Advanced Finite Element Analysis
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Lecture - 24

The next part of continuum mechanics which we are going to study is the constitutive equations.

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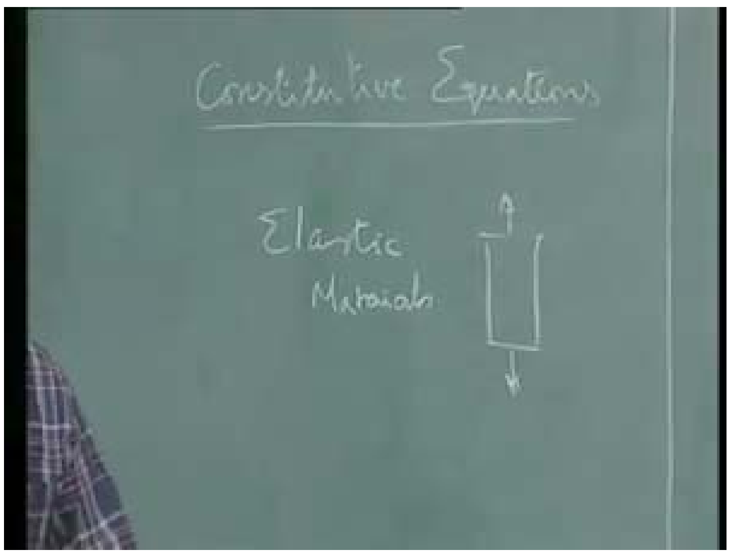

We have studied all the background that is required for understanding the constitutive equations. We have talked about the thermodynamic principles that should be satisfied by the constitutive equations. We have talked about the objectivity and we have not talked at length about other principles like determinism and the effect of the neighbourhoods and so on. Though we are not going to go into the details of the theory of constitutive equations per say, we are in a good position to understand the available constitutive equations. Now, let us look at as a first part, what we call as the elastic materials. Though you would have studied the elastic materials to be a, I would say a, material where when you remove the loads, then suppose I have a say, material like this and then when I remove the load, it regains its original position. This is what you would have studied as probably, as the elastic material. You will find that the way we are going to define elastic materials especially under this kind of large deformations, finite deformations are going to be very different.

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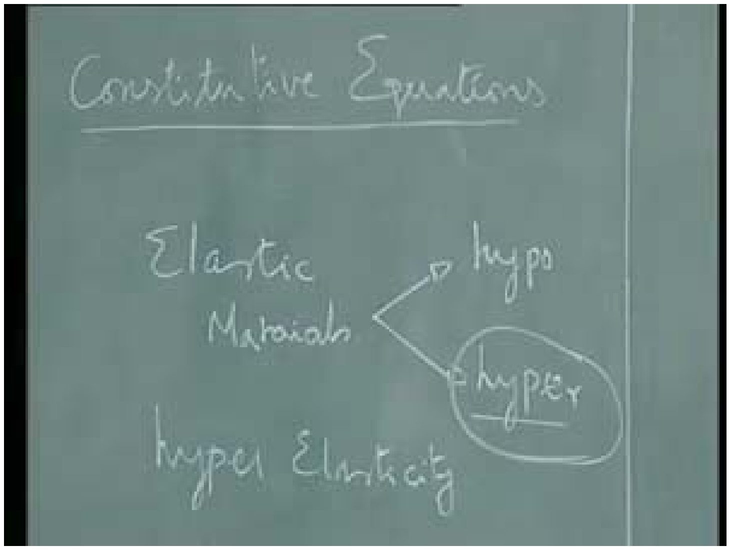

Now, let us see how we are going to define. We are going to say that the elastic materials can be represented, note this carefully, can be represented by means of two different formulations or two different schools of thoughts. One is called as hypoelastic or hyperelastic. So, the first thing you have to notice is that hypoelastic materials are not something, separate elastic materials. It is not like hypoelastic materials are rubber, hyperelastic materials are say, steel or something like that; no. Hypoelastic and hyperelastic materials are a way of representing elastic materials; they are two schools of thought.

Hypoelastic materials are also called as Cauchy materials and hyperelastic materials are also called as green elastic materials. So, this is school of thought, which was put forward probably by Cauchy; we do not have historic background for it and then carried over and the other one is by Green, Hay Green and so, they are called as hyperelastic materials. Hyperelastic materials are ones which respect thermodynamic principles or in other words, which follows thermodynamic principles. The hypoelastic materials start at a different level where it just puts down the relationship between stress and strain. Many times you would find that hypoelastic approach, if I can call that as an approach, or a hyperelastic approach you will find that both of them take you to the same result for many of the materials. But, since the approach of hypoelasticity is not that very sound, in fact, purists do not agree with hypoelastic material, most of us look at only hyperelastic materials for further analysis. So, we will abandon the hypoelastic theory for the time being. I should not say that everyone has abandoned it; there are situations where people use hypoelastic theory.

But, let us now proceed in this course with hyperelastic theory or hyperelasticity. Note that hyperelasticity is just a name that is given for elastic materials and usually we refer to them when they undergo large deformations. It is not that elasticity is different hyperelasticity is different and so on. When an elastic material undergoes large deformations, then we classify or we call them as hyperelastic materials, because these materials have to undergo certain rules and regulations which we are going to see now. One of the easiest, even more than that, nice way of representing or calling an elastic material, a material to be elastic, is to say that a material does not dissipate energy or in other words, if you had noticed or if you go back to our thermodynamic class, you would find that we had written, we had written the dissipation to be consisting of two terms.

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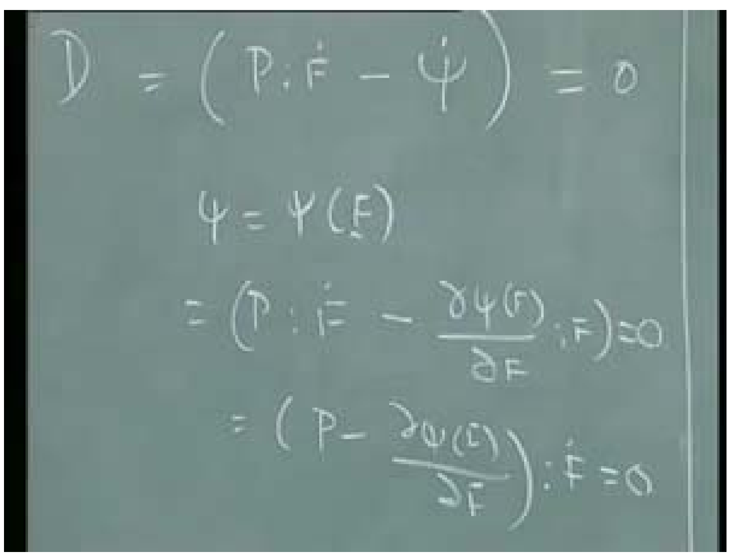

One is P colon F dot where, of course you know that, P is the first Piola-Kirchhoff stress and F dot minus psi dot. What is this psi which we had introduced? Helmholtz free energy, but usually this free energy is called as strain energy - the terminology that is, most of the time you use that as strain energy itself. Strictly speaking what we had introduced is a free energy at that place and if I remove all the thermal terms from it, then you would see that this term should be greater than or equal to zero.

Now, if this term happens to be equal to zero, if this term happens to be zero, then there is no dissipation of energy which means that the process is completely reversible, is completely reversible. Now, through the other two, I would say the principles, you can write actually psi to be a function of deformation which is represented as F here, so that this equation now can be written as P colon F dot. I hope you know this is a contraction product, this is a contraction and P colon F dot is nothing but P i j F i j, P i j F i j and you can in fact write it in a matrix form also as trace of P transpose F dot and so on. So, minus psi dot, psi dot can be written as dow psi by, chain rule, dow F colon F dot is equal to zero. In other words, P minus is equal to zero and which leads us to an important relationship which can be written like this.

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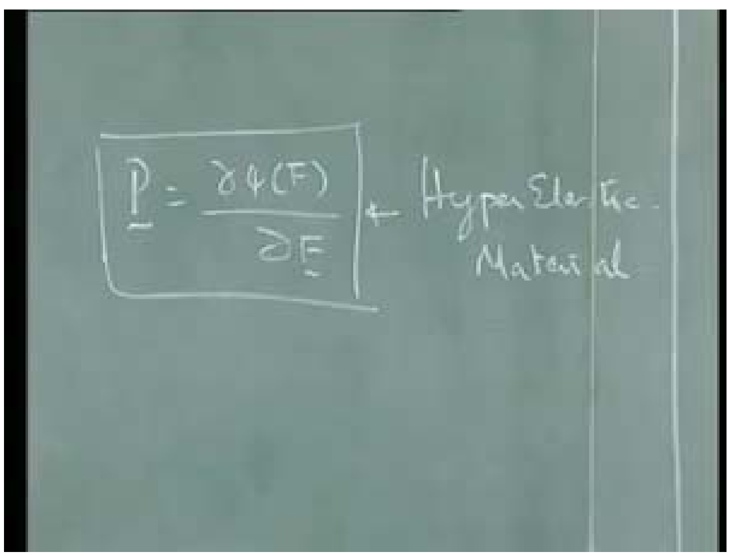

First P-K stress is a gradient of or is the dow by dow F of the strain energy. So, that is a very important, probably the most important step in the whole development of constitutive equations. Now, how do we define different materials? We said that an hyperelastic material ultimately is one which follows or which respects the thermodynamic principles, which means that a hyperelastic material is one which follows all these steps and which of course means that a hyperelastic material is one which has the relationship written here. Is that clear? So, that is the hyperelastic material.

The question that may come to your mind is I know of elastic materials say for example, elastomers or rubber that is an elastic material. We know most of the biological tissues say, skin or your heart muscle, calf muscle, all these things are again elastic materials and again you know that see, even some of the cell structures, they are all elastic materials. You know a number of practical, if I may call practical within quotes, materials which are elastic. Hence, according to our definitions should also be classified as hyperelastic, because they undergo large deformations and of course, a set of metals in the elastic region which are again are elastic materials, again elastic materials, if that is the case do they all undergo or do they all respect this law?

Yes, they all respect this law. But, then what is the difference between them, what is the difference between them? On one hand, I have this equation; on the other hand I have practical materials, which satisfies this equation. Of course, I know that the deformations induce different types of stresses in different materials. So, where do I actually draw the line and say that look, the elastic behaviour of rubber is like this, the elastic behaviour of steel is like this, the elastic material of biological tissue is like this and so on.

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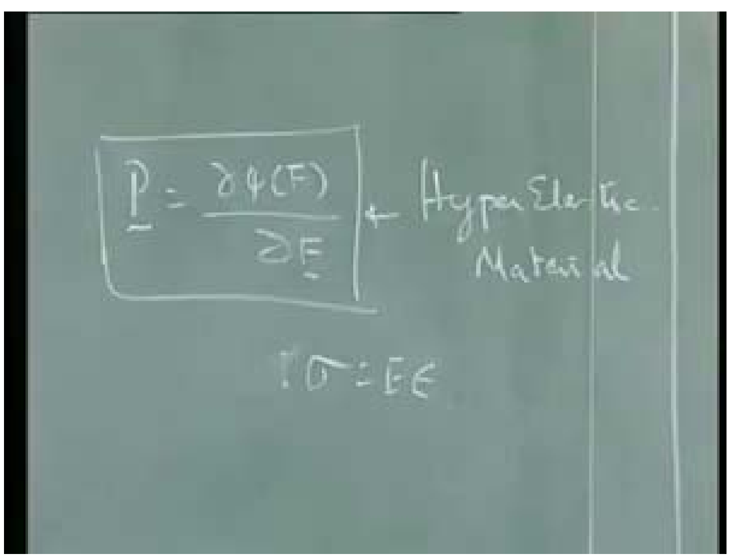

Where do I draw a line, where or in other words, to put it in a more simple fashion, I know that from a very, very early class that sigma is equal to E into epsilon and that this kind of simple elastic, linear elastic rule is valid for all metals whether it is in the elastic region, of course, whether it is steel or whether it is copper, whether it is aluminium and so on and I also know that the value of E changes from one material to the other; value of mu is also different when I say for example, titanium and steel and so on.

So, what is the analogy now between this and this? Now, the analogy is not very straight forward, it is a two step process.

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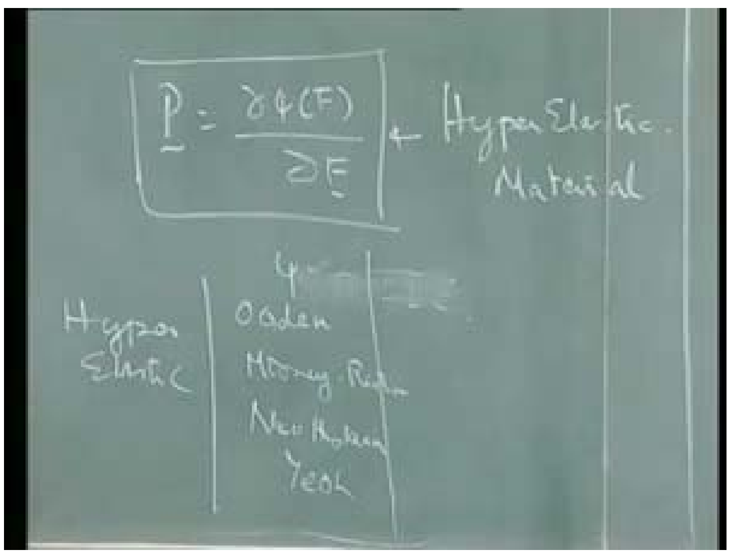

In other words, we have on one hand hyperelastic material satisfying this equation; on the other hand, we have a number of types of what we call as strain energy functions. This is the function; psi function of F, strain energy functions. These strain energy functions may be say, Ogden, Mooney-Rivlin, Neo-Hookean, Yeoh and so on. So, these are the strain energy functions that are available. This may vary and let me write down one function.

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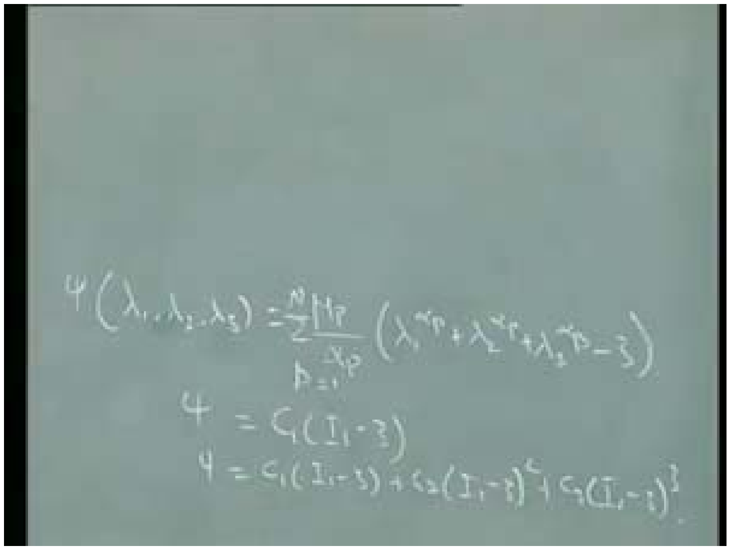

Say for example, if you look at a Mooney-Rivlin function or say, Ogden function, then I can write down the strain energy function actually in terms of stretch, usually you write this in terms of stretch say, lambda$_1$, lambda$_2$ and lambda$_3$ and then this is, though I am pre-empting some things, you may not understand it completely; just I want to put down this because why did I get stretch here, we will see that later, but all of you know intuitively that stretch means deformation. So, I can express it in terms of lambda$_1$, lambda$_2$ and lambda$_3$. So, this will be defined as say mu$_P$, sigma mu$_P$ by alpha$_P$ into lambda$_1$ alpha P plus lambda$_2$ alpha P plus lambda$_3$ alpha P minus 3; something like this, where alpha$_P$ and mu$_P$ where P varies from 1 to N. N can be 2, 3, 5 and so on. These kind of equations are the ones which define what we call as the strain energy function.

If you look at another, say for example, Mooney-Rivlin, then Mooney-Rivlin, psi can be written as a C_1 I_1 minus 3, where I_1 is the first invariance and if you, sorry, this is for a Neo-Hookean; if you look at Yoeh's model, then Yoeh's model has the strain energy function to be written as C_1 into I_1 minus 3 plus C_2 into say I_1 minus 3 whole squared plus C_3 into I_1 minus 3 whole cube and so on. You can write down a set of equations like this for different size, put forward by different people looking at the physical behaviour at the same time saying that certain other conditions are satisfied. Please note that most of these theories, which we are dealing with, are macroscopic theories.

There is no attempt right now, in this course of course, to link this macroscopic theory with the microscopic theory of what happens inside. For example, it is very clear that C_1 is some constant, alpha_P is some constant, C_2 is some constant and so on. How do you get these constants, what are they? They are obtained more from a phenomenological approach and right now, we do not link this up with the microscopic aspect, we will come to that later. So, the approach, which we are dealing with are what are called as the macroscopic or phenomenological approach. So, these are the materials that are available. Note the word, common word, hyperelastic material, Ogden material, Mooney-Rivlin material and so on. It does not mean Ogden material means physically there is a material. All materials which can be expressed in terms of Ogden strain energy function is called as the Ogden material. All materials which can be satisfied or which can be written in terms of Yoeh strain energy function is called as Yoeh material and so on. Is that clear? All of them can be written in different types of things.

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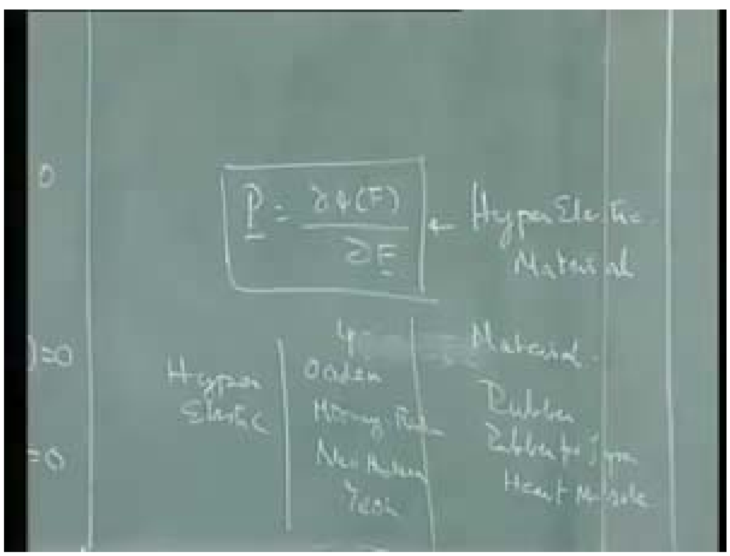

See, on the other hand, the third column is the actual material. This can be rubber, a carbon filled rubber used in tyres; a rubber for say, tyres. It has a special chemical composition and compound. Then you have say for example, biological materials, say, heart muscle and so on; list of actual materials. The big picture is, all of them are of course, hyperelastic materials. The second picture is that each one of these materials can be represented as one or more of the middle column of materials.

Rubber can be represented as Ogden material, not very, so very accurately, using a Neo-Hookean material. Some of them, for example, rubber for tyres can be represented exceedingly well with Yoeh as a Yoeh material, not so very accurately as Mooney-Rivlin material or Ogden material and so on. In fact, when you come to heart muscle, the type of strain energy function is totally going to be different.

As a next step, what you do is you now build connection between the actual material and the mathematical representation of its behaviour put forward by Ogden, Mooney-Rivlin and so on.

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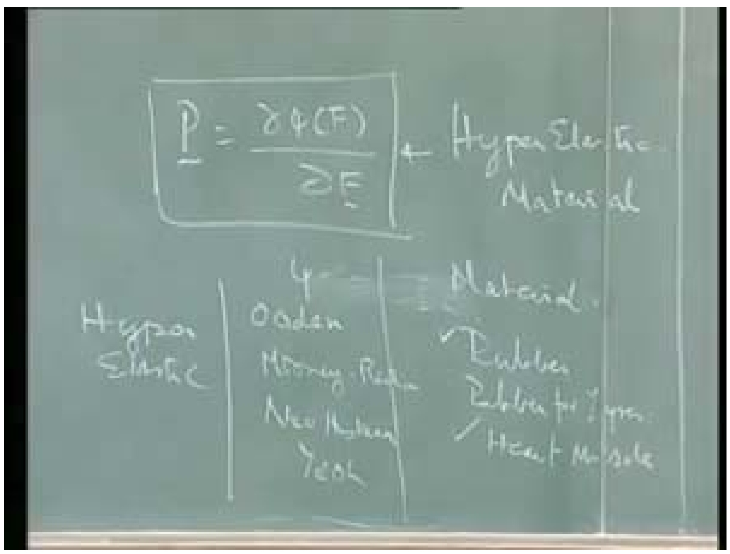

Now, let us say that I can represent it using Ogden material say, rubber. Even heart muscle, there has been a paper 1983-84, where or why 84, even in recent times where the Ogden material is used to represent the heart muscle. Then, how do I distinguish these two materials? Come back to this place. Yes, that is true, so, both of them have the same relationship. But, these constants differ from one material to the other. So, I hope the whole complex structure of hyperelasticity is clear now. If there is any question, I will answer that.

Student: How can I represent it, how can I know that it can be represented by Ogden model or Mooney-Rivlin model?

Yeah, see, when you have a material you should know certain of its behaviour definitely, before you can attempt to represent it using one of these materials. Now, there are reasons why for example rubber for tyre is exceedingly well represented by Yoeh's model, because the shear modulus with deformation it varies and is supposed to have a double peak. So, that kind of mechanical behaviour can be represented in a much better fashion using Yoeh's model. For example, if you take a heart muscle, there is a, there is a strain energy function by Hunter. He has looked at the mechanical behaviour of the heart muscle and said that, look if this is the behaviour, then this is the type of strain energy function which represents, which is closest to the behaviour or in other words, all classes of heart muscles can be represented now using this type of behaviour or this type of strain energy function. So, there have been people who have been working on this for a very, very long time, the constitutive equations, how to put forward strain energy functions.

When you take a material and when you want to analyse it using non-linear finite elements, the first thing you should do is how do I represent this material? Suppose you go back and look at packages like say, Abacus, you will see that they have different materials that are there. I mean, they have different materials to represent the actual physical materials or material models to represent actual physical models or physical materials. They call this as material model. For example, if you take Abacus, you will have Ogden material, you will have Yoeh material, you have Mooney-Rivlin material and so on, material models. So, when I, when you want to take say, want to go and analyse tyre, the first thing to do is now what is the appropriate representation, model for this tyre; rubber which goes into the tyre? You have to really understand how it behaves. May be go back and look at literature; you will find that people have put down Yoeh's model specifically with the purpose in mind. Yoeh had put this down, may be in the, if I remember right, it is in early 90's. So, then you go and apply Yoeh's model.

In order to apply that, one of the very, very important things in non-linear finite element is that you need constants. This testing of many of these materials are quite, I would not say difficult, but time consuming and in fact, it requires a lot of effort. Now, there are actually routines that are available. If you take Abacus, routines that are available even to fit these constants. How do you get these constants? You do a series of tests. What are these tests? Uniaxial tests; if you do uniaxial tests, you can fit certain of these constants. May be, you have to do a biaxial test; you can improve some of them. Then, you may have to do a shear test. Then you will improve, you will get a much better understanding and so on.

In other words, the testing of these materials which means that we are interested in the mechanical behaviour, not in how say, the structure inside, the isomers or whatever is inside - how it behaves, we are not interested in that; mechanical behaviour on a gross scale, on a macro scale. So, you have to understand them using all these kind of test results and then fit these kinds of constants to a particular equation. That is why I said there are routines available. But, for some of them it is very simple. Neo-Hookean, it is very simple; it is not very difficult to fit this, you need much less tests. When you need much less tests that means all the different types of mechanical behaviours are really not covered by that material. That is what it means. Is that clear? Fine; so, that completes, that gives you an overall picture of where we stand with respect to the hyperelastic materials.

With that in mind, let us look at other considerations that are, that should be there in order to define the strain energy function. I am not going to go into lot of details here, because constitutive equation itself is a big topic and you can have, offer a course on the constitutive equation and strain energy functions. I am not going to do that, but I am going to give you an overall flavour in the next two classes as to how these representations actually are there for different materials. If you look at the constraints that we may have to put, all these things are very straight forward; they are not very difficult to understand. Let us see what the constraints are. The constraint is that F, of course, F cannot be zero. If it goes to zero, what does it mean? That means you are trying to annihilate or shrink the material to almost or make it almost disappear; almost disappear, which means that the energy that you require in order to do that is enormously large.

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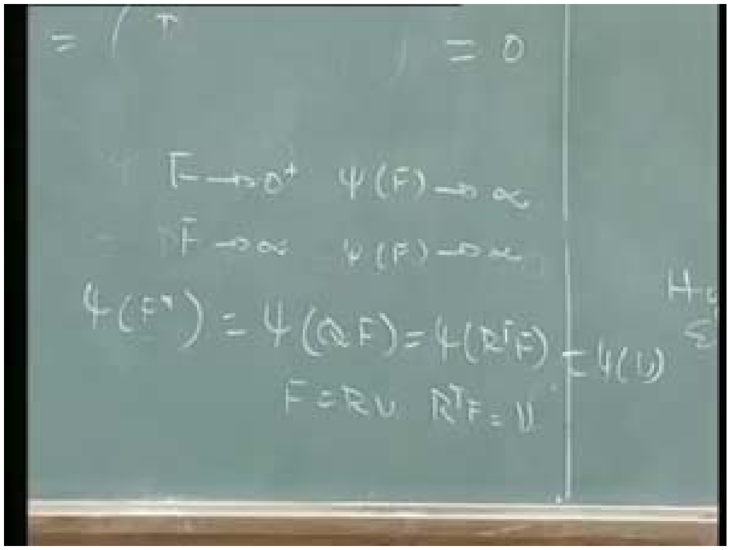

The one thing is that when F tends to zero, then psi of F tends to infinity. We use infinity to mean that it is very large; we use zero to mean that they are very small. Same fashion, when I make F go to infinity, when I make F go to infinity, what do I mean? I mean to say that I am going to increase or increase the volume material enormously, which means that F at infinity means that psi F also has to be extremely large. So, either way this psi F, this strain energy function, sorry, strain energy function has to be large. That is one thing.

The other is with respect to objectivity. Let us say that I am looking at or you are looking at it as a new observer. The function will not change. It is an Ogden function, it has to be the same. If it is to be a, if it has to have those constants, then the constants have to be there and so on, but your objectivity means that there is a relationship between the F star or your F and what I observe. We already know that both of us should have the same function, so that this can be written as, the function can be written as Q F, Q F. So, necessarily, necessarily the function should be written in such a fashion that the function is not changed, when you post multiply F by an orthogonal tensor Q.

If you choose this orthogonal tensor to be R transpose, this happens to be and from the relationship that F is equal to R U, R transpose F is equal to U. Of course, R R transpose is equal to I, which means that we get a very interesting and very important result that the strain energy function is now a function of only the stretch part of F, stretch part of F. Remember that F had two things in it. One is the rotation part represented by R, the stretch part represented by U. I would have been very worried if I had stopped here and that since R is already present there, if I had to now have my function to be a function of R as well as U, I would have been a bit worried. Why because, then I can just rotate a piece and if this strain energy function respects it, then I would get a stress, after all P, the Piola-Kirchhoff stress is function of the dow by dow F of this function. Fortunately, it does not happen like that and so, actually this function is a function ultimately reduces to a function of only the stretch alone. In other words, function of U. Is that clear?

Before we proceed, we look at one more aspect of the material behaviour and this objectivity, what we call as anisotropic. The question that comes to our mind after looking at this is that, what is the connection between anisotropy and objectivity or in other words, what we said here is that when there is a rotation, the rotation does not affect, does not affect the behaviour of the material or the response is not affected by the rotation.

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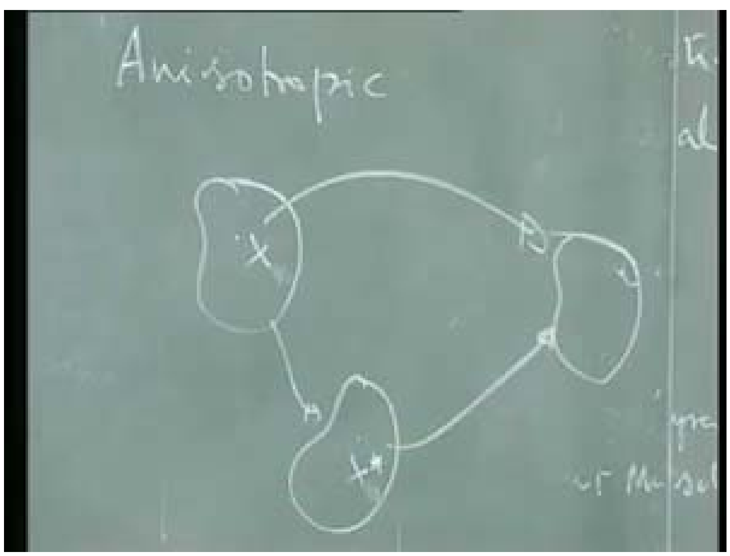

We have not completed the story yet, but we know that there are materials which are anisotropic, which are anisotropic. Typically, most of the composite materials are anisotropic materials. What do I mean by that? I mean to say that there are fibres which run in these composite materials and that the deformations are function of the relationship between the fibres and your loading and so on. Suppose I take this, I apply the load. Composite material, it behaves in one fashion, but load applied in another direction behaves in a different fashion and so on. That means that you have a vague notion that when I rotate the specimen and apply the load my deformations are going to be different.

But, how is that related to this objectivity? In other words, what we mean to say is that suppose I have a material or I have a component and then there is a deformation due to some loading and let me call this as the reference configuration with X and that now there is a new reference configuration say, let me call this as X_0 or no, X is enough and so, this is X, say, X star and there is again a deformation, let us say like this. If it happens to be like this, then we call this material as an isotropic material, as isotropic material. Now, what is the condition for isotropic materials? Simple; it is not very quite easy to understand that.

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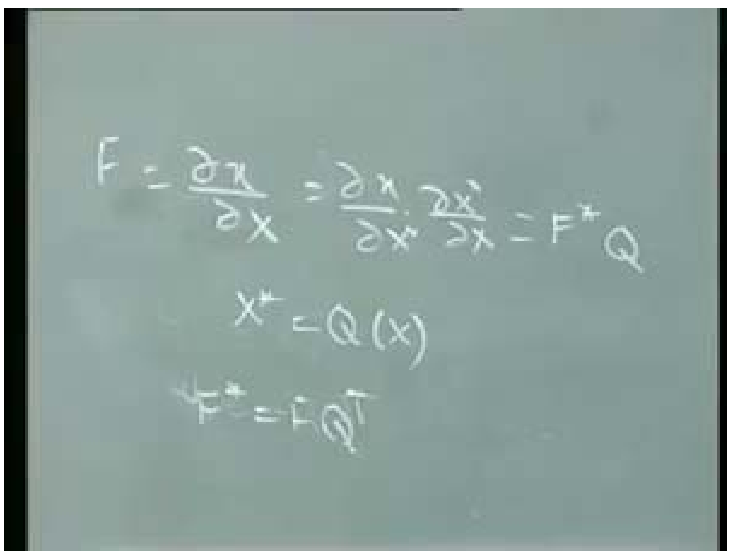

In other words, F which is say, dow x by dow capital X should be the same whether I start my journey from here or here, when there is a difference in my reference frames; that is what I said when I rotate it and then deform. So, I am going to define first isotropic material and one which does not satisfy the law which I am going to put down are called as anisotropic materials. That is equal to say, dow x by dow X star dow X star by dow X and noting that the relationship between X star and X is quite straight forward. Say, for example, I can write X star to be Q X, so that dow X star by dow X can be written as Q, so that the second equation here, second part of the equation here can be represented as Q, so that dow x by dow X star is F star. So, this becomes F star Q or in other words, F star is equal to F Q transpose.

But, note the subtle difference between F star here and F star what we had for objectivity. So, this is the condition for isotropic material, condition for isotropic material. Note the difference between the two. In this case, I have F to be post multiplied by Q transpose, note that. So, I have F to be post multiplied by Q transpose. In the previous case, I had F to be pre multiplied by Q. So, there is a difference between anisotropic material and sorry, isotropic, anisotropic and isotropic material. Is that clear? But, physically what does it mean; physically what is this post multiplying and pre multiplying and so on? That is very important to understand what it physically means. Physically it means that, what it means is that the isotropic materials are ones where if I take the reference configuration, rotate the reference configuration and apply this loading or without rotation if I apply the loading, in both cases, in both cases my deformation is going to be the same. That is what I mean by saying that this equation is satisfied.

On the other hand, on the other hand, what do we mean by objectivity? What do we mean by objectivity? Objectivity means that, if I now load the material, come to this state with the deformation, with the loading in place, if I now rotate this which is equivalent to an observer rotating, because observer, two observers have the same reference configuration. When they look at the reference, they are at the same level. That is why, if you remember, we got F star is equal to Q F. If you go back and look at how we got for objectivity, you will remember that we got objectivity to be F star is equal to Q F, basically because we assumed that the observer looks at reference configuration in the same fashion; both the observers look at the reference configuration in the same fashion.

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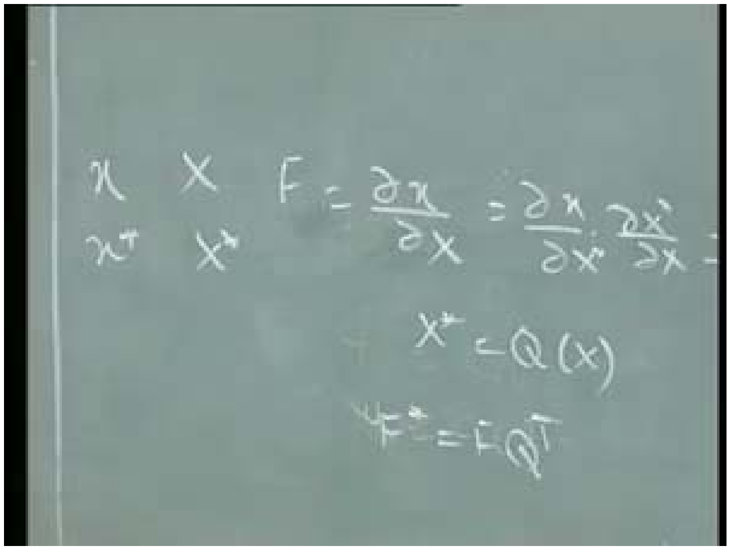

If you now take this deformed fellow here and then rotate it, which means an observer observing the deformed configuration separately, so, deformed configuration, the observer will get x star. Then, my result is objective if it follows the three conditions which we had put down in one of our earlier classes. So, the difference between anisotropy and objectivity is that, in objectivity we are looking at x and x star, in anisotropy we are looking at X, capital X and capital X star. Is that clear?

Now, having studied this, let us get back to certain other issues which are important for us and see how we can represent this. Now that we know that the strain energy function is a function of U alone, which means that the strain energy function is a function of C, which means that you can also say that the strain energy function is a function of E, so, you can say that function of C function of E.

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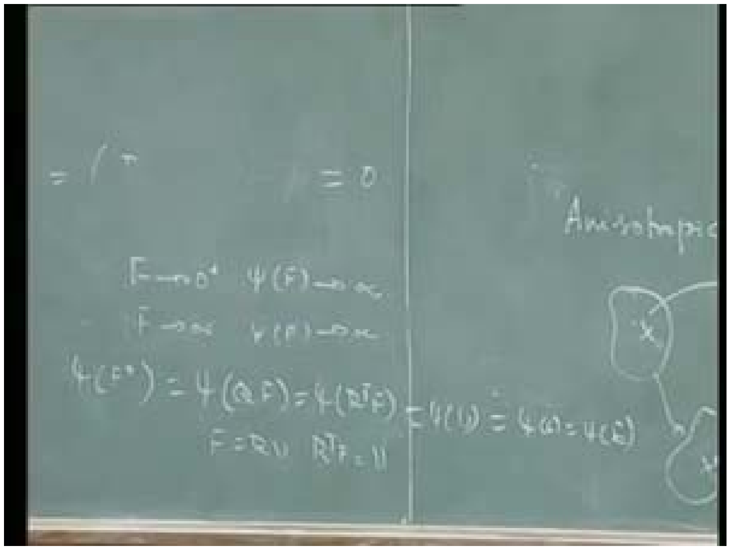

Note that usually you use the same symbol function, you know, the same symbol, whether it is C, E or whether it is U; we use the same symbol here. C, what is the C? We defined, you remember, defined F transpose F, F transpose F to be C and we had a relationship between E and C. What was the relationship? Half into C minus I is equal to E or 2 E is equal to C minus I and so on, all those things; because of these relationships, you can say that the function can be in terms of C or E and so on.

Yeah, it is, since is a function of U, I am saying that it can be looked at as a function of C, it can be looked at as a function of E and so on. As we have done in our earlier classes, I have to …

These, no, no; we are right now talking about, please note that we are talking about material objectivity. These results are due to material objectivity. So, if psi F is equal to or psi F star which is psi, I mean, let me write that.

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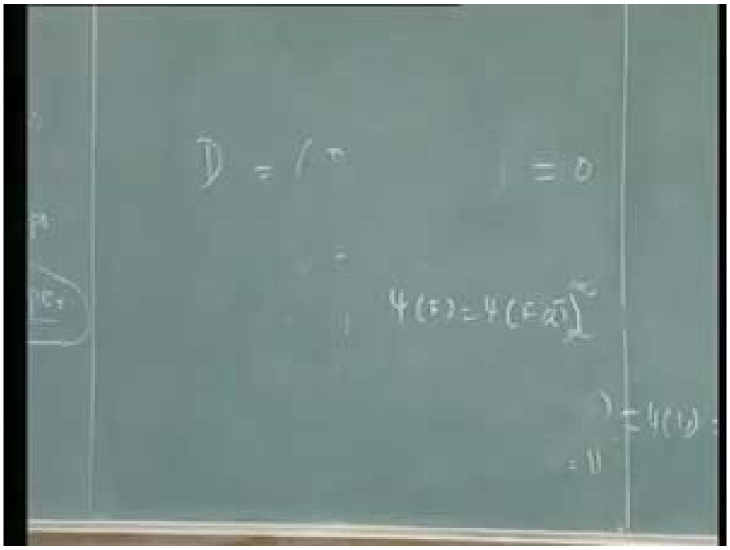

If it so happens, then we say that the material is isotropic. Having done all this work, let us see how we develop the relationship between P and sigma and so on. See, because, ultimately if you look at all these codes, you would see that they express the results only in terms of the Cauchy’s stress, because Cauchy’s stress has the most meaning as far as our interpretations are concerned. So, we always talk about the quantities in the material frame as well as the quantities in the spatial frame; talk about both. Let us see how we develop the relationship between them.

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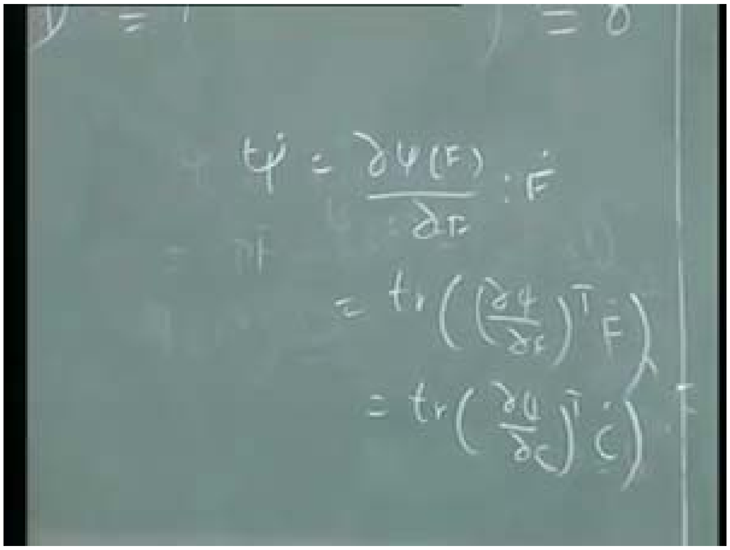

Let us now go back to psi dot and psi dot can be written as F dot which can be written as trace of dow psi by dow F transpose F dot in terms of the matrix notation. If I now express psi in terms of C, then you can write that down also as trace of dow psi by dow C transpose C dot. Now, what is the relationship between C dot and F dot, because I need this; we will come to that in a minute, but what is the relationship between the two? Let us see what is C dot? Can you just write down?

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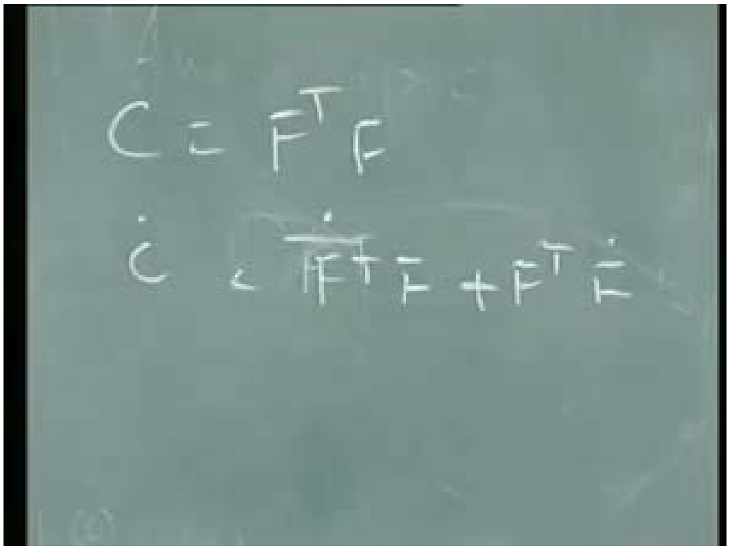

Since I asked a question what is C, can you write down? what is C is equal to F transpose F. So, C dot is what and then substitute it back here and see how we can represent psi dot.

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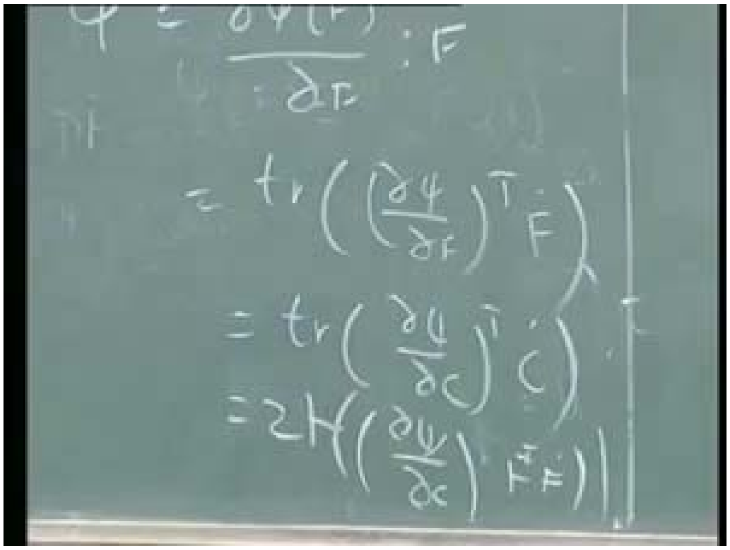

This is equal to, so, substituting that C dot into this expression and noting that C is symmetric, we can write down the equation here to be equal to 2 times trace of dow psi by dow C transpose F transpose F dot. Since both, since transpose is the same, since it is symmetric, I can remove that and say that 2 times dow psi by dow C F transpose F dot.

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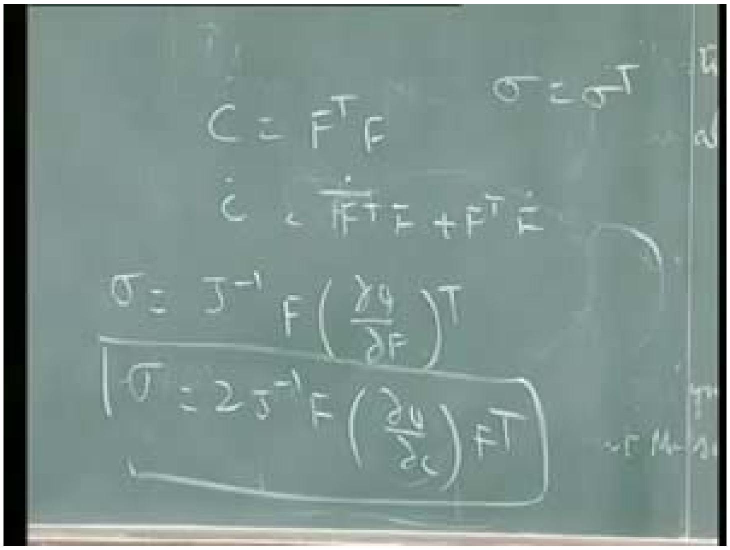

Noting the relationship between P and sigma, I can write down sigma to be, let us say how we write down the sigma to be, in terms of P, yeah, in terms of, sigma in terms of

P. J inverse F P transpose. Write this, go and have a look; dow psi by that is in other words, dow psi by dow F transpose, which now using this relationship, you can write that down as, look at this here, the two relationship between them. So, you can write that down as dow psi by dow C F transpose, so, sigma, 2 times of course, is equal to 2 J inverse F dow psi by dow C F transpose. J, J inverse, J inverse P F transpose. But, P transpose, both of them are the same. So, that is why, because sigma is equal to, note that sigma is equal to sigma transpose. Hence, whether you write it as P F transpose or the other way, it will be the same, because sigma is equal to sigma transpose. So, that is the relationship between sigma and and dow psi by dow C.

Yeah, what I did was to replace from here. Look at this expression, this and this; compare this and this, so, F dot is common. So, the rest of it with 2 should be equal to dow psi by dow F. When I express it in terms of C, then actually it is 2 into dow psi by dow C F transpose and that is what I have substituted.

Our next goal is to now develop what the relationship is between S, second Piola-Kirchhoff stress. See, I have already said this that the first and the second Piola-Kirchhoff stress are very important. Though the first Piola-Kirchhoff stress has a lot of physical meaning to it, we know what it is, it is a nominal stress. Unfortunately or fortunately, we do not use first Piola-Kirchhoff stress to that extent in computational engineering. Tell me, why?

Very good; it is not symmetric. So, most of the situations we use only second Piola-Kirchhoff stress and it is very important that we develop what the relationship is between second Piola-Kirchhoff stress and these quantities. By the way, P is of course, as someone said P can be now, P is equal to what is it from here? In terms of, if psi is expressed in terms of C, then what is P?

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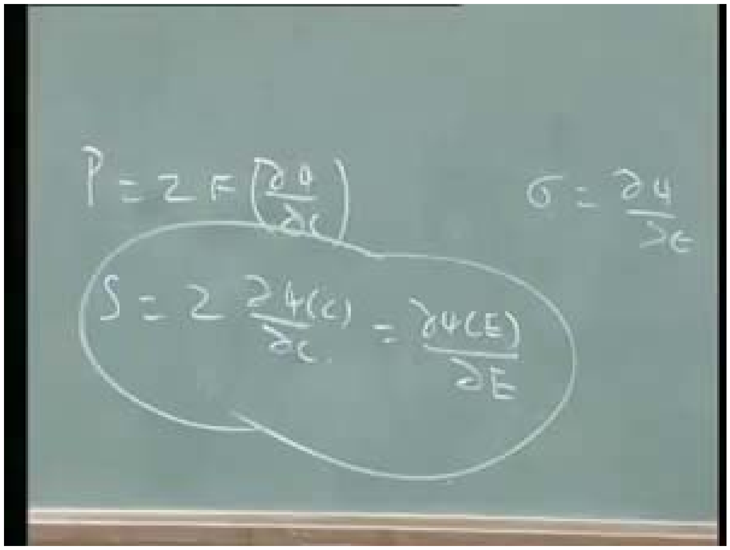

2 F dow psi by, now what is the relationship between S and P and so that you can express this S as, note that what the relationship between S and P is. Can you just write that down? Note that down, what it is. You will write this down as 2 of dow psi by dow C and noting the relationship between E and C, we say that this is equal to and that is a very, very important relationship. So, S is equal to, if I express this strain energy density function in terms of E, then dow psi by dow E. If you remember we had approximated S to be sigma and E to be epsilon and in fact, in our, in the last course we had just said sigma is equal to dow psi by dow epsilon, strain energy function, because of this small deformation case that is involved there. So, you get that important relationship between the strain energy density function and the second Piola-Kirchhoff stress.

As a next step we now look at, having studied the background, so, we can be now fast, we can move forward. As a next step we look at now work done in a closed cycle and why hyperelasticity? Just with that one small goal, we can move forward and look at various functions, how they are written and so on. We will stop here and will continue with the rest of our discussions on constitutive equations in the next class.