Advanced Finite Element Analysis

Prof. R. KrishnaKumar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture – 26

Yeah, last class, we were looking at the constitutive equations and we are going to continue that. May be we will finish it, summarise the whole thing and finish this topic. Topic is very, very vast; in fact, it can be a complete subject on constitutive equations and the relationship between so many different quantities which we have mentioned in the past about 4 or 5 classes. We are not going into the details, basically because we have to understand all the terms that are defined. Tomorrow, if you want to look at any other new material and or any other constitutive equations, what you have to know is only the terms and we are also, I hope you understand the big picture of where we are coming from and then fit in the smaller pictures accordingly. In other words, if you remember, we came from the isotropic or anisotropic and isotropic materials. We then branched off the isotropic material, then we said that for isotropic materials we have a special rule. We said representation theorem and then from there we developed a very, very general relationship between the stress and deformation.

If you remember, this was the result of my representation theorem. It stated that for a scalar value tensor function if isotropy is going to be satisfied, then it would be possible for us to write this in terms of the invariance. From there we wrote a very general equation. So, that equation is valid for all isotropic materials. From here we are going to again branch off.

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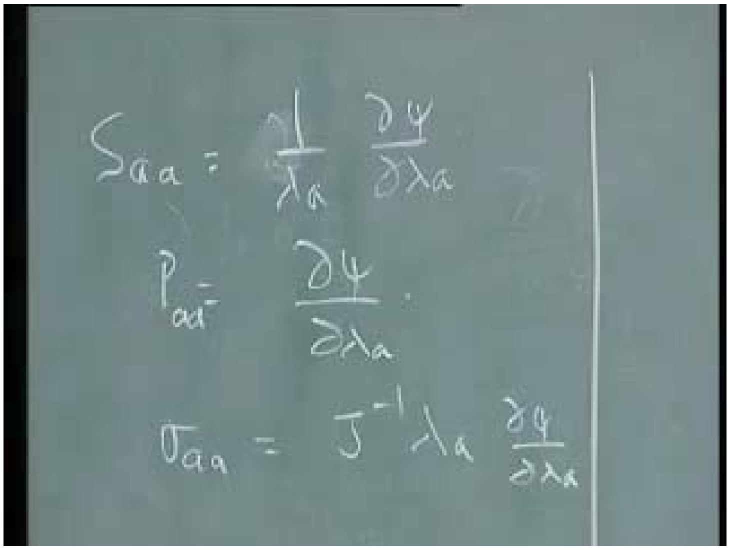

Before we branch off, ultimately when we left the last class we said that the stresses, different measures of stresses for example, second Piola-Kirchhoff stress, the first Piola-Kirchhoff stress, the Cauchy stress and so on, can also be written in terms of stretch which means that, note this carefully that, stretch is stretch are nothing but the Eigen values, square of them or the Eigen values, C and so on or in other words, U and so on. In other words, when I write these stress measures in terms of stretches, automatically the stress measures do have only the diagonal terms. That is why we had written it like this, diagonal terms.

It is very common today to work, most of the the numerical analyst work, with stretch based constitutive equations. That is the reason why I had put this down. It becomes much simpler to implement it using finite element analysis, in finite element analysis instead of looking at the constitutive equation as we did, you know, before this in terms of just I_1 I_2 and I_3; not that that is not correct. I mean, it is not that it is wrong, but it is easier to work when we look at this kind of things. You know, we will come back to this in a minute; but, before we do that, let us now see where in this picture we are. So, we have just finished isotropic materials. This isotropic material can further be classified in to two categories. What are they?

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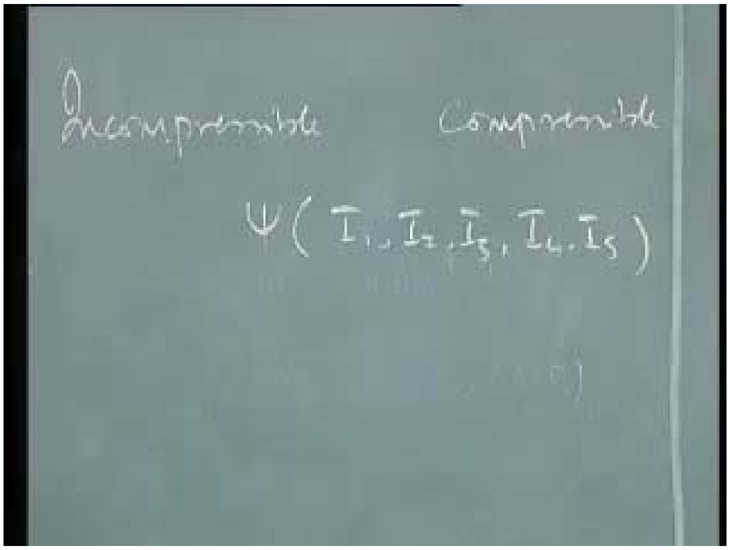

We call them as incompressible isotropic materials as well as compressible isotropic materials. Same way, the other branch also goes of, you know, anisotropic materials and so on. Now, what is the difference? Let me just point out the difference; ….. whether you have understood that. The difference is that the representation theorem is no more valid. In other words, we cannot represent this psi which I had done before in terms of I_1, I_2 and I_3. In fact, interestingly what we need is, for an anisotropic material what we need is more number of this kind of invariance and these invariance are sometimes called as say, I_4 I_5 and so on. So, there are more number of invariances to take care of say for example, fibre orientation.

How the fibre is oriented, that becomes important when material becomes anisotropic, for example, in a composite material or for example, if you take heart muscle where the collagen fibres run at a particular angle, these become important. So, in which case what is that we do? We take a very similar expression, but the only thing is that we now have more than three invariances. Some of the purist still question whether this I_4, I_5 and so on, you know what does it mean, do I put this down, are we correct; but, people use it I think. It is mathematically quite sound, so, people do use this putting down and number of other invariance when anisotropy is encountered. It is fairly a new topic, not much work has been done till very recently. In the 90’s people have started working on this kind of anisotropic strain energy function with I_4, I_5 and so on. Though in the 70’s and 80’s there are few papers, but basically people have right now started putting this kind of thing. This I_5 may keep, you know, increasing depending upon the anisotropy and so on. But, I am not going to deal with this. I am not going to write down what is I_4, what is I_5 and so on; suffice it to say that these are introduced and these depend upon, say for example, fibre orientation.

Let us now look at the incompressible and compressible isotropic materials. What are they? Now, what is, what is incompressible material?

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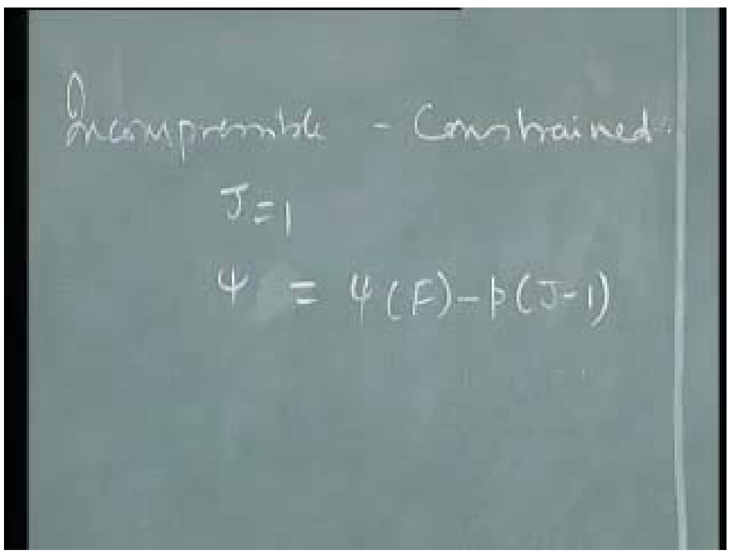

Incompressible materials are ones where J is equal to 1. Let us remove compressibility for a minute. I am going to just tell you the path which is followed here. Just that, incompressibility means that J is equal to 1 and what does it mean for us? It means that I have to satisfy this constraint. So, these materials are also called as constrained materials, constrained materials. Constraints are very important. It has, again there is lot of theory, lot of mathematics associated with it in pure continuum mechanics, but for our study it is sufficient to say that sorry, the strain energy function now has one more, one more term added to it. In other words, it is not only a function of this or gradient, deformation gradient, but also is a function of what is called as minus p into J minus 1, where p is the Lagrangian multiplier and is identified to be the pressure term p; very common thing that you do usually when there is a constraint.

When there is constraint condition called J is equal to 1, what you do is, you have this J minus 1 multiplied by a Lagrangian parameter, Lagrangian multiplier and then determine the Lagrangian multiplier as well. In other words, what it really means is that this p which is a scalar quantity, though identified with respect to pressure is one which can be determined only by means of kinematic conditions or in other words, boundary conditions. In other words, this is not supposed to do any work, it is a non working scalar quantity which actually comes, because of the condition that sorry, J should be equal to 1 and that this has to be determined. We will, we will do a small problem later and has to be determined only by a scalar quantity. In other words, what does this give us?

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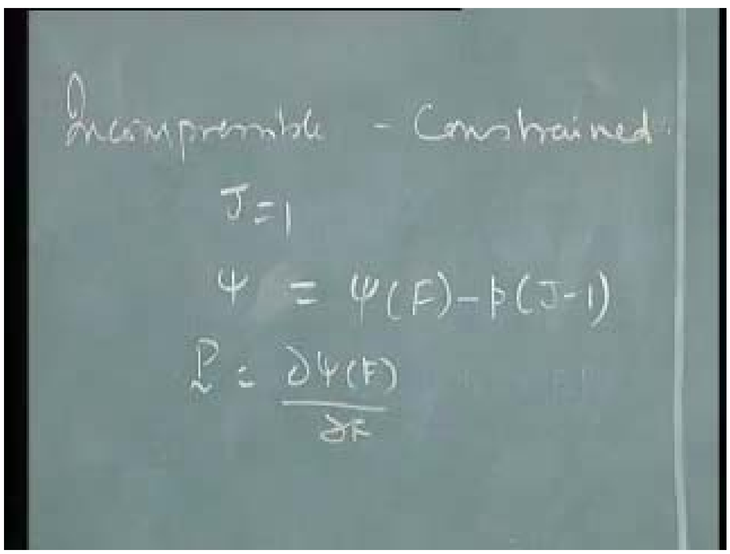

This gives us, going back to my first Piola-Kirchhoff stress equation …… relationship remember that we wrote this like this, did you remember that?

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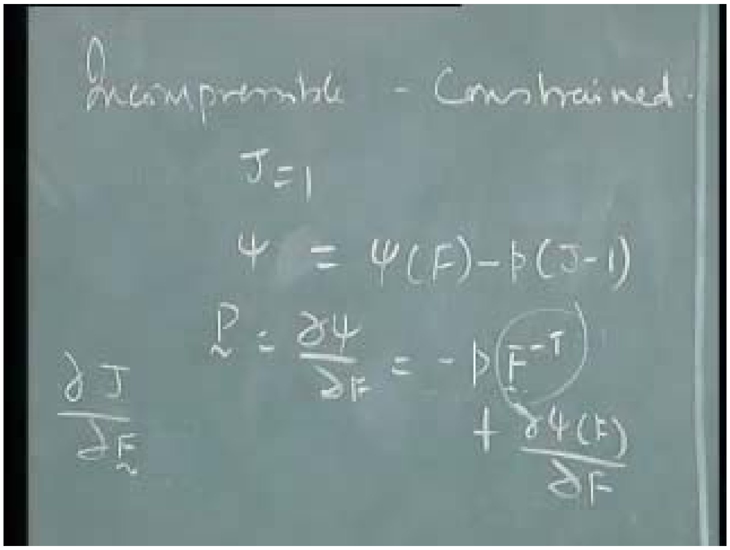

Instead of that, we have to now write down this as, this strain energy function, the derivative of this strain energy function which is put forward here and so, that becomes minus p dow J by dow F; remember that yesterday we derived this. What is that? Look at what we did yesterday. Dow J, what is J? Determinant of F dow of determinant of F by dow F; did that yesterday, yes, what did we get? Good way to look at what we did in the last class; determinant of F into F inverse transpose. So, what do, what happens here?

Determinant of F should be equal to 1, so, minus p F inverse transpose plus, note that we have just deviated a bit from our original expression, in the sense that there is an addition term here and please also note that, there is some question, please also note that dow of J by dow F, J is a scalar quantity, dow of J of, dow J of dow F where F is a second order tensor, results in a second order tensor, is very important. So, when I differentiate a second order tensor with a second order tensor, the result will be a fourth order tensor and so on, obviously because, dow J by dow F_{11} you can write, 2 2 you can write and so on; so you get F inverse transpose. Let me see. Can you write down what is there for S?

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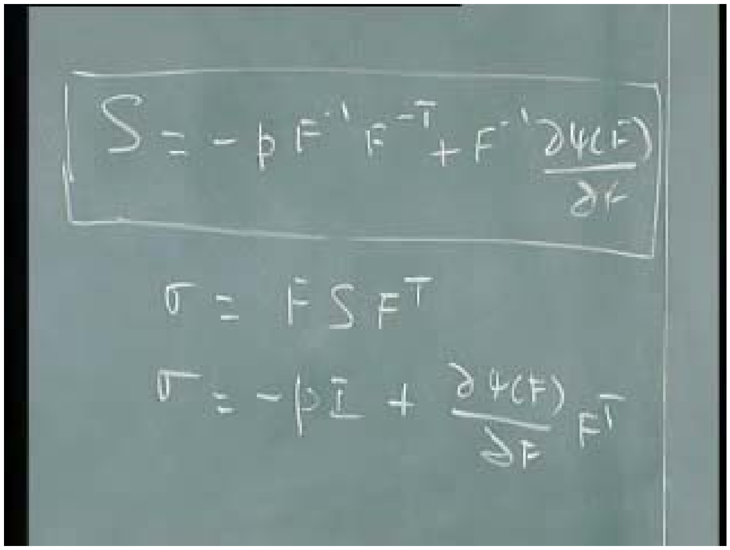

What would be the expression for S? Because I understand that there are lot of equations and examples; that is a nice thing to do, to look back and see what the relationship is between S and p and see whether you can determine that. Yeah, S is equal to minus p F inverse F inverse transpose; that is the relationship. In this case, it becomes S is equal to F inverse p, F inverse p, so, plus F inverse dow psi of F by dow F. Go back and refer to our relationship between p and S. In this case, it becomes S is equal to F inverse p. Then, that is the second one. It is very important relationship. These are again general relationships for incompressible material, general relationships for incompressible material.

When you want to define this for isotropic incompressible material, we again go down and then express this; it is very simple, express this quantity like what we did before. See in that whole map where we are, that will give you an excellent picture. We are not, we are not deviating from what we did. In other words, this is another special case. When I say isotropic, then immediately my attack is on this. In other words, this can be expressed now in terms of the invariance or we can also express it in terms of the lambdas. Can someone tell me what should be the relationship or what should be sigma? Let us see. So, what should be the relationship? You know the relationship between them; minus, of course minus p is going to be there, minus p, what is the relationship between sigma and S? Go back and refer to it. Yes, J inverse becomes 1;

so, sigma is equal to, sigma is equal to J goes out, because it is equal to 1, F F into F S F transpose J inverse; J is equal to 1 in this case.

Apply that there, so you will get sigma is equal to minus p I first, plus, right; we know that this commutes. So, you can write this as F dow psi by dow F transpose. Now, just to understand what we have done, I want you to, is this clear, any question? p into F inverse transpose. Yeah; yeah, A inverse transpose becomes F inverse transpose, so that is exactly what we did here. No, no, no, no, determinant of, what it means is determinant of A dow J by that is dow J by dow F is equal to determinant of J into that is determinant of J into F inverse transpose. The determinant of J became 1. So, that is the doubt. Why I left out J is because J became 1, because that is the incompressibility condition. That is why J is not here, so that is why I had written straight away minus p F inverse transpose. So, determinant of J into F inverse transpose, determinant of J became 1 and that is it.

Suppose now I reduce this to isotropic. I have not yet reduced it, now I am going to reduce this.

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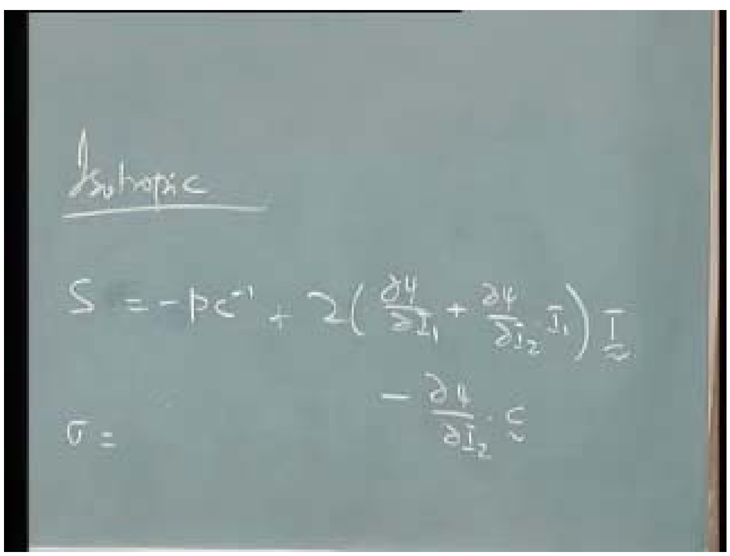

If I reduce now this to the isotropic case, in other words, isotropic incompressible elasticity, how do I now write this term? Let us see. Go back to what we did in yesterday's class and let us see. You write down general equations quickly. How we write? We take this and then write down I for isotropic case. In other words, what isotropic case means as I told you, this can be expressed in terms of, this can be expressed in terms of I_1, I_2 and I_3. Go back, look at your expressions in yesterday's class and then now write down what should be the, what should be the expression for say, S and sigma. Let us, I mean, of course you can write it down for p also. Let us, let us say, let us say that you just write it down for S; let us see how you write it down. Go back to yesterday's class and look at that expression there. What are the expressions you had? 2 into dow I_1, yes, into I minus into C. Yes, so, where does it now stand? What is the term that is not going to be there?

There are terms involving I_1 differential of I_1, I_2 and I_3. What is that you are going to get? Simple, it is simple; that I_3 is not there. I_3 is equal to 1. You will not have that term, so you can write that down to be minus p. That is what is F inverse, F inverse? J is equal to 1. That is all. So, F inverse F inverse transpose can be written as C inverse. That is equal to minus p C inverse plus all the terms that go with it. That is 2 into, read out, 2 into I_1 plus I_1 I minus into C; third term will not be there. So, that is it for that is the general, look how we have very neatly reduced. Now, this is the isotropic incompressible elasticity.

What will happen now to sigma? Sigma, I mean from, you can write it in terms of b or b inverse. I am just not going to write that, you can do it. You can write that also in terms of sigma and p. That is the one aspect of it. I told you right in the beginning of the class that it is possible to express this whole relationship in terms of the Eigen values as well, lambda_1, lambda_2 and lambda_3, in which case the relationships just reduced to this. How does it, let us see what you do?

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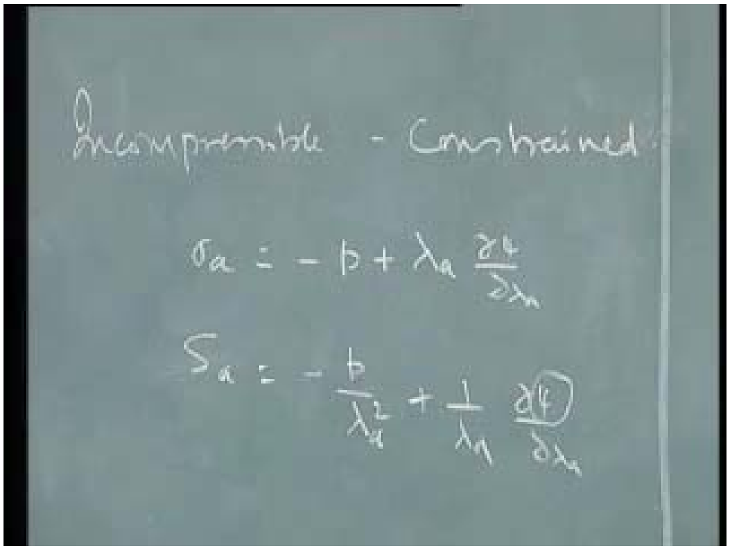

So, $sigma_a$ I have written it here; it is here. What would be the, so, J inverse will go off here, so $sigma_a$ or a or a we have seen or will see? what it is, is equal to minus p plus, minus p plus $lambda_a$ and so on. But, there is a small change for P_a. I am not, as I told you I am not going to derive this. P_a will have and in fact, S will have, see or C?, note that, note this sigma had a minus p I here and S had a minus p C inverse here. So, that C inverse term, I did not derive p; let us not, let us not worry about p now; very simple, you can convert it.

What will be the S_a term now, first term? Say, when we had minus p I, we had minus p; so, minus p C inverse C inverse. So $lambda_a$ is the, $lambda_a$ squared is the Eigen value for that. So, this will become minus p by $lambda_a$ squared. Look at this term here minus p the C inverse term, first term. If I want to write it in terms of the stretch, becomes minus p into $lambda_a$ squared plus 1 by $lambda_a$ ……. That is the first thing …… that is the thing which we had previously. I have not derived this completely. I mean derivation is quite involved, but the second part I did not derive it. Just note it down; we can do that, standard derivation. People who are interested can look at Non-linear elasticity or Non-linear Solid Mechanics by Holzapfel or a book by Wood. The derivation is there. If there is time, if I have enough time today, may be we will just indicate how to do it. So, that completes our picture on incompressible elasticity.

What does it mean? It just simply means that that is the crux of the whole story. So given, these are general expressions; so, given a function here, this function here, I will be able to get these things. Of course, for an incremental formulation which we are going to follow, I am jumping two steps which we are going to follow. We need something else as well, we need the relationship between delta S and delta e. In other words, dow S by dow e is what is also required. These formulations are slightly different. I will indicate to you some of the final forms may be in the next class, when we enter again into finite element analysis. But, what I want you to follow is the theory, is to how this overall picture emerges.

We now move into compressible elasticity, again a general compressible elasticity followed by isotropic compressible elasticity.

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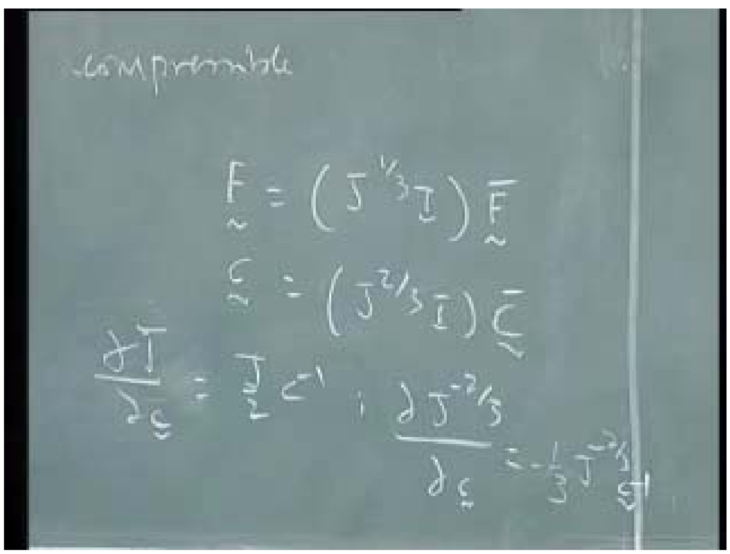

One of the things that is done in compressible elasticity is to define F in such a fashion or split F in such a fashion that we have a volumetric part and an isochoric part. In other words, a deviatoric part and dilatational part. Before I forget, one of the things which I want you to remember is that, though we have clearly demarcated saying that materials are either incompressible or compressible, we had demarcated the material into two, many of the rubber materials that are used are actually nearly incompressible; but, most often in numerical analysis, this is treated as an incompressible material itself. Many people treat it as an incompressible material itself, though actually we say that they are nearly incompressible or in other words, mu in a linear scale, in a linear case, mu is equal to 0.499 or 95 to 99; not equal to 0.5.

Once mu is equal to 0.5, we say that the material is completely incompressible. But, many formulations and many examples if you see, people continue to work with the incompressible part and say that it is incompressible as well. So, for a compressible material, F is decomposed into a volumetric part and a deviatoric part and in fact, F is written as J power, I think we did that before; anyway I am just writing this down, J power 1 by 3 I F bar, so that you can see that the determinant of F bar is equal to 1 and same way you can write C is equal to J power 2 by 3 I C bar. Note that both of them of course, are second order tensors. This is due to very early work by a very famous chemist, in fact, Flory. So, this is called as Flory decomposition; it is called as Flory decomposition.

What we now do is that, so, there is a deviatoric, dilatational part. What we now do is to say that there are two versions of psi, the strain energy function. There are two things. One is due to dilatational part, volumetric part. The other is isochoric part; there is no volume change or deviatoric part. So, we put down two sizes. Before that, let us write down some expressions which may need, which we may need later. Dow J by dow C rather, is equal to, we know that J by 2 C inverse, so some of the expressions and dow of J power minus 2 by 3 by dow C is equal to minus 1 by 3 J power minus 2 by 3 C inverse and then a few more which we may need; in fact, you may need them for doing problems.

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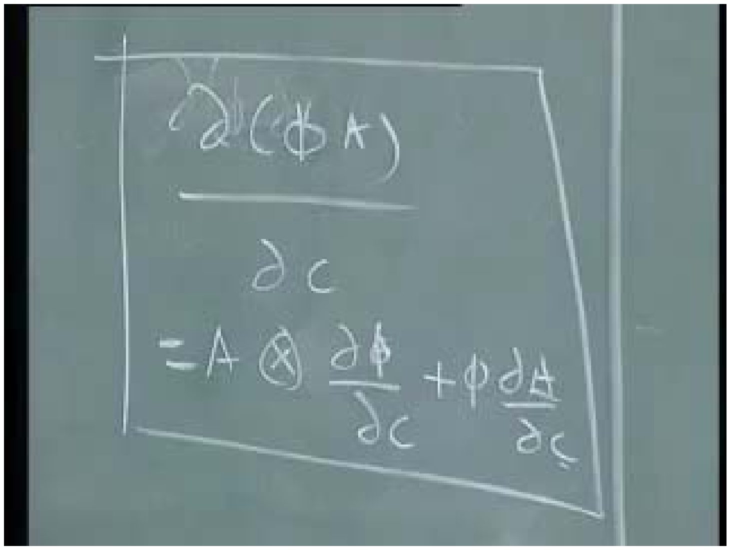

Suppose I have dow phi, sorry dow of phi A that is a scalar quantity and a vector quantity of dow C, how do we write down this differential? Dow of phi A scalar and a vector quantity? It is written as A dyadic, note this A dyadic, dow A, sorry, dow phi by dow C plus phi dow A by dow C. Of course, these are second order tensor. So, what is the result? Look at this, dow of phi A by dow C, what is the result of this? Second order tensor, second order tensor, so, this is a fourth order tensor and that is the relationship. No, any second order tensor. This relationship may be important to you.

Now I need, I need one more quantity. You will see why I need this.

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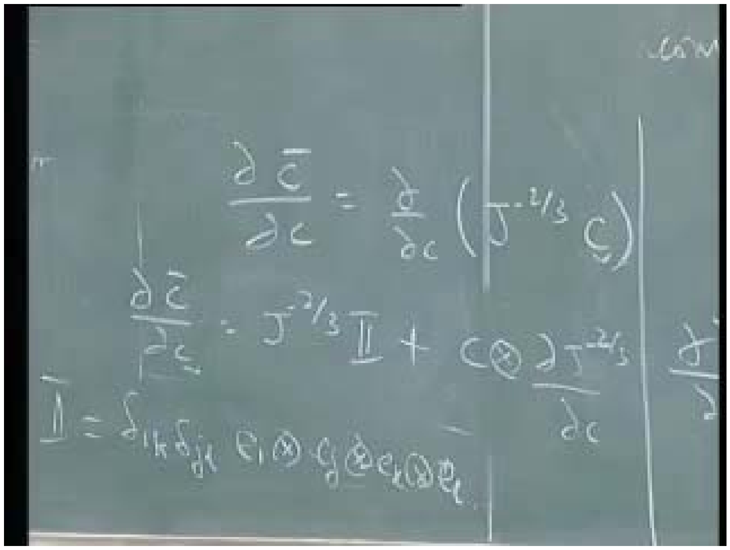

I want dow C bar by dow C, dow C bar by dow C. How do I get that? Dow of, from here C bar is equal to J. That is equal to dow of dow C J power minus 2 by 3 here, C, I C or in this you can remove that and put straight away, I into C is C, so, you can put that as well. Compare this with what you did before or what I wrote down before and let us see what this gives rise to. The first term is phi dyadic dow A by dow C we had, so, phi dyadic here phi is J power minus 2 by 3. So, this is equal to J power minus 2 by 3 into dow C by dow C.

What is dow C by dow C? But I, fourth order I, beautiful answer. So, that is the, so, it is a fourth order I. Fourth order I is like written like this. So, fourth order I is written as actually delta_{ik} delta_{jl}, because there has to be i k j l; delta_{ik} delta_{jl} and what are the basis for this? e_i, e_j, e_k, sorry, e_l. So, that is the fourth order I, fourth order unit tensor. That is the first part of it. So, what is the second part? Refer to what I did and tell me what the second part is?

C dyadic of dow J power minus 2 by 3 by dow C. So, substitute this dow J power minus 2 by 3 dow C from what I had written down there, minus 1 by 3 J power minus 2 by 3 C inverse.

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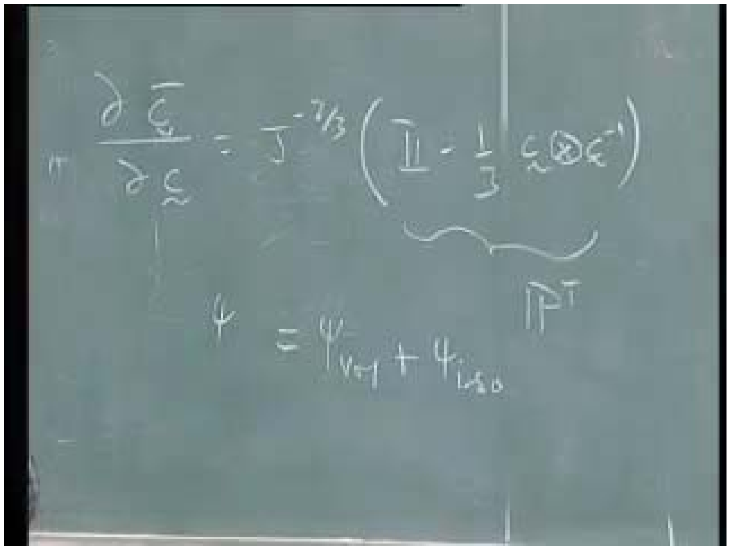

Substitute that into this expression, substitute that in that expression and ultimately, I am just going to remove all this, you can write down dow C bar by dow C to be J power, you can take that out, minus 2 by 3 fourth order I minus 1 by 3 C dyadic C inverse, where people identify this as a transpose of what is called as a projection tensor. We have already seen what a projection tensor is. It is a fourth order projection tensor. Does not matter, just it is identified; let us not go into the theory of projection tensors, but this is what it is. We will see in a minute why we need all these kind of things? Again, all of them are algebra. So, I am not, that is why I am not very perturbed by it, but the theory comes from here.

In fact, if you really look at it, the whole theory can be very easily understood in not more than one class, the rest of it is only algebra. There is nothing more than that. Now, the theory is very simple. What we are going to do is the total strain energy function is split into two parts. That is all is the theory. One is the volumetric part plus the other is the isochoric part. The next two sentences are theory. Then, again we will come to algebraic manipulations. In fact, I want you to summarise all the theory. Algebra is always there. The tensor algebra is, I mean may be, it may put you off; you have so many equations, but I would like you to understand the theory behind it. Once I write like this, again see that we are going to use that old theory, what we know. What is that I am going to do now?

Let us see, someone, so ultimately, I want to develop a stress strain relationship or stress deformation relationship. What do I do? All of them are inelastic materials. So, my choice is very simple.

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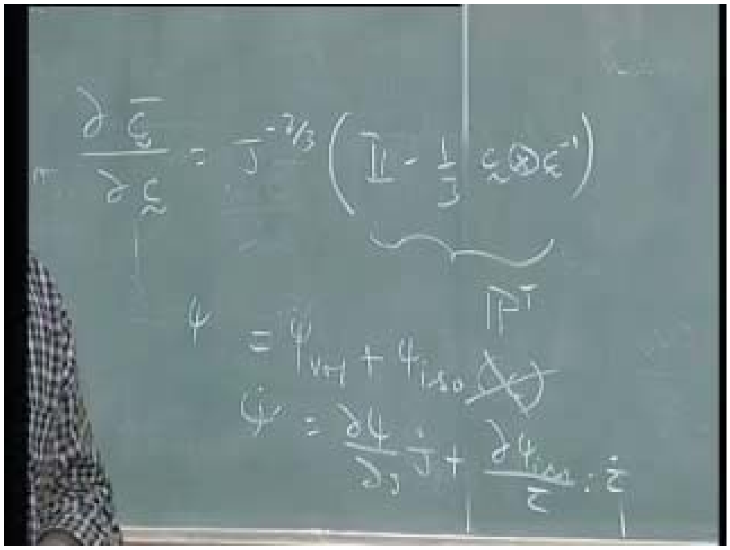

I find out psi dot, so, where volumetric part J dot plus the isochoric part which is written in terms of C bar. Now, that is what, this can be written in terms of C previously, now it is written in terms of C bar, so that this can be written as C bar C bar dot. Is that clear? Yeah; yeah, because, please note I had split C into two parts: one is C and C bar. This C bar, what is C bar? I have removed from C bar the volume changing part, because we are talking about compressible materials. So, I have removed that part and C bar now represents the isochoric part; C bar represents the isochoric part. When I want to look at the free energy, sorry, the strain energy function, then I have to write this in terms of the isochoric part or C bar. I cannot write this obviously in terms of C, because C has that compressible part and the compressible part has been removed and is put here. That is all.

So, what do I do next step? Having done this, what is that I do? Very simple, there is nothing great, exactly what we did in the previous class. Let us see, how did we get the, can any one answer? Why did I go to psi dot, previous class, previous couple of classes back, because I wanted to substitute this into my, no, where, where is that I want to substitute this? CDI, Clausius-Duhem inequality. Only thing is that I am removing that inequality part, because I am still in elasticity.

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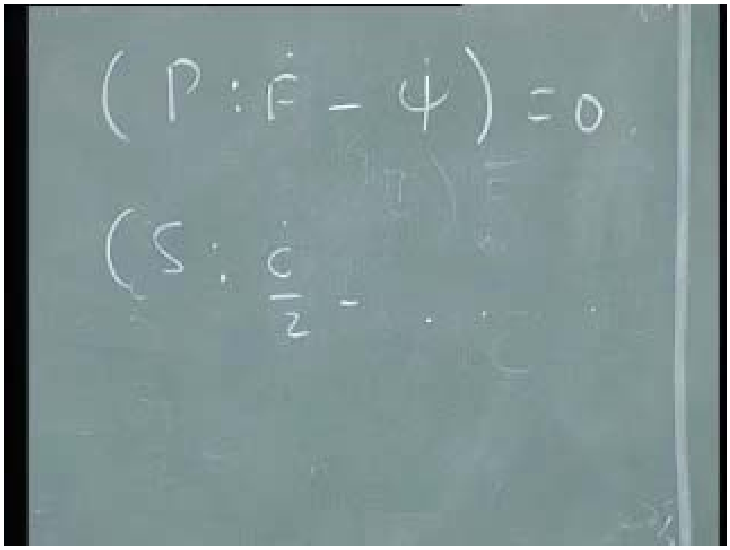

Remember that we had an equation of the form F dot minus psi dot equal to zero. So, that is where we are still there, nowhere else. What we are trying to do now is S colon, now, P I already told you that we are going to work most of the time with S, because S is, though S lacks physical meaning, I am very comfortable with S, because it is a symmetric tensor. We can, we can write that in terms of S and C dot. So, C dot by 2 minus a number of terms; I am not going to and I am not going to write that down there. Probably you can write it down, because you know many of them; what is J dot, what is all these things in terms of C? We can, yeah, this is, there is a small change that we should do I think. One minute, yeah, there is a small change. Just look at this, because if I want to substitute it back into this expression here, so, this is written in terms of C bar, no doubt; this is written in terms of C bar.

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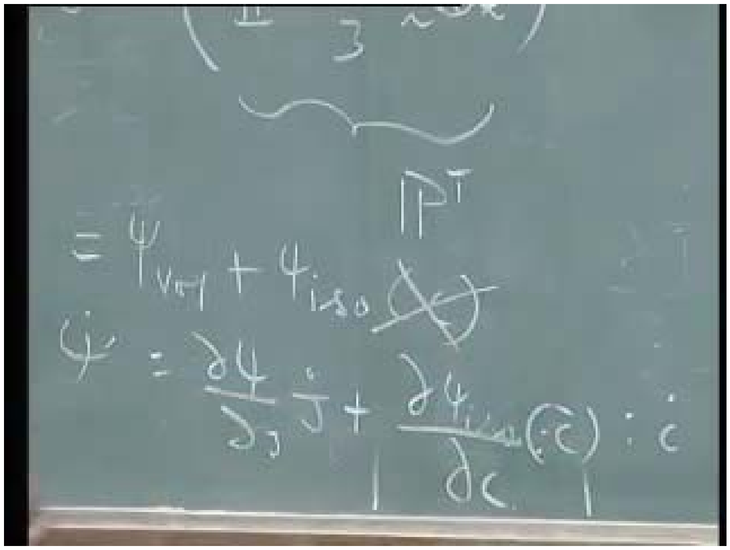

But when I want to substitute this, it has to be in terms of C, because this is C dot. So, colon or into in fact, I have to write this down as dow psi iso C bar by dow C bar into dow C bar into dow C, this should be colon C dot.

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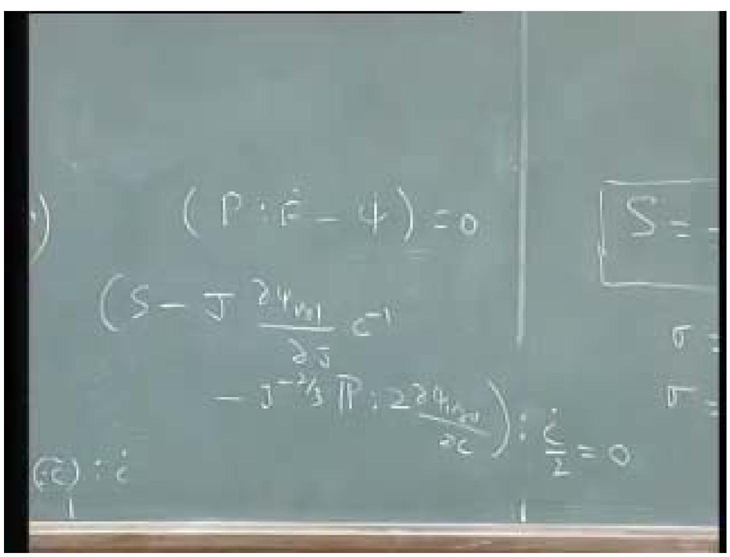

I mean, because I have to substitute it back into that expression, so that ultimately this expression what I have written here, when I substitute it back here this can be written now as S minus J, you know that before, dow psi volumetric. I think I wrote that as volumetric, yeah, dow psi volumetric by dow J C inverse minus J power minus 2 by 3 that is dow C by dow C bar is what I am writing. That we derived that, that is here, into all these things. Now, it becomes transpose of it when I convert this into, when I convert this double dot in the matrix notation, it becomes transpose. Ultimately, you can write that as actually P colon 2 times dow iso by dow C colon C dot divided by 2 is equal to zero. Remember, this is very similar to or exactly the same as what we did. Only thing is you have to be careful with all the terms, whether you have included all the terms; again that is algebra and just see that you have, all the terms are there.

In other words, what is this, what we call, you remember? Coleman Noll procedure. In other words, what it means is that S is equal to this plus this. So, S consists of two parts now. In other words, S, just check up whether I have included all the terms or else you can just check that or else you can replace P by this whole thing; you can write that down.

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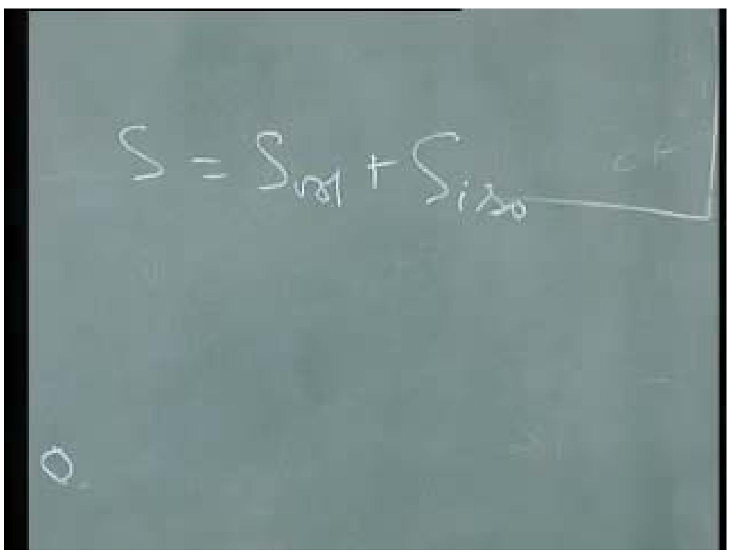

In other words, S has two parts - S volumetric plus S isochoric. The first part is the volumetric part the second part is the isochoric part.

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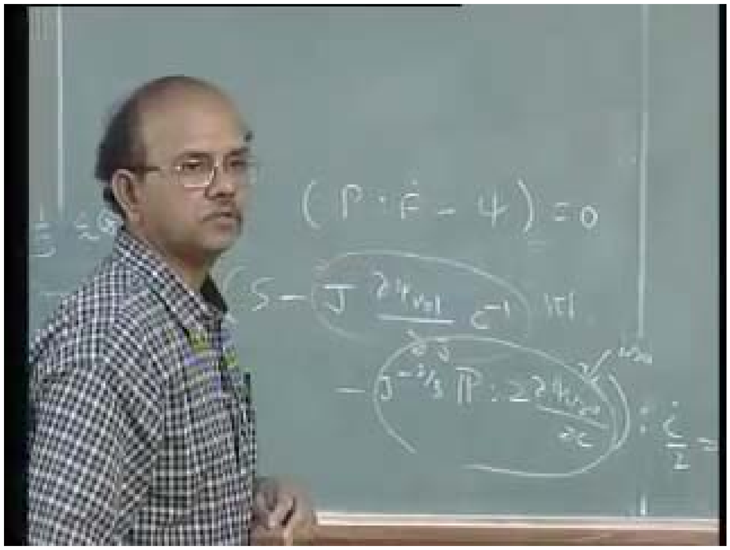

That is the volumetric part; that is the isochoric part. The major thing in compressible elasticity is to split psi into two parts - volumetric and isochoric. Yeah, you can keep, substitute it. Now, do lot of algebra to express all these things in terms of various quantities. I am not going to do that, I will leave it at this that this is the concept that we usually follow. Further than this, I think you can see many of the standard text books on Continuum Mechanics or you can, most papers themselves derive many of this relationship. We will not continue further than this. So, that is compressible elasticity. Now of course, I can, I can again say compressible isotropic elasticity, in which case I can, I can express that iso in terms of I_1 bar I_2 bar, you know, I can keep on extending this. The concept is the same. Is this clear?

Let me summarise this, so that you are not lost in the whole thing. What we did? The first thing what we did was to write down Clausius-Duhem inequality; remember that we did first Clausius-Duhem inequality. We finished that and then came to the constitutive equation. In constitutive equation, we defined an elastic material as one where there is no dissipation, so that that right hand side inequality sign becomes equality sign. So, we wrote down the expression for P to be just dow psi by dow F. That is concept number two. Concept number three what did we say? We said that materials can be isotropic or anisotropic. We went to isotropic. We said that it can be or the psi can be written in terms of I_1, I_2 and I_3 - concept number three. Then four, we expressed in a more general form S, P and sigma. The next concept is that we said it can also be expressed in terms of stretch and we expressed sigma S and so on, in terms of stretch, stretch. This is for general isotropic hyperelasticity.

Then, actually we realised that isotropic hyperelasticity can be now classified into incompressible and compressible hyperelastic. We have to now reduce that general thing to these two conditions. When we, when we came to isotropic hyperelasticity, we realised that we have to put that J as a constraint, put J is equal to 1. What we did was to add to that strain energy function a constraint condition, through a Lagrangian multiplier P …. into J minus 1. We did that and then the rest of it followed in a very similar fashion, as we did in the first case.

Then, when we came to compressible elasticity, what we have to do is to now distinguish between or split this into volumetric part and the isochoric part. So, then we again developed equations, S using Clausius-Duhem inequality and realised ultimately that S has a volumetric part, S itself has a volumetric part and an isochoric part. So, the only thing we have to now do is what are the forms of this strain energy function? We have already indicated what are the forms of strain energy function, but we have to go slightly deeper into it. May be, in the next half a class we will finish what are the popular strain energy function and then move back into the finite element formulations. We will stop here and we will continue in the next class with the finite element formulations.