Advanced Finite Element Analysis

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Lecture - 25

We were discussing the constitutive equations and we had discussed quite a bit of both what is hyperelastic material, the philosophy behind it.

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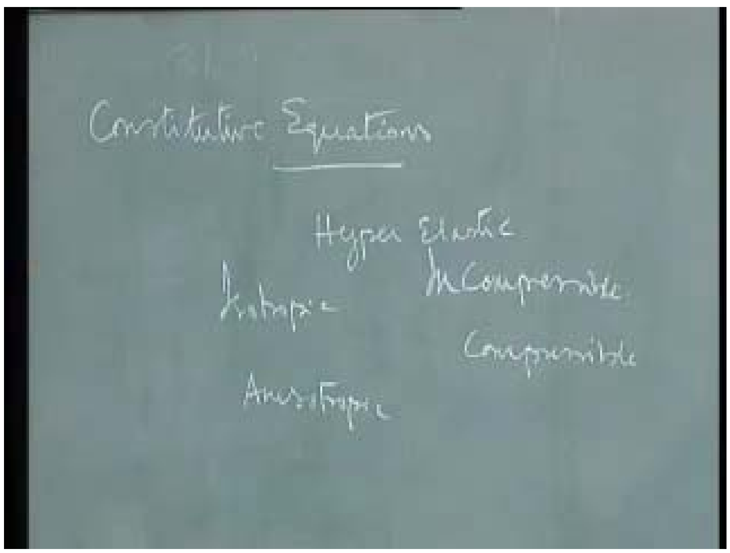

The next two classes, we will quickly summarise the equations which are necessary for us to go forward to look at the finite element implementation of the hyperelastic case. I told you right in the beginning that I may not be deriving all the equations, may be this class I will derive a few of them; may be next class, I may not derive the equations, I am going to give you the final versions of it, so that we will use that for our finite element work later. Let us look at some specific things in hyperelasticity. In fact, if you look at the materials that come under or that can be classified under hyperelastic materials, they can be classified as of course, isotropic and anisotropic. I am not going to cover the anisotropic part in this course, because that is again a huge area. You need to know lot more of continuum mechanics in order to cover anisotropic part of it. May be towards the end if there is time, I will indicate how we approach the anisotropic problem, but the applications of anisotropic hyperelasticity is not very high. Except biological materials, you really do not use anisotropic hyperelasticity to that extent. In fact, I will just comment on that in a minute, let me complete this, the other classification.

The other one is, of course, isotropic hyperelastic materials. The other type of classification comes from whether the material is compressible or incompressible. So, you will say that incompressible and compressible. Essentially what we are going to do now is that for hyperelastic material which undergoes say, isotropic incompressible we are going to put down some equations in terms of general size that is general strain energy function. Then, we will go to specific strain energy function and state some of them.

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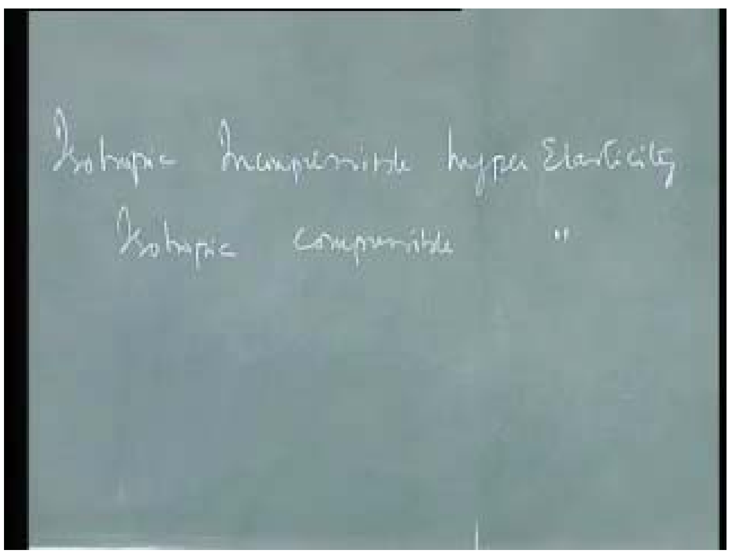

Essentially, what I am going to cover is isotropic, isotropic incompressible which has the maximum amount of, I would say, applications, so, isotropic incompressible hyperelasticity. For sake of completion also I am going to state the isotropic compressible hyperelasticity. I am not going to the details, but I will also complete in the next class what are called as isotropic compressible hyperelasticity. What I am not going to cover, though I may state one or two sentences, is the anisotropic incompressible hyperelasticity and anisotropic compressible hyperelasticity or in other words, anisotropic hyperelasticity I am not going to deal with in this course.

Let us, just before we go further, because we have made this assumption, let us just check what this anisotropy is, physically what it means and why we can, for many, many problems, we can do away with anisotropy and use isotropic materials. For many practical problems, for example, if you take say tyres, tyres is basically, I had talked about hyperelasticity; most of them are incompressible hyperelastic materials and of course, you can also classify another category of materials called isotropic nearly incompressible materials also, but usually do not put down a separate constitutive equation for it. You can also say that there are some materials which are nearly incompressible, but many times we treat, because of the finite element formulations we treat, incompressible material to be nearly incompressible and so on. That we will discuss when we come to the finite element pattern after two classes, but what I want to specify is that, say for example, if you take tyre, then there are reinforcements running that makes the material composite or that makes in a gross sense, the behaviour to be anisotropic.

In other words, here are reinforcements which can be identified say that, this is the steel reinforcement which can be identified and then you look at its behaviour along with the rubber matrix. If this is the case, then you do not write down separately or you do not classify this as anisotropic on a micro scale. What you usually do is to write down isotropic behaviour or isotropic constitutive equation for the matrix material and then take that reinforcement separately and usually put special elements which are called as reinforcement elements or re bar elements or in other words, you treat them as two different materials, as two different materials and there are ways of handling them in finite element analysis.

On the other hand, you have biological materials, where there are collagen fibres – tens of thousands of millions of them. If you take a piece, there will be so many of them. You cannot one by one pull up and say that, yeah, this is the collagen fibre; this is the fibre that is running and this is the other fibre. So, I will put one special element for this fibre another for this fibre and so on. In other words, it is so very intermingled at a scale, at a much lower scale, that it will not be very easy to separate them out and treat them as a separate material and say that I have a base material which behaves in an isotropic fashion and I on top of it I have material which behaves, which makes it to behave anisotropic. I cannot say that take a piece; the whole piece behaves as an anisotropic, as an anisotropic material. There is always a problem with that an; anisotropic and when I say an isotropic, it looks as if it is anisotropic. So, I said an anisotropic material.

In this case, if I have collagen fibres that are running and I take biological specimens, I want to treat it using finite element analysis; I have to necessarily treat them as anisotropic material. Now, I will also indicate one or two other things with anisotropic materials, but the, but the major good thing is that, the major thing is that the procedure that I am going to put forward now is exactly the same for anisotropy. Just that we do not have time enough, you know, time to, enough time to complete this whole anisotropic material behaviour. It is only that, but the, but the procedure is exactly the same. So, that way if you want to go back and read a paper, it will not be anyway different. I will also indicate what they are in a minute.

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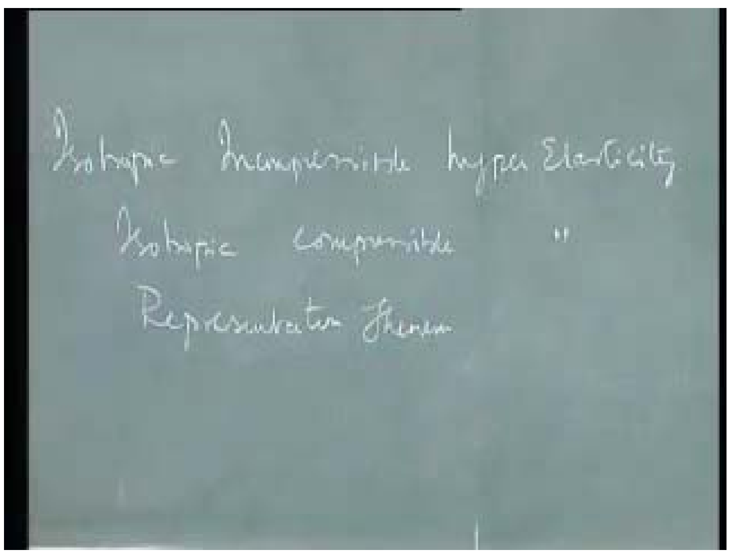

One of the most important theorems that we are going to look at is what is called as representation theorem; representation theorem or sometimes people call this as representation theorem of invariance. What it means is very simple.

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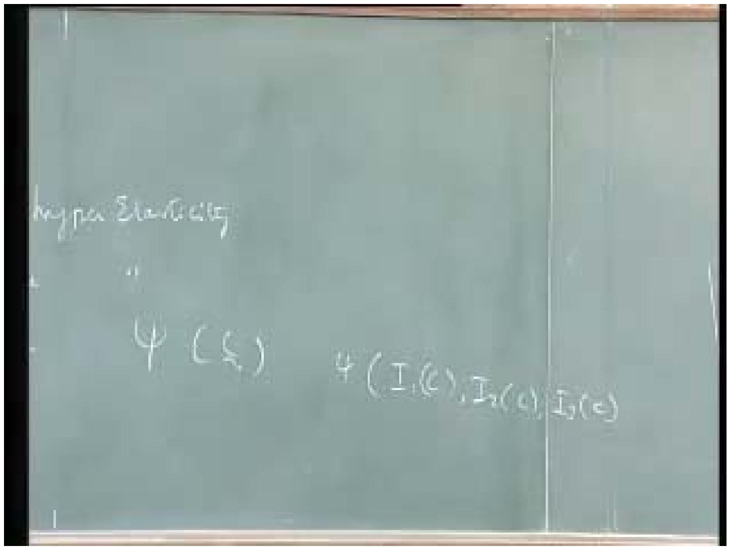

What it means is that, if you have a scalar value tensor function, nothing great, your strain energy, scalar value, tensor function which means that it is a function of tensor say C, such that it is not affected by rotations or in other words, if it is isotropic, if it is isotropic, then this function can be written in terms of the invariance of the tensor valued function. Though the proof is very complex, are quite involved, I am not going to prove this, but it is very simple to understand. What it means is that for isotropic materials, now how do I express when I, when I say C and how do I express this as isotropic materials.

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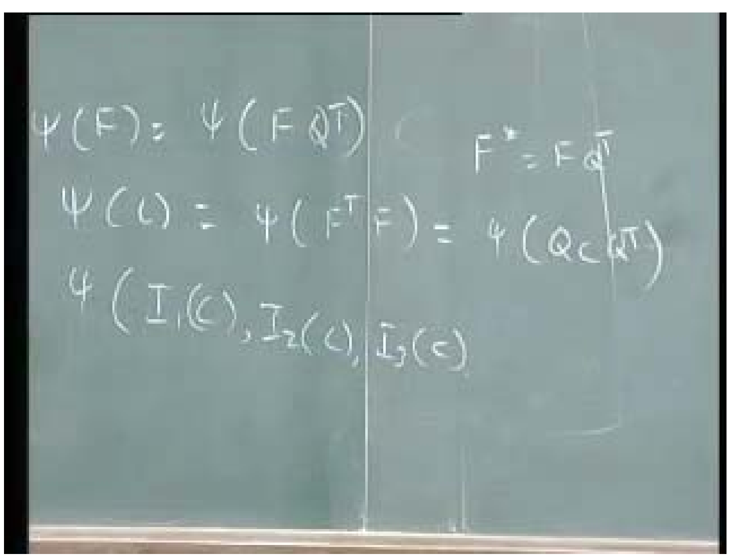

Remember, yesterday we wrote or last class we wrote that this, if I have to have this to be an isotropic material, then this has to be written as or this is equal to F of Q transpose. We already said that the strain energy function can also be written in terms of C and E, because they are all functions of say, U and so on. Suppose I write this as C, let us see in a minute, tell me how this should behave, this function should behave if it were to be an isotropic material. What is C? No, no, what is C in terms of … Yes, yeah, F transpose F, so, just check how this should be. Note that F varies as or if I just for a minute call that as F star, so, F is equal to F Q transpose. Right Cauchy tensor or in other words that is F transpose F. So, this should be in terms of star. How it should be, the stared co-ordinate? Just substitute from here, you will get that, that has to be Q_C Q transpose, just verify that.

If you can have or if you have a strain energy function like this, with this equation, then the representation theorem states that such an equation can be written in terms of the invariance of these tensors. What are this invariance by the way? Yes, they are different for different things.

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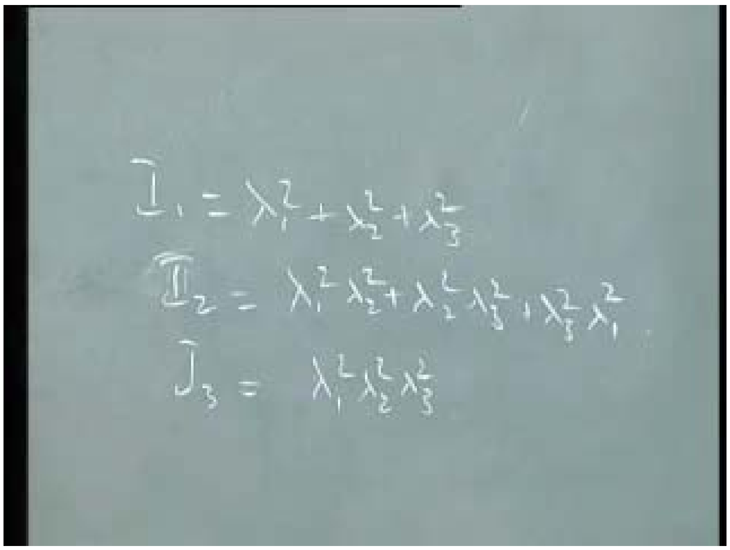

Say I_1 is equal to the invariance of C and so, if you can, if you want to express this say, in terms of lambdas which are the principle values or lambda squared, which are the principle values of C, then you can express them as I_1 as lambda$_1$ squared plus lambda$_2$ squared plus lambda$_3$ squared. Lambda squares are the Eigen values and then

I or I_2 rather can be expressed as $lambda_1$ squared $lambda_2$ squared $lambda_2$ squared $lambda_3$ squared plus $lambda_3$ squared $lambda_1$ squared and 3 is equal to the determinant of that. So, it is equal to ….. of this diagonal matrix and that is equal to $lambda_1$ squared $lambda_2$ squared $lambda_3$ squared. Some of the text books like Wood expresses this as $lambda_1$ power 4 plus $lambda_2$ power 4 plus $lambda_3$ power 4. So, if you see this in a different form, do not worry about it. That is another form of writing the second invariant.

Yeah, it will not be different, because that can also be shown to be an invariant. You have to know some invariant theory to say that it can also be written like this. No, no, see, invariance, what are invariance? What are invariance? Yeah, this will not change with the co-ordinates. So, you can also show that if you write this in terms of $lambda_1$ power 4 plus $lambda_2$ power 4 plus $lambda_3$ power 4, they also can be formed as an invariant. Just as a caution, I am just telling you, if you see different things do not worry about it. That is another way of writing the second invariance. First and third invariance are usually what I have written here, they are the same. So, that theorem gives us a lot of advantage of writing down the stress strain relationship.

There are two ways in which you can write down the relationship. One is straight away in terms of invariance. Of course, if I do not, if I do not want to write it in terms of $lambda_1$ squared, I have to, I mean for, this is one way of writing it; simple to write, but of course, if this is not given, you can also write in terms of C itself, I_1 I_2 and I_3 and how do I write that?

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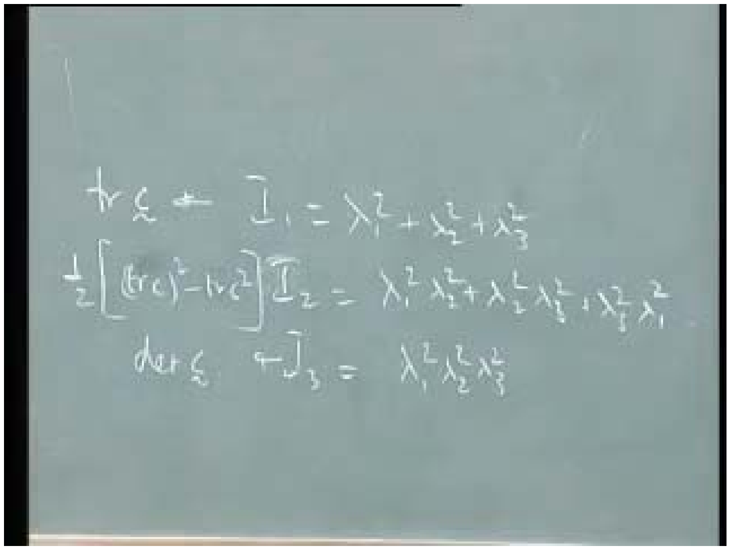

This is trace of C. This is one way of writing it. But, that is also equal to, I am going to use that later, trace of C and this is equal to half of trace of C squared minus trace of C, square of C that is I_2 and I_3 becomes determinant of C. These are the three ways in which it can be written and now my procedure is very straight forward as to first, so, what is my goal.

My goal is just to express, so, I have come one step down saying that this strain energy function which is a function of C, can now be expressed in terms of this invariance. You know that is the one step which I have come towards writing down the relationship between deformation and the stress for an isotropic material. What is the next step? Next step is just to substitute this into my expression, which I had given you in the last class. What is the expression that we saw in the last class?

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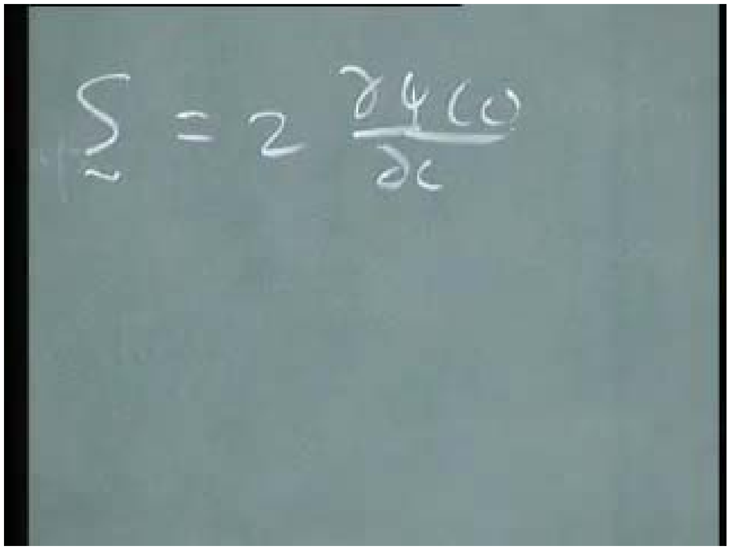

P say for example, if you look at S or if you look at P, then we had written that in terms of dow psi by dow C or in other words, if I remember right, we had written that as 2 into dow psi by say, dow C and note of course, one of the important thing which I am not, I may not continue with that line again, this is in the, this is in the reference configuration; this equation is written in the reference configuration. Since the Eigen values happen to be the same whether I use C or whether I use b, this whole equation can again be written in terms of b as well. That is where F is in the left of F transpose that is in the spatial co-ordinate.

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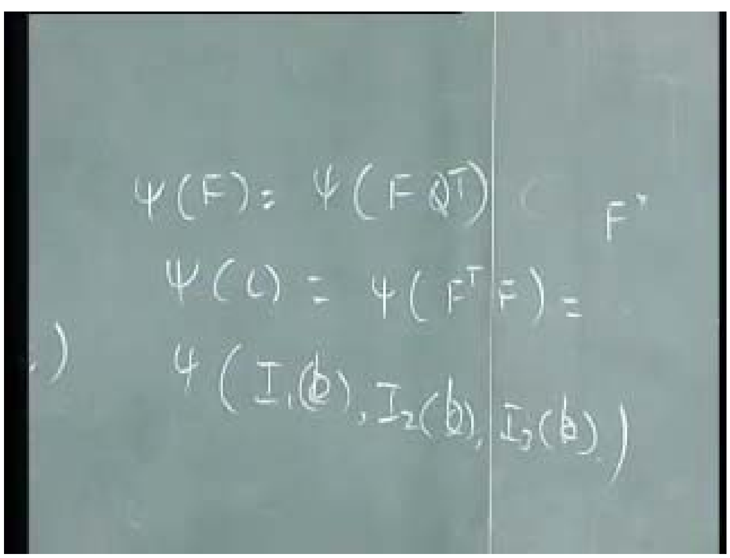

So, I can write this down say, as like that as well; either way I can write it, either in terms of C or I can write that in terms of P as well.

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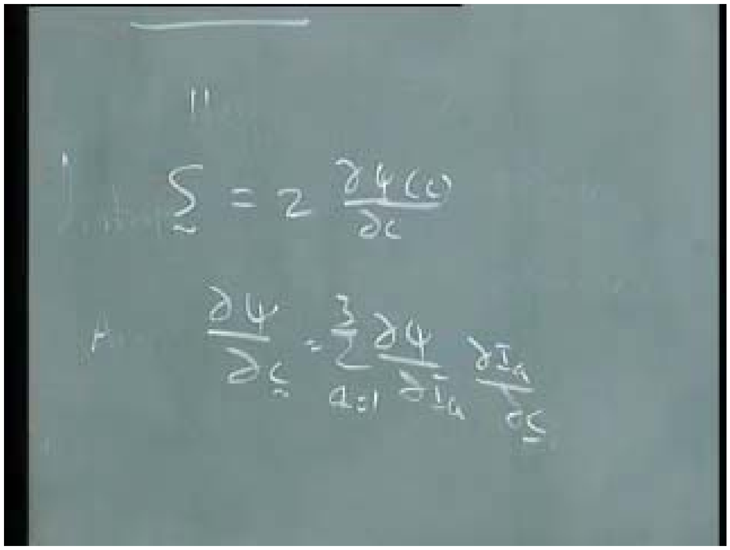

So, my job becomes very simple, because, now that I know how to write this for S, what I need is only that quantity. In fact, in fact you will see that what all we expand after this is only algebra. Only one more concept is there, after this it is algebra. Let us see. That is why I said I am not going to derive all the equations. Let us see how we write this, this strain energy function psi with respect to, differential with respect to C. Very simple, how do I write it? Chain rule or chain rule that is all; dow psi by dow I_a dow I_a dow C or in other words, dow psi by dow I_1 dow I_1 by dow C plus I_2 plus I_3. Let us keep this, because this is what will vary from one material to another material. This function is one which is going to vary or in other words, this function here, what we have put here, is going to vary from one material to another material or one type of material to another type of material.

Let us keep that as it is, but let us now look at a general formulation or general equation. From there we can quickly reduce it to different forms. In fact, that also I am going to reduce in a minute. So, what is the general equation which everyone writes? Simple, I need to know, what are the things I need to know? I need to know dow I_1 by dow C, dow I_2 by dow C and dow I_3 by dow C.

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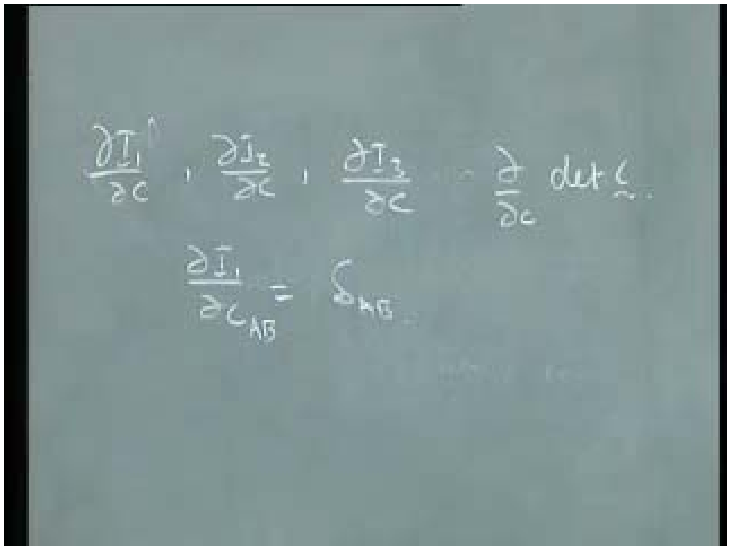

If I know all these things, that is dow I_1 by dow C, I_2 and I can get first one expression for S, no problem at all, substitute it here. Now if I know this function, I can get straight away the relationship. Let us see what they are. Let us say what is dow I_1 by dow C? Have a look at this. Anyway I am, I am going to write the results. May be I will give you couple of minutes, at least the easier part of it. Let us check whether you are able to do it. The first two, third one I will derive it. It is slightly more involved and that is a very important relationship. So, I will derive that; that derivation is quite, it is not very straight forward; so, I will just derive that part alone, but the rest of it, let us see whether you are able to do. That is dow by dow C of det C, determinant of C, is the only expression which requires some amount of manipulation.

Let us write down these two. So, dow I_1 by dow C is equal to what? Fantastic; so, that is I, sorry, that is I, basically because trace of C that means C_{11} plus C_{22} plus C_{33} that is what you will get for trace of C. So, dow I_1 dow C of say AB is equal to delta_{AB}. In other words, what it means is dow I_1 by dow C_{11} is delta 11 which is equal to 1 and so on.

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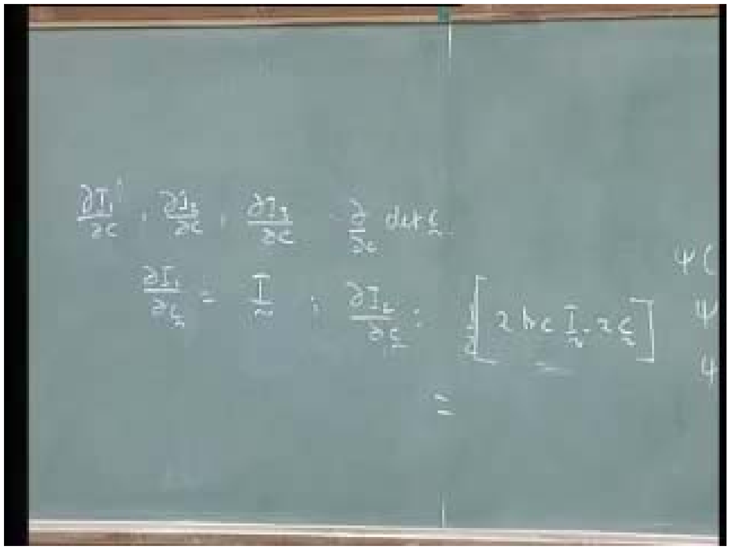

So, obviously it is an excellent answer that this is going to be I. Let us see how you do. dow I_2 by dow C, have a look at this here and see whether you will be able to do that. Chain rule, do not forget the chain rule. So, first term is what? 2 into trace of C into, so, first term is half 2 into trace of C. Is it C? Then, dow of trace of C by dow C, which is I minus, simple, no, dow of 2 C; dow of trace of C squared, which is equal to 2 C. So, that is the first part of it and now most important thing, how do I get and this and this of course, can be written in a slightly different fashion. Trace of C I can be written as I_1. This is I_1 I minus 2 C or minus C. The most important thing is, hey is that correct? One minute, trace 2 times trace of C, yes, into dow trace of, yeah, that is correct. So, C squared 2 of 2 trace of C squared dow by dow C squared, no, yeah, C is there; correct and lastly dow I_3 by dow C; there is dow by dow C of determinant C. What is that expression?

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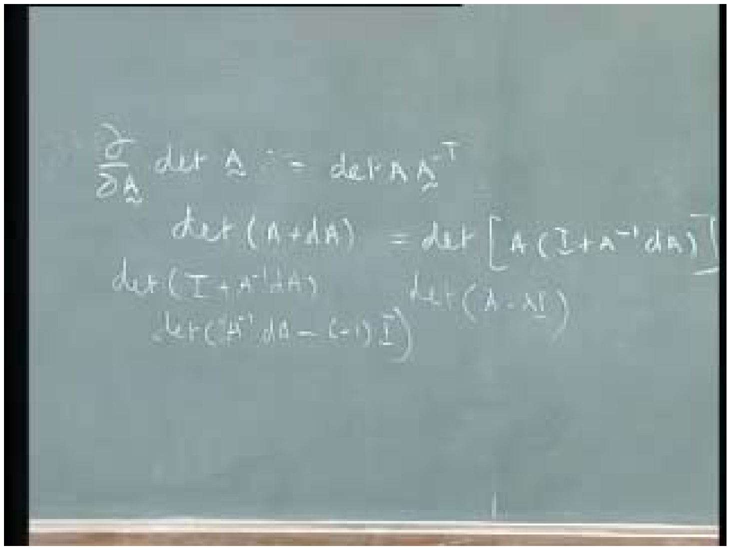

How do we get this, dow of any quantity? So, dow of dow A say, determinant of A. It is actually equal to, the result is determinant of A A inverse transpose. Now, to start with, this is a very standard procedure, actually. Let us say determinant of A plus dA can be written as determinant of A into I plus A inverse dA which can be written as determinant of A into, determinant of AB is equal to determinant of A into determinant of B; so, determinant of A into determinant of I plus A inverse dA. Now, what is this? Let us take that second term determinant of, let us take determinant of I plus A inverse dA. Does it ring a bell? This is very similar to the determinant which we use for characteristic equation, as an Eigen value problem.

What is that? The determinant of say, if I want to write down the characteristic equation for A, how do I write down? Determinant of A minus lambda I; yes, compare that and this and say what is the lambda? Say, for example this can be looked at as if lambda is equal to, if you want you can write that down as determinant of A inverse dA minus of minus 1 I. Compare these two now. So this is the characteristic equation for A inverse dA, small jugglery; you will see most of the derivations are like this, because you want a result and a very nice result. So, usually what you do is this kind of small mathematical jugglery; that is what you will be doing, nothing else.

Now, having done that what is my next step? Write this down in terms of the polynomial expression that you usually do. How do you write this down in a polynomial expression? Yeah, no, no, usual, yeah, in terms of invariance, usual expression that you write down will be say, lambda Q minus I_1 lambda squared plus I_2 lambda plus I_3 is equal to zero. This is how you would write it. If instead of that, you just write this down as, so, let us see how you will write it down.

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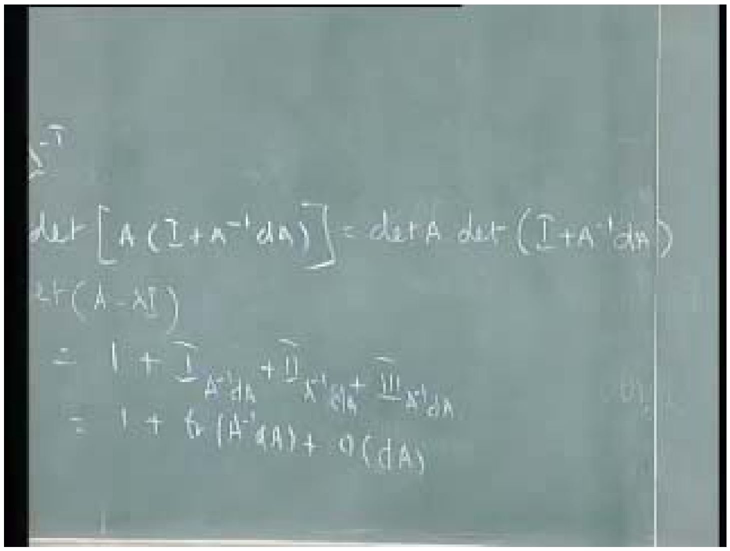

Substitute for lambda and then, simple, you will get 1 plus the first invariant of A inverse dA; I am just changing the notation slightly, because I_2, if I write two here you will get confused. So, I am just saying that it is two A inverse of dA plus three A inverse of dA, where I have substituted into my expression of lambda Q minus I_1 lambda squared plus, no; yeah, minus 1. So, lambda, everything will be minus, so, lambda Q when I substitute, that becomes minus. So, everything will be minus, so, we can write that down as something like this. This is the characteristic equation which you can write it down.

Now, what is this? This is 1 plus trace of A inverse of dA plus the second and the third invariance have higher order terms of dA, squared terms of dA and so, I am going to, because this is, this determinant is very close to dA, so, I am going to remove that. I am saying, I am going to say that these are order of dA.

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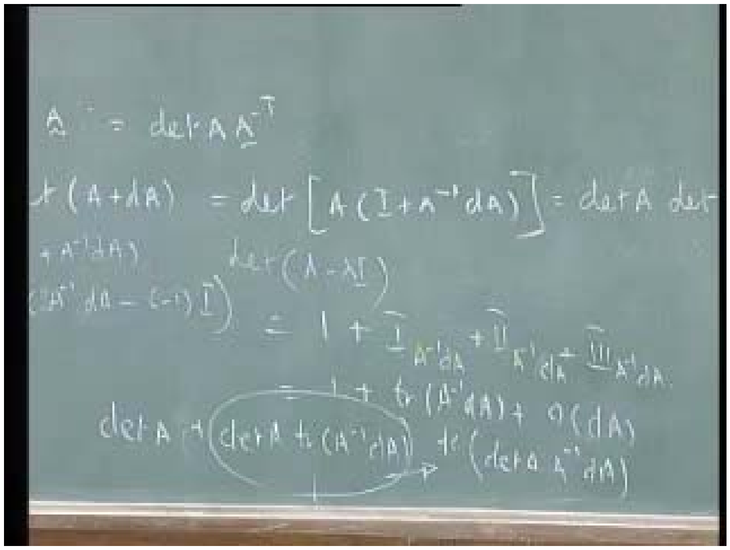

Substituting this into this expression here, you can write that down as determinant of A into this expression 1 plus or in other words, you can say determinant of A plus determinant of A trace of A inverse dA and we will stop with that, because other terms or d squared terms we can neglect or in other words, we are linearizing it. In other words, it brings out another equation. Of course, this determinant of A trace of A inverse dA can also be written as, because alpha of determinant of A is equal to determinant of alpha A, so, we can, we can write this down as trace of that is, sorry, alpha of trace of A is equal to trace of alpha A. You can say determinant, this will become, determinant A A inverse dA.

Now, that is the first thing. Since I have normalised it, I can also express the determinant of A plus dA in terms of the Taylor series expansion. Let us see how we do the Taylor series expansion. Now, what I am going to do is simple. Let us see how many of you do it. I will give you two minutes, write this down in Taylor series expansion determinant of A plus dA, compare the second term of the Taylor series expansion with this and you get the answer. That is all. So, the procedure is very simple. Then, the next step is to write this down in terms of Taylor series expansion. Write it down.

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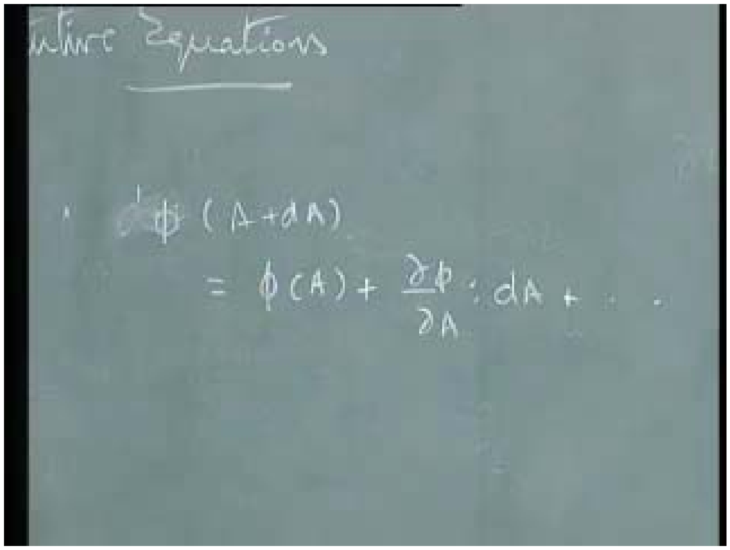

Suppose I have phi of A plus dA, how do I write that down? Suppose I have some function phi is equal to phi of A plus dow phi by dow phi by dow A dA plus higher order terms. Here, instead of this phi what is that you have?

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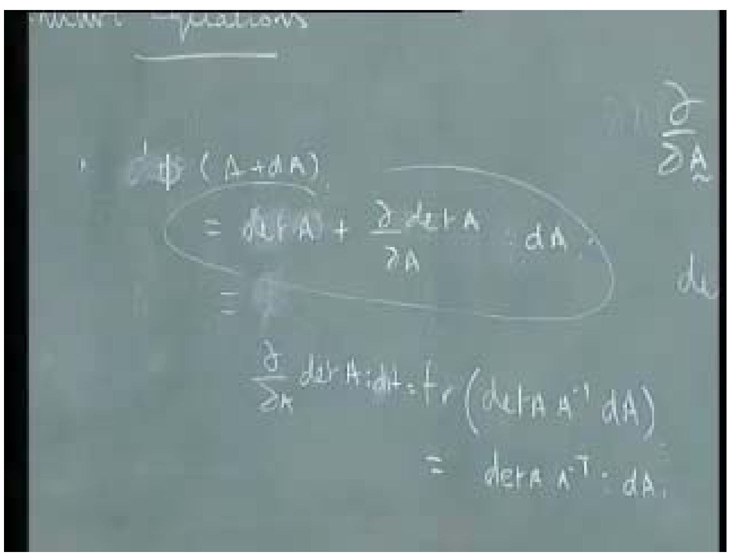

Determinant of C or here, sorry, determinant of A; so, this will be det A dA. Now, let us see, compare this and that. Yeah, dow of, simple; this becomes, actually this becomes determinant of A. Compare this expression with this expression what we have here. The second expression is what we are interested in. So, dow of dow A determinant of A is equal to trace of what? Determinant of A A inverse dA. No, no; I have not yet come to that. That is, no, no, dA, right; yeah, sorry, this I left out. Just I have added that, but what is this? This is nothing but I want to write it as a double contraction or double dot product. So, this becomes determinant of A A inverse transpose, noting that determinant of A is equal to A transpose dA. Yeah, this is simple; we had done that yesterday. What is A double dot B? This is trace of A transpose B is equal to B double dot A trace of B transpose A and so on. Yesterday we, we had done that in the last class and so, comparing the left hand side and the right hand side, very simple, if you, if you have any difficulties, it is quite straight forward. You say, that B you can write it in terms of indicial notation. A_{ij} B_{ji} or A_{ij} B_{ij}, then you can go on look at the trace of this. You will see that they are the same.

Comparing the left hand side and the right hand side, you will see that dow by dow A of determinant of A which we are interested in is determinant of A A inverse transpose. That is the expression we have been looking at. Substitute that or in other words, A is replaced by C and that is what you get. So, dow of dow C determinant of C is equal to determinant of C C inverse and determinant of C is nothing but I_3, so, I_3 C inverse. I have removed many of those things in order derive this. Go and substitute that back. Now, go back and substitute all these things into my expression for S and tell me what that expression is.

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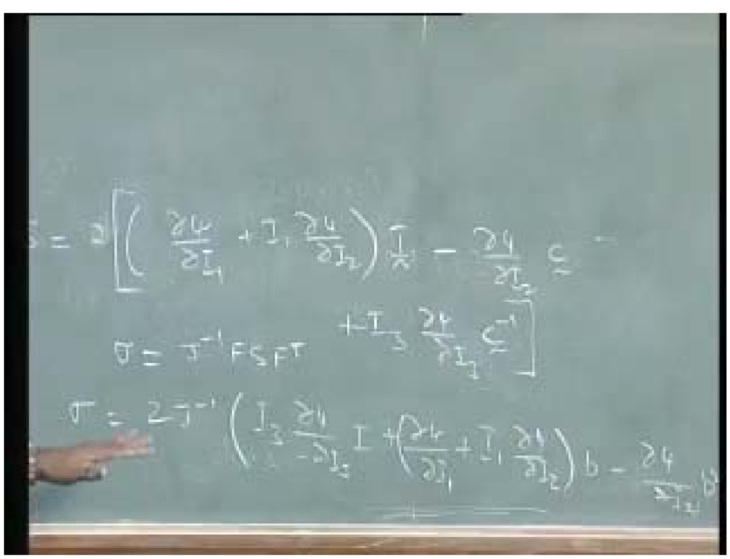

So, that is the fundamental expression. That is why I am taking time to derive it. Go back and substitute, you have all those things before; go back and substitute and say what that S should be. Remember that we had written down this as 2 into, sorry, this is how we had written it down. So, this will now become, let me write down, let me substitute for all that things which we did; 2 into dow psi by dow I_1 plus I_1 dow psi by dow I_2 into I_1 sorry, into I minus dow psi by dow I_2 C. What I am, what I am essentially doing is to re substitute back what all I did plus I_3 dow psi by dow I_3 C inverse is my final expression. This is the relationship. Once I give this, you will be able to find out S. You can also write this in terms of sigma, please write that down. So, in terms of sigma, sigma is equal to J inverse F S F transpose J F F, I mean, J inverse F S F transpose. Substitute that back, you can write the expression for sigma to be 2 J inverse multiplied by or pre multiplied by F post multiplied by F transpose, you will get that to be I_3; do that. I am not, as I told you, I am not going to derive all these things. Dow psi by dow I_3 I, it is a very important expression, dow psi by dow I_1 plus I_1 dow psi by dow I_2 into, if you want to express this in terms of b straight away instead of C it is more appropriate, so, you can write this as b minus dow psi by dow I_2 b square.

Note that, note this carefully that we have got this expression from here, which means that the psi's, this free energies are still in terms of C, they are still in terms of C. Though we have written here b, they are in terms of C. We can also write down sigma if psi happens to be written as, the free energy function happens to be written in terms of b as well; you can, you can write that down also.

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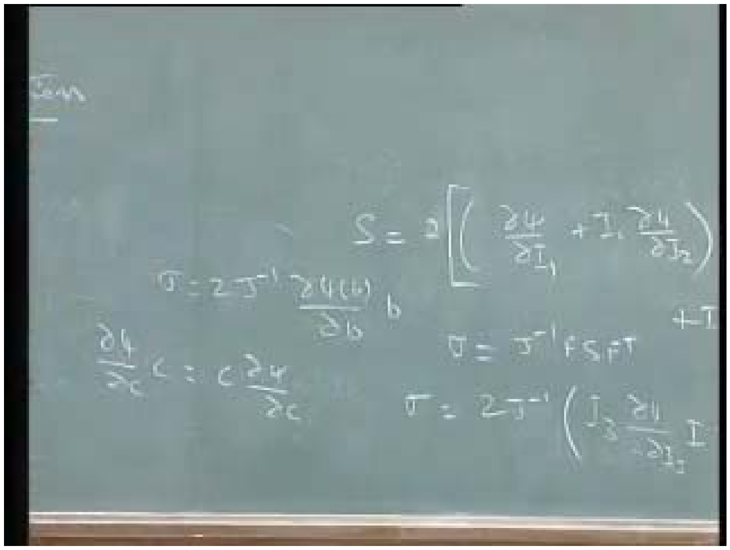

If that is the case, in fact you can write down sigma to be 2 J inverse dow b b. One of the beauty of this is that you will see that this expression commute; same way this expression that you can verify this that also commutes. You can verify this, please verify it; so, that also commutes. We have now grand expressions for S and sigma. Now, all other expression that we are going to use actually comes from this. They are simplifications of this expression. What is the assumption that we have made? What we have made is that the material is isotropic, material is isotropic. Having made that assumption, we get this. From here, you can follow through for incompressibility, compressibility and so on. Now, it is usually customary to write down not this free energy function, not in terms of I_1, but in terms of λ_1, λ_2 and λ_3.

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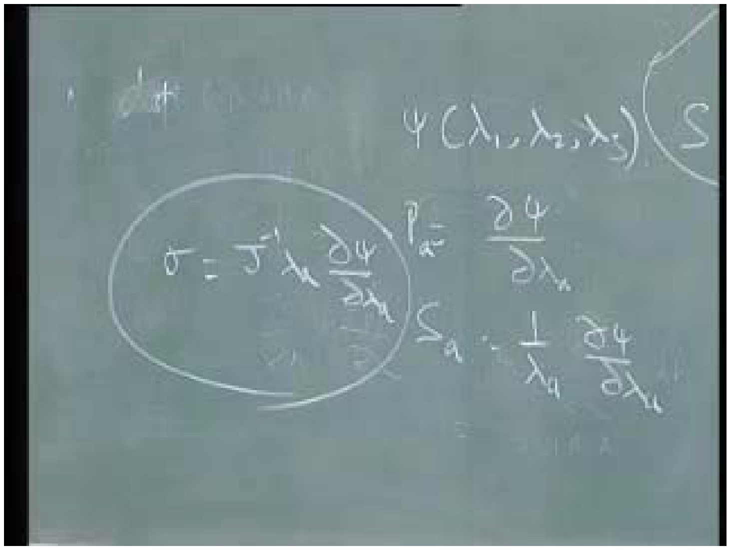

You can write down in fact this in terms of lambda$_1$, lambda$_2$ and lambda$_3$. That is the usual way of writing down this expression. I am not going to derive this and again it is a big derivation. We can write down P_a, the principle stress of or in other words, a corresponding Piola-Kirchhoff stress, a varies from 1 to 3, to be dow psi by dow lambda$_a$, often used expression. S_a, which is the second Piola-Kirchhoff stress is written as 1 by lambda$_a$ dow psi by dow lambda$_a$ and sigma J inverse lambda$_a$. These are the expressions in terms of the principle stretch values. In other words, when I express these strain energy functions in terms of principle stretch values, which I did yesterday, for example, for Ogden model what we did was to express these things. Remember that sigma sigma of alpha$_p$ by mu$_p$ lambda power alpha$_p$ and so on; you know we had expressed that in terms of an expression.

So, lambda$_a$ lambda$_1$, lambda$_2$ and lambda$_3$ and if you remember that we had used for Neo-Hookean and for the Yoeh's model in terms of I's, I_1 and I_2; remember that we had used that as well. Now note that, one more small thing which you can, which you can see that when F is equal to 1, that in other words, it is necessary that when there is no deformation, lambda's are what? 11.

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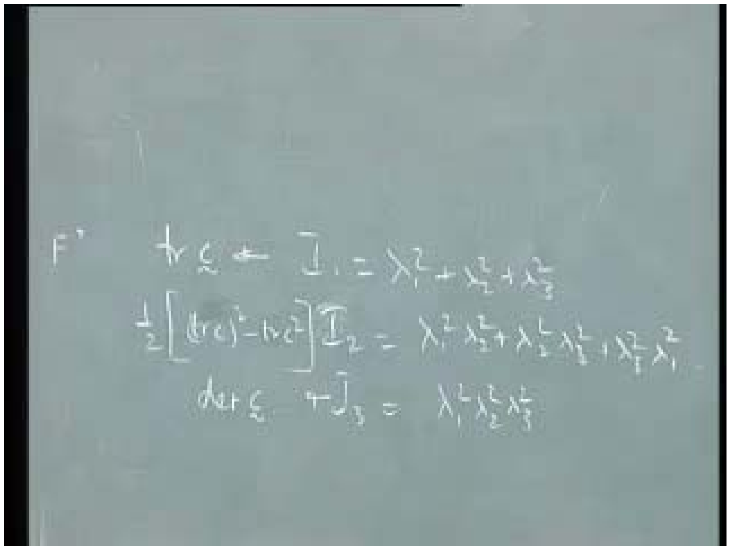

So, I_1 becomes 3, I_2 becomes, sorry, yeah, I_2 becomes 3 and I_3 becomes 1. With this background, let us now move over to what we call as the incompressible hyperelasticity, incompressible hyperelasticity.

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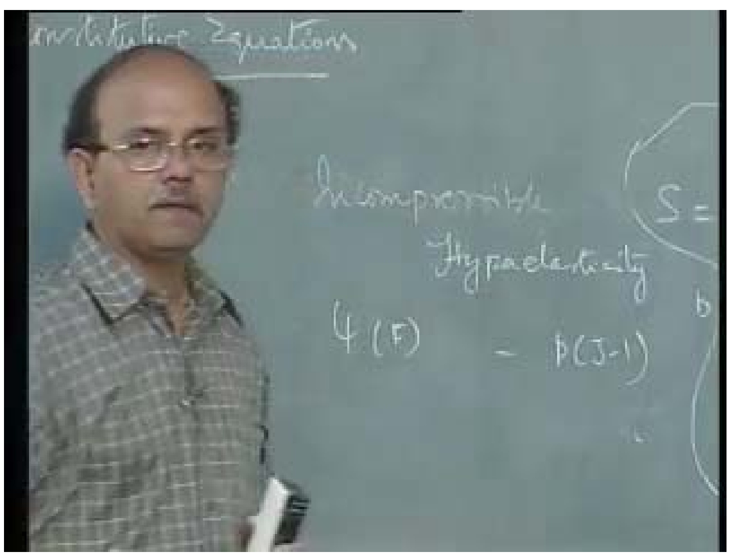

What do we mean by incompressible hyperelasticity? What is the condition that I have to … Very good; so, determinant, not zero, so, we said that it cannot be, yeah, we had put down important conditions that determinant of J is equal to 1. We have to also install or implement that condition to the or in other words, I should have that to consist of two terms, two terms to take care of the incompressibility as well and the second term would be in terms of p into J minus 1. We will talk more about that in the next class. We are now moving to incompressible hyperelasticity, which will have two functions. Fine, we will stop here. It is a nice time, good time to stop, because we will start incompressible hyperelasticity in the next class. Is there any question on what we have done, we will answer that and then close this. Please revise whatever we have done. We will stop here and will continue in the next class.