 1 you get $sigma_1$. So, $sigma_1$ can be written as, of course, minus p plus p is equal to 1 to N mu_p $lambda_1$ power $alpha_p$, which say in this case p happens to be this and that when substituted here you will get this.

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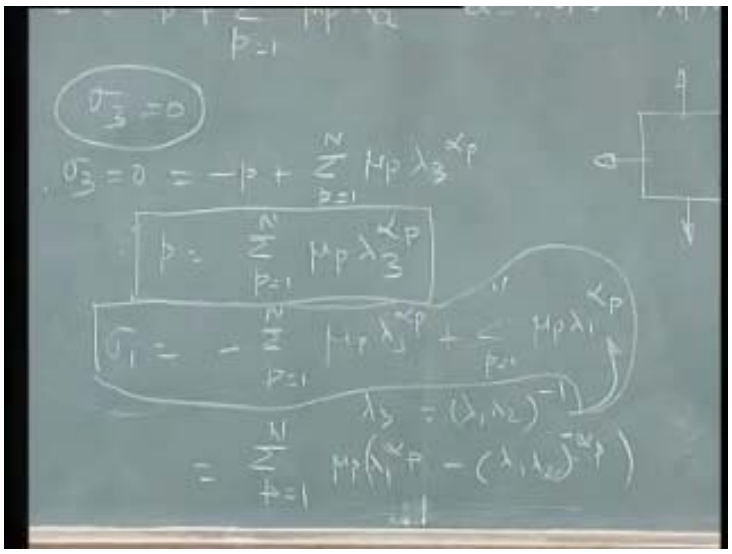

If I now put $lambda_3$ is equal to $lambda_1$ $lambda_2$ whole power minus 1 or 1 by, yeah, this is because I had put as, condition as, why is this is equal to 1, because we had put incompressible. I had told that they are incompressible, incompressible hyperelastic material and that is the reason why we had put $lambda_1$ $lambda_2$ $lambda_3$ is equal to 1. Yeah, that is correct. So, from there I substitute back into this expression and then into this expression, this expression and I get this form. This is a straight forward calculation of stress knowing the type of strain energy function. Is that clear?

Now having studied, yes I will not claim fully, at least to a great extent what different terminologies mean in the constitutive equations as well as continuum mechanics, you will get back to the finite element part of it. The whole idea of teaching this, may be in the last 10-12 classes, is to understand all the terminologies that will be necessary in order to even talk the language of finite deformation elasticity. Now, I do not claim that what I am going to teach is good enough to implement completely a non-linear finite element code, because that goes a long way still. There are things which you have to work out, but you will have all the background; if you want to look at a paper

and study, you will have all the background to understand how non-linear finite element analysis works.

In my opinion, one of the major problems with non-linear finite element is that people do not even understand the manuals that are given with the software. Whether you use Abaqus or Marc or Ansys or whatever it is, for non-linear finite elements, people do not even understand the terminologies that are given in the manuals, people do not even understand. If he says stretch based element, people do not even understand what stretch is. So, the reason why I did all this is because, at least you will be able to understand, you know the commercial codes, even if you want to use it. Commercial codes for non-linear finite element cannot be used blindly; you cannot just take a code, because if he asks you are you are using Ogden model, if you do not know what Ogden model is, where it comes in the theory, where it fits in, what is this $alpha_p$ mu_p, if he asks for $alpha_p$ and mu_p and he is going to ask you, are you going to use N is equal to 3 or N is equal to 5, if you do not even know that, you are not going to use, forget about writing codes, you are not even going to use a finite element code and definitely you have to use this kind of Mooney-Rivlin, Neo-Hookean or this thing and you will call general polynomial expression for psi, for the strain energy, then you should know at least what these things mean and he would also talk about what we are going to say now for total Lagrangian and updated Lagrangian.

Having completed all this, we will get back to the finite element form. Yes, I have not done one aspect. Yeah, I know it will be very useful, but due to lack of time one is what we require, what we are going to require now. I am going to say that what we call as the incremental reduction of many of these things. I will, I will tell you what it is as I go along, but we do not have time to get that. Again that is a topic where we have to go through for next 4-5 classes. Because of lack of time we are not going to cover it, but I think it would not be very difficult; it would not be very difficult for you to derive or catch that. But, one thing again I would like to warn you is that, there are very many algorithms in non-linear finite elements. It would be impossible in any class to cover all of them; it is, it is almost impossible with so many papers that are coming out recently on so many formulations.

Now, if you look at stretch for example, people also work with logarithmic stretch and so on. Because of the ease with which you can implement certain things, people work with C_{ij} or people work with lambda or the natural logarithm of lambda and so on. So, people work with different types of strain measures in order to express the strain energy function, but does not, does not deviate from what we have done, but the expressions, the algebraic expressions that are involved are going to be different, different. Again you have to calculate the same gradient with respect to lambda or whatever it is, but the expressions will be different. So, that is one of the things that I want to warn you, because it is almost impossible in a course to cover the whole gamut of non-linear finite elements.

The same way say, we are going to talk about mixed formulation. There is so much of theory behind mixed formulation that it would be impossible for us to cover all those kind of things. We will any way by the end of the course, you will be aware that these are the things that exist. When someone or when you read a paper someone gives you that this is the formulation, you will understand where you stand; may be algebra you may have to work out separately.

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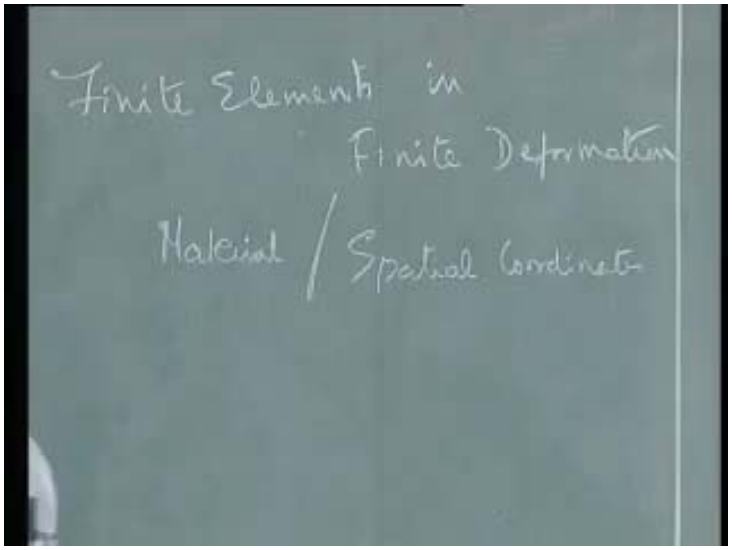

With that background, let us come back to the finite element formulation, in finite deformation. In fact, if you look at the finite element formulations in finite deformation, whatever we have done with respect to finite deformation you can

immediately say that I can approach this problem in two fashions. One is to look at what we called as the material co-ordinates or to express all these coefficients or functions or whatever it is, in terms of material co-ordinates or in terms of spatial co-ordinates. In other words, it may be possible to express the whole of these things in terms of a Lagrangian formulation or Eulerian formulation.

Eulerian formulation is the current configuration and Lagrangian formulation is the reference configuration, fine. But, we are going to deviate a bit from these two and have some sort of a middle path. Note this carefully. The Lagrangian formulations straight away are called as total Lagrangian formulation or in other words, if you formulate the whole problem in terms of the reference configuration, you call the finite element formulations based on all the quantities which are given in terms of reference configuration, you call that as the total Lagrangian formulation; it is total Lagrangian formulation, number one.

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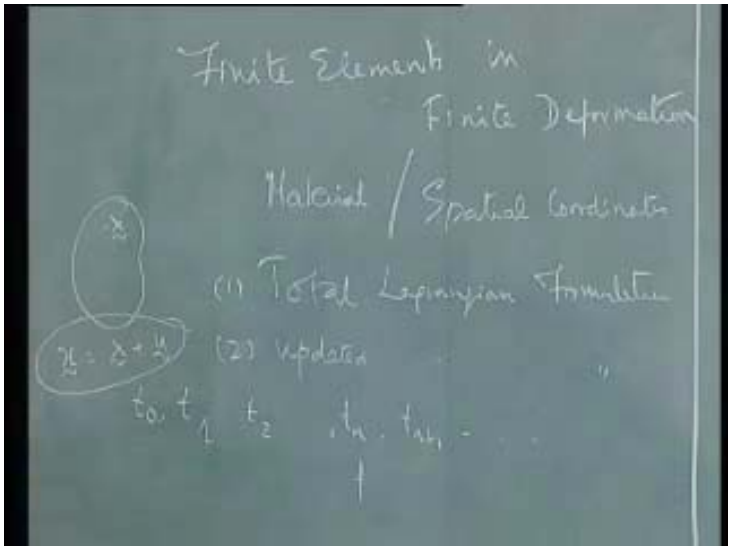

That means that the reference configuration is not updated, you have one reference configuration and that reference configuration which is defined in terms of x_1 x_2 and x_3, the points there, that configuration stays; you do not update. There is another formulation which is not actually spatial co-ordinate formulation called as the updated Lagrangian formulation; people give different names to this. If you look at books by Belasco and Crisfield, they call this as an updated Lagrangian formulation. If you

look at books like Zinkevichs, he calls this as current configuration formulation. But, there is a difference between the current configuration or updated Lagrangian formulation and Eulerian formulation, though the difference is very subtle. In other words, you may be wondering why I did introduce an updated Lagrangian and a total Lagrangian, one based on the reference configuration and the other based on the current within quotes, we will define what current is, why we have introduced new terms, why not I just call this as Lagrangian formulation and why not I call this as an Eulerian formulation.

There is a subtle difference. Of course, Lagrangian formulation is called total Lagrangian formulation. In the updated Lagrangian formulation, the configuration, the reference configuration is updated to get to the converged value of the configuration. All of you know now, that we march in terms of time steps, of course, we are going to use Newton-Raphson method, there is no doubt about it. So, we will march in terms of time step, which means that we are going to be, the loads are going to be given in terms of pseudo times t_1, t_2 say, up to t_n, t_{n+1} and so on, till we exhaust the whole of the loads for a static problem. Then, if I am say, in t_n and then I want to go over to t_{n+1}, then let us, let us say that we start with say, t_0. t_0 is the place where we start this reference configuration; t_0 is equal to zero is the actual state, then in the updated Lagrangian formulation, we update this configuration by adding to this x say, calculating x, adding to this x, we say the displacements that has happened up to t_n, this is updated. Though this is called current configuration, actually it is not a current configuration. I hope you understand the difference. It is not current configuration.

Current configuration is an instantaneous configuration at time say t, when I am going from t_n to t_{n+1}, actually in any time t between these two, that configuration is current configuration that is the Eulerian configuration. But here what you do is you keep updating the configuration, you will see why you are doing it in a minute; you keep updating the configuration and treat as if the reference configuration at t_n is equal to this updated configuration. So, this is called updated Lagrangian formulation. Yeah, any question?

Can you use the mike?

Student: Sir, we are updating the displacement …..

Not displacement, we are updating the co-ordinates.

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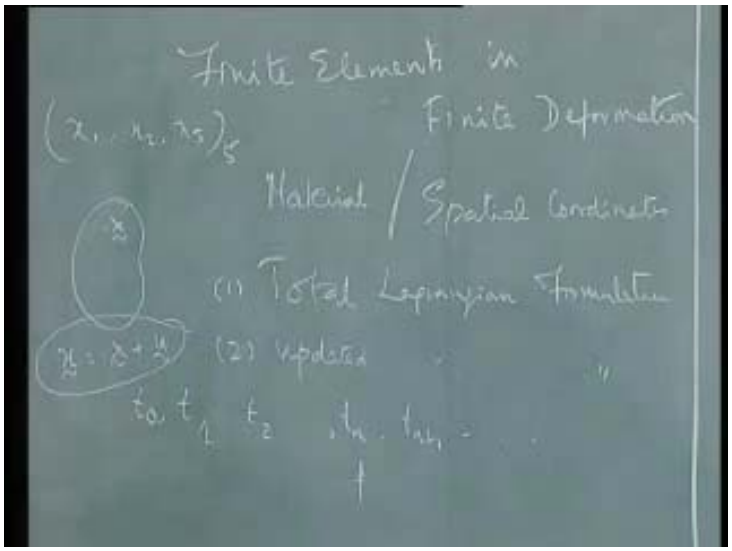

Yeah, actually since we are into finite element regime, you can say that we are updating the co-ordinates say x_1, x_2, x_3 of say a node, say 5. Yeah? Yes, the elements shapes are different that is in the sense that element may get deformed. So, you are into another configuration where the element shape has changed. Shape function is, anyway we are using iso-parametric formulation, so we are using all the time iso-parametric formulation. Let me, let me tell this very clearly. This is usually the confusion many people have. Though, if you look at a textbook, the iso-parametric formulation comes in the sixth chapter, you talk about so many other things before you come to iso-parametric formulation, let me assure you that every code uses iso-parametric formulation only.

So, whenever we talk about any of these things they are straight away extension of iso-parametric formulation, which you had studied in the earlier class. There is no doubt. Let me make it very clear. When you talk about integration schemes here, they are the same as that you had studied in your previous classes. When you are talking about shape functions here in terms of psi, eta, tau or whatever it is, 2D or 3D, then again they are the same as what you have done in your earlier classes. So, the iso-

parametric concepts are valid and are just extended to the finite deformation form, but the B matrix and all those things have, are going to change. So, strictly speaking updated Lagrangian is not an Eulerian formulation, but just that the configuration is updated.

There are several advantages with this. We will see what they are as we go on, but it does not mean again that all packages use updated Lagrangian. Packages also use total Lagrangian or they use updated Lagrangian, many of them they use. Can you, can you just use the mike?

Student: When we converge at a final value that will be exactly equal to the Eulerian co-ordinate or updated Lagrangian, because updated Lagrangian co-ordinate formulation we are assuming ….

No, actually, let me explain that again. Updated Lagrangian is not in between, it is not in between t_0 and t; it is not in between. So, suppose I have arrived at t_n that is in other words, I have converged value at t_n, what I do is I update the co-ordinates of the configuration, of the configuration and get to a new, within quotes, reference configuration. It is actually not a reference configuration, but, within quotes, reference configuration which has stress, strains and all that in built into this. So, that is the difference. That is why people are afraid to call this as an Eulerian formulation. In Eulerian formulation what happens is that, the mesh remains the same. Here, mesh gets updated, but still it is current, so that is why there is confusion between Eulerian and updated Lagrangian. So, I spend some time to explain it.

In fact you look at Zinkevich's, you would not call this as updated. All formulations are with respect to the current, you would call this as current Lagrangian formulation. In other words, updated Lagrangian is a current configuration formulation. Now, you may have a question whether we have to use a total Lagrangian or updated Lagrangian and whether there is going to be any difference between them, how do you choose one over the other and so on. Now, if you look at the formulations, in fact I am going to go through first total Lagrangian and convert it into an updated Lagrangian, you would see that both the formulations are the same. The only thing is to which configuration you refer to, whether for example, the integrations that are

involved are done with respect to the reference configuration or current configuration or in other words, theoretically speaking, both of them should give the same result; there should not be any doubts about it. But the difference is for a person who actually does the coding.

Specially if you have a small deformation code and you want to convert this into a finite deformation code, then the amount of effort you put in, in total Lagrangian formulations are usually much more, much more than in the updated Lagrangian formulations. Hence many people prefer to use updated Lagrangian formulations, though there are claims and counter claims that in some situations one is better than the other and so on.

Now, let us look at first the total Lagrangian formulations. If you, if you see what we are going to go through, it will be exactly the same as what we had already seen.

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In other words, here again we are going to see, we are going to look at what is called as displacement based formulation, which is our regular formulation which we had used in our earlier course, as well as mixed formulations. Both of them are used. Why does mixed formulations or why do mixed formulations become important in finite deformation? The answer is very simple, because many of the materials that you use in the case of finite deformation are ones which are incompressible. In other words,

these guys, these materials when used with displacement based formulations have a tendency to lock. In order to avoid that, we go to mixed formulations. We have already seen what mixed formulation is. Now, only thing is that the way we are going to look at the strain, the stresses are going to be different.

Again, to answer an age old question whether if I use mixed formulation for a situation which does not warrant it, where there is no mesh locking and so on, will I get a better result than displacement based formulation? No, you will not get; the results should be the same and this mixed formulation should reduce to a displacement based formulation if it does not warrant that situation. So, let us go through now the, may be the first two steps or three steps and we will continue in the next class, see, how to do a total Lagrangian approach.

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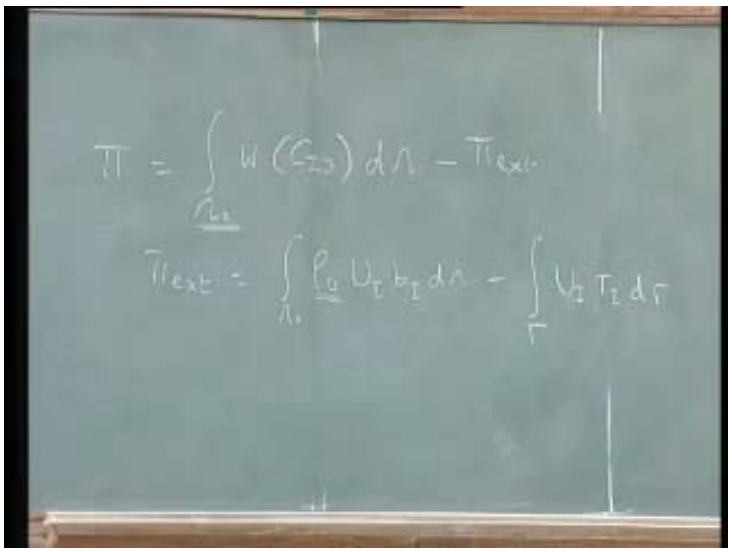

The starting point is the same. We write the functional, but of course, we assume the existence of strain energy and as I told you the whole difference is what is it that you use here? The integration is done over the original volume and write this as this strain energy. So, if I want to, I will just change this notation to W. It is easier to understand from many papers, so that that will give you the strain energy term $omega_0$ minus the, what we call as pi external term. This is due to the external work done by the external forces or the potential that is lost rather due to the external forces and the pi external term of course consists of two terms again. One term is due to body forces. See,

where I am integrating again; that is with respect to the original configuration into rho_0 U_I b_I d omega. So, this is the first term. This is due to the body force term and the second term you have gamma. This is the boundary again referred to the original boundary and this is equal to U_I T_I d gamma.

Now, note two things. One is that, look at this. This is the original rho_0, so original rho_0 and look at these here, capital, which means that we are referring to all these things with respect to the original configuration. So, what I am going to do is to use small letters i, j, k, etc to mean the current configuration and we are going to use these capital letters to mean the reference configuration. Again, you would see that this is the work done by the traction terms or the potential loss due to the traction terms, lost by the traction terms and that you will see that this is referred to the original surfaces, surface area.

So, right away the concept is very simple. If I have to convert this to the current configuration say, omega, current configuration, what I have to do is to replace these terms by relevant terms which refer to the current configuration. That is the only difference.

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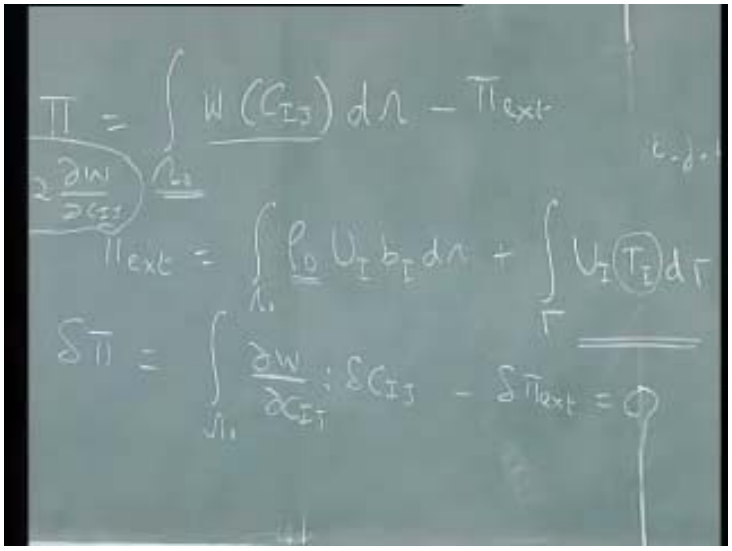

Now, let me write the variation, first variation of del pi. Now, before that let us also remember, you know it; I am just saying that this is the, this is a very general

statement and in fact, all of you know how to write S_{IJ}. S_{IJ} is the second Piola-Kirchhoff stress. Note that we had derived this already, is equal to dow W by dow C_{IJ}. You know it, but anyway I am just writing it. So writing this, it is quite simple, $omega_0$ dow W by dow C_{IJ} colon delta C_{IJ}. That is the first term here, which is due to the internal or the rather due to the strain energy term minus delta pi external term and that first variation has to be zero. Note that again that, this would be the starting point for our Newton-Raphson scheme.

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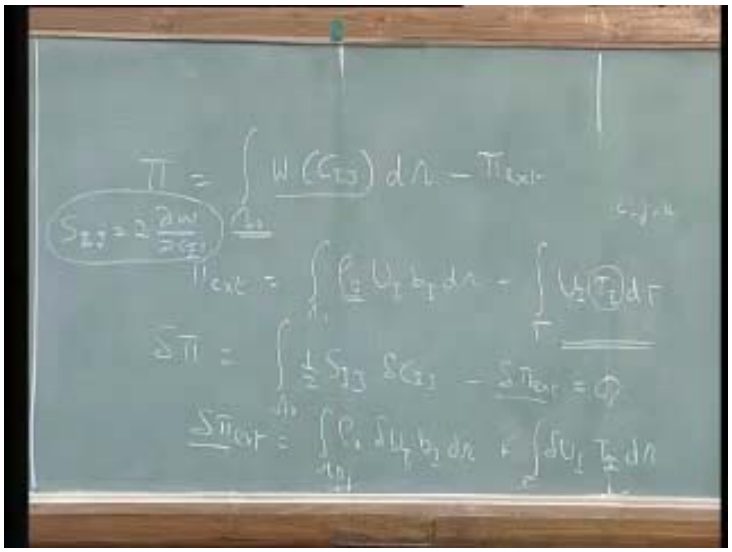

Now, this can be replaced by the relationship between the second Piola-Kirchhoff stress and W, so that that can be written as delta C_{IJ} half of S_{IJ} delta C_{IJ} minus delta pi, external terms. Of course, you know how to write the delta pi external terms is equal to $omega_0$ rho_0 delta U_I b_I d omega plus ……. So, what you do is substitute this here, is equal to zero. Now, I hope this is clear. As I told you this would be our starting point for our Newton-Raphson scheme. Now, before we go to the Newton-Raphson scheme, we have to look at certain new things. That it is nothing new, but in the new terms that we had introduced how the variation of these terms can be given?

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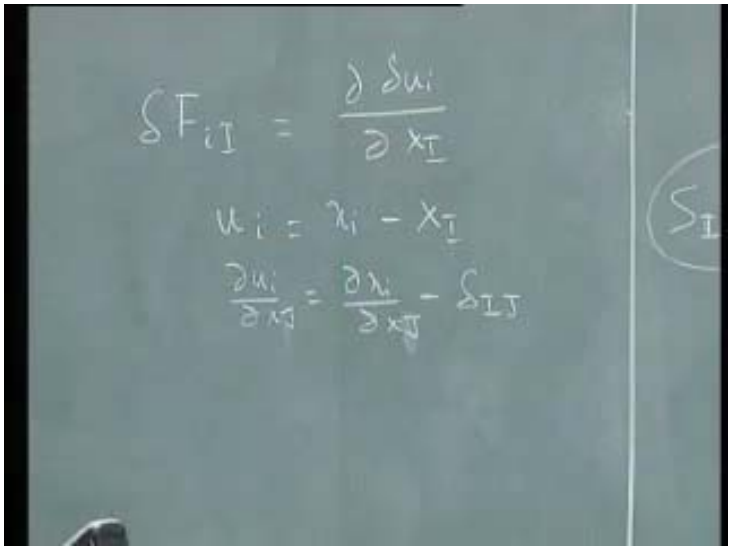

For example, we know that F, the deformation gradient tensor is a two point tensor. That means that it is written in terms of small i as well as capital. That is in other words, this is due to or this has contribution from both the current as well as reference configuration. But note again, I know, note this here that the S_{IJ} has everything. This is an, this is a quantity which is belonging to the Lagrangian co-ordinate system; S_{IJ} is with respect, both of them is with respect to the capital letter. If I want to find out delta F, how do I find out?

Now you know for example, that you can write for example u_i is equal to x_i minus capital X_I. In other words, if you want to write dow u_i by dow x_i, you can write this as dow x_i dow capital X_i minus $delta_I$. That is depending upon, let me write generally as J, so that $delta_{IJ}$ and so on. So, delta F_I now can be, from this you can write this as in terms of delta u_i as dow of delta u_i by delta X_I. That would be the delta u or delta F_{iI} term.

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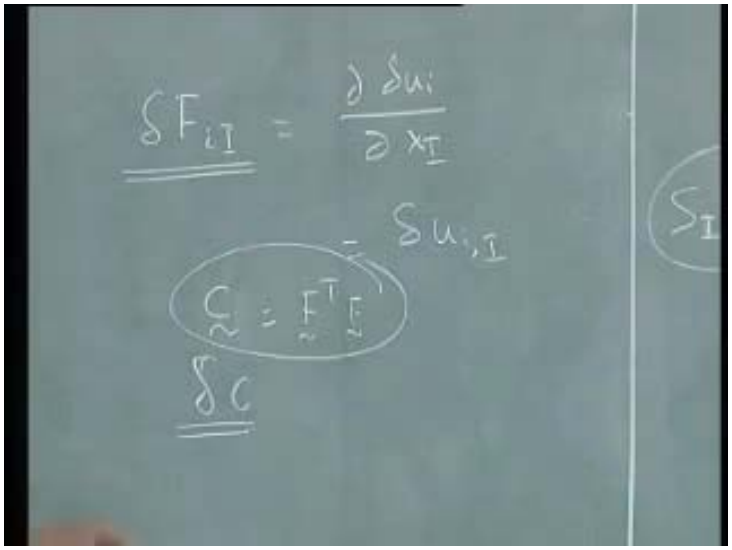

In other words, this can be written as delta u i comma capital I, so that now delta pi which consists of these two terms, the strain energy term and the external work term, these two terms can now be written as by substituting this expression into and of course I mean one more thing I forgot, may be we will write that later. One more thing I have to do that what is C_{IJ} or C? Remember that C is the F transpose F, so that we have to also get delta C as well. How do you calculate delta C? What I am going to do is that, having known how I write delta F and having known the expression for C, what I am going to do is to write down delta C in terms of delta F from here, substitute that into this expression and then do an integration by parts, in order to get the final expression for delta pi. You will, you will see just now why I am going to do that or may be since it involves a couple of steps, we will stop here and we will continue this in the next class.

What essentially I am going to do is to derive first the fundamental equations in terms of the indicial notation, then go over to the matrix notation after putting forward the fundamental relationships between the discretized quantity and the continuum quantity. So, we will stop here and we will continue it in the next class.