Advanced Finite Element Analysis
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Lecture - 23

In the last class, we were looking at objectivity.

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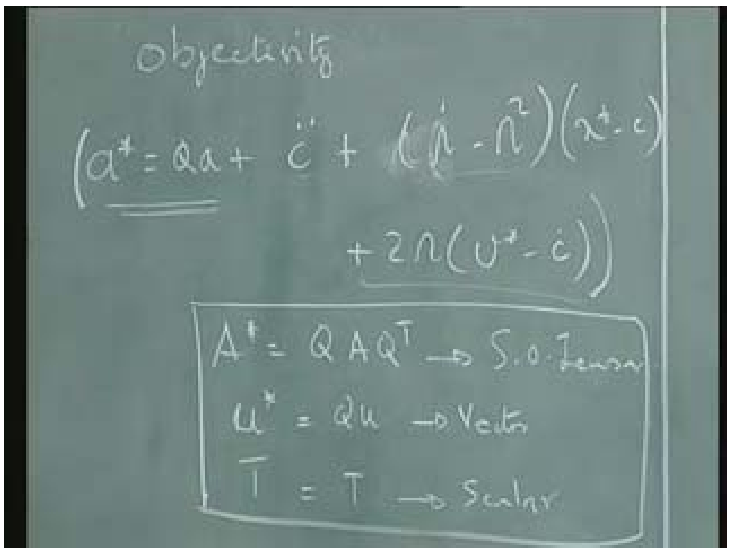

If you remember we had looked at, for example, how acceleration behaves when there is a change of observer or under two observers how acceleration would look like. We had seen for example, a star in terms of a and we recognised that, number 1, that the accelerations and of course, we did before that velocity are not material frame indifferent quantities. If at all they have to be material frame indifferent quantities, then a star should have gone through the transformation like this - a star is equal to Q a, but since they are not like that and then you have additional terms, we recognised that acceleration is not material frame indifferent quantity. In fact, if you had noticed, the major reason why acceleration is not material frame indifferent is because of the fact that both C and Q, both of them are functions of time; both of them are functions of time.

In other words, we have allowed the observer, the second observer, the starred observer not only to be different, but also allowed him to do what he wants - move with respect to time and rotate that is the Q, to vary that Q also with respect to time. That is very important. So, that is the reason why we have acceleration to be, to behave in that fashion and we recognise that some of them are quantities which are known, say for example, this is the Coriolis component and that is the centrifugal force and the other one is called as the, this is called as the Eulerian accelerations and so on.

Yes, there are, before we proceed the step, a point of comment; but, there are some constitutive equations being put forward, for example, which analyses asphalt and other things where you make some conditions. You say that the constitutive equation is not valid if Q as a function of time is given to the observer. In other words, you say that I am violating material frame indifference, I am violating it. But, violating and putting it in this fashion and that what I am violating is actually the observer's motion with time and so on. That is also done, but I am not sure whether that is the right approach, because there are difference of opinion as far as this is concerned and anyway people have done that as well. So, if you come across such kind of work, please understand that there has been a violation of material frame indifference and so, you have to take that a bit more carefully and when you do analysis that also should be taken into account and we also recognise for example, that if this part is zero, then we say that the transformations are Galilean. These are Galilean transformations and there are what are called as Galilean frame invariance; let us not worry about that. But, we will only worry about material frame indifference as it is applied to continuum mechanics. Let us just summarise with the fact that these are the types of changes that could take place between the starred and the un starred co-ordinate system.

Now, let us look at certain quantities of interest to us and see whether these quantities can take part in the type of constitutive equation which you want to build.

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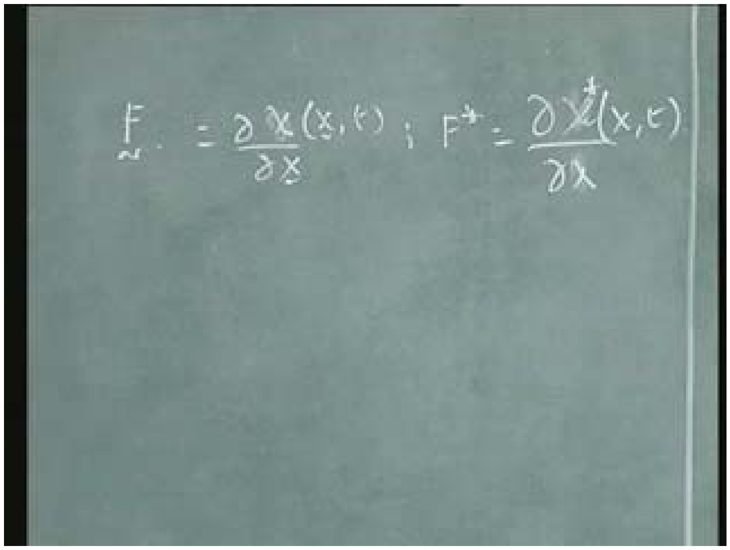

First of all, let us understand how F would transform or in other words what the relationship between F star and F is. What do you think would be the difference or what you think would be the relationship. I had made a comment yesterday and that comment is very much important. Note that F is defined as dow x by dow capital X or deformation like that and note that when we define F star and this as a philosophy, is defined in such a fashion that the reference configuration is the same for both the observers or in other words, F star is dow chi dow capital X or in other words, this is a function which is written like this.

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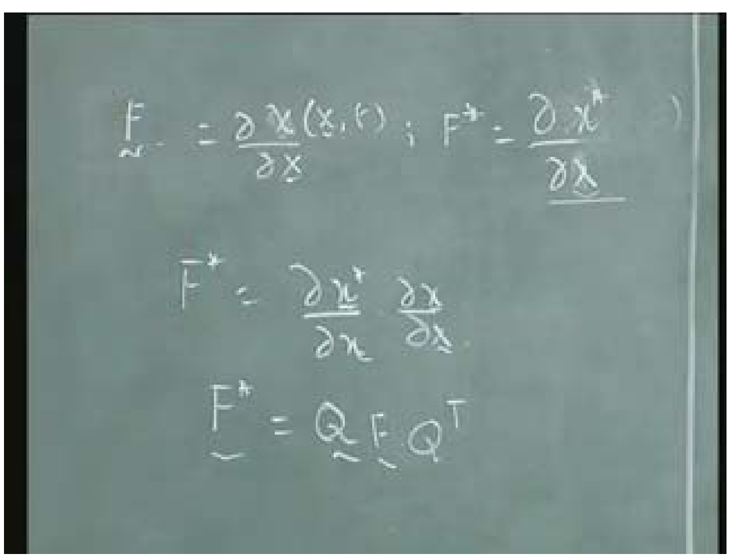

In other words, this is written as dow small x star by dow capital X and note that both of them are the same. So, how does this transform? Say, you can write say for example, F star to be dow x star by dow x into dow x by dow capital X and knowing the fact that x or I mean, of course, you know all of them are vectors and knowing the fact how x star and x vary, we had seen already x star is equal to C plus Q t of x. Knowing that F star is equal to Q F, actually this is one of the things which I want you to notice that F star does not vary as Q F Q transpose, which you would have expected, because it is a second order tensor, but varies as Q F only and this does not exist. There has been variance of this definition. For example, people who follow Ogden, Non-linear Elasticity book, you would have found that he does not except the fact that both the observers have the same reference co-ordinate.

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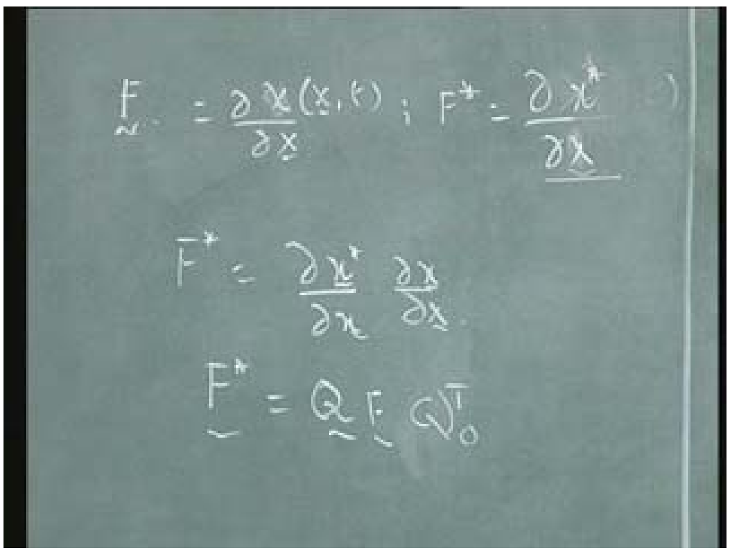

In other words, he says that it is possible for you to have two different reference co-ordinates for two different observers, in which case you will get one Q here and Q transpose here, but we call as this is Q T or Q transpose, initially or at the origin. But, in most continuum mechanics literature, you will find that the observer has the same or two observers have the same x or same reference configuration, very important to understand that and hence Q varies or F star is equal to Q F or Q is the only thing which is responsible for variation of F. This is also called as two point tensor, if you remember and so, two point tensors transform in this fashion. What is another true two point tensor, which probably you remember?

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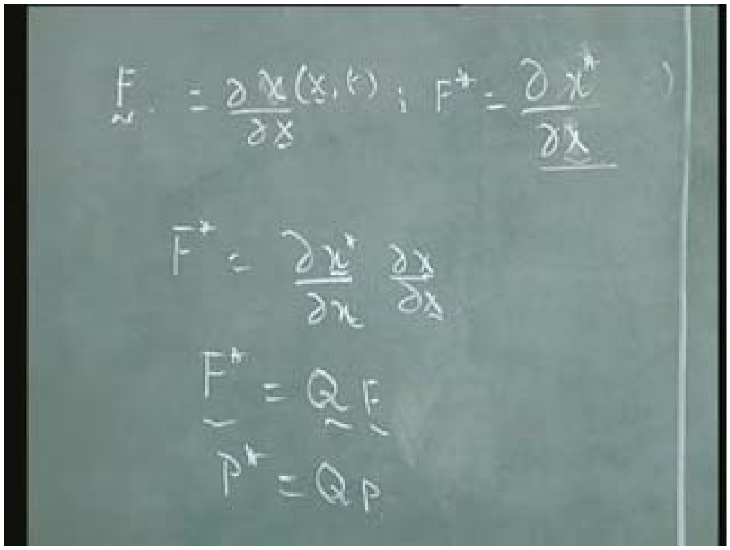

First Piola-Kirchhoff’s stress, first Piola-Kirchhoff’s stresses, so, the same way P star is equal to Q P is how it varies. Is that clear and what about determinant of F, what would happen to determinant of F? Must be same, very good; because, it is a scalar quantity and one of the things which we have said is that, the scalar quantities are not affected.

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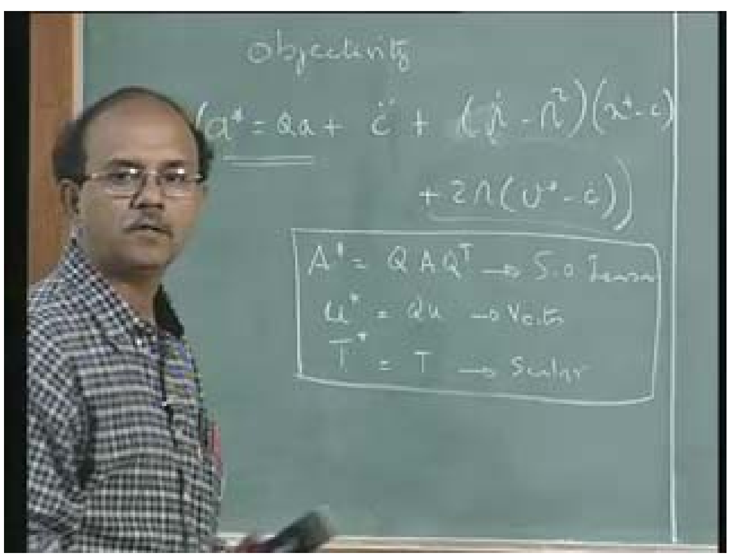

Suppose this is what we call it as temperature, so, T star is equal to T and so, the temperatures whatever or whoever takes it at a point is the same. So, T star is equal to

T. The same fashion determinant of F does not vary and so, you have the same factor. Now, the other thing is there are other important quantities. Let us see how that works out.

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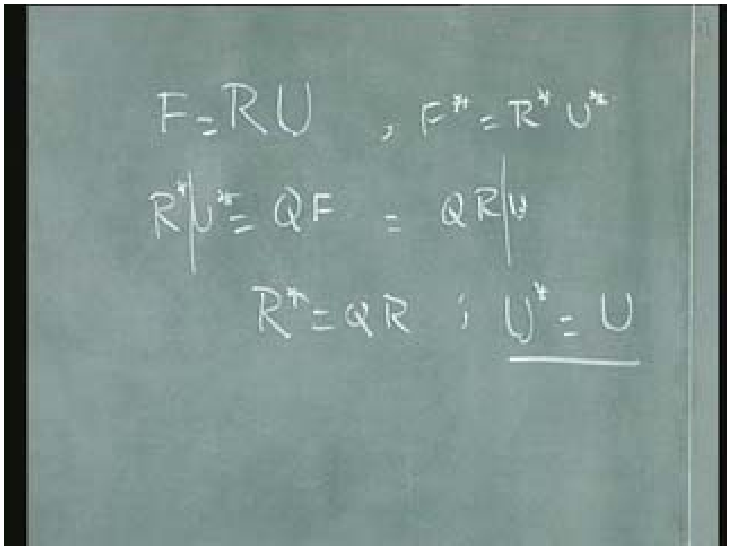

How does, say, we had already seen say for example, F is equal to R U. How does R vary and how does U vary with observer? Let us see how, can you work it out? In other words, we have F is equal to R U and F star is equal to R star U star. What is the relationship between R star and R, U star and U, so, that is the question. We have, start here, F is equal to or F star is equal to Q F and that is equal to Q R U or in other words, R star U star is equal to Q R U. Look at that carefully and can you pass some judgement on that; R star U star is equal to Q R U. Very good, so, from this what would you, where? R star, very good; so, R star is equal to Q R. It also stems from the fact that that there is, this can only be decomposed in the unique fashion. If you remember we had talked about this decomposition theorem and we said that the decomposition has to take place in unique fashion.

This also means that since the uniqueness means that there has to be a line which separates out the rotation part, the stretch part, rotation part being unique, so, R star has to be equal to Q R and that throws up a very important relationship that, what? U star is equal to U. Note that yesterday we had already commented that the quantities which are to be referred to the Lagrangian co-ordinates transform in such a fashion that they remain to be the same. So, for example, we said that the Lagrangian quantities, starred quantities, is equal to the Eulerian, sorry, the un starred quantities, not like the Eulerian transformations. This means that U star is equal to U and that is what we are going to use in E. You will see that E star is equal to E. So, what is the, yeah, yes, why it is, why is U star is equal to U? You can see right away from the uniqueness theorem that R star is equal to Q R.

You can, there has to be unique decomposition. Hence, you see that R star U star is equal to Q R U. Both of them if they have to be the same, then the rotation parts have to be the same. So, R star is equal to Q R and the stretch part has to be the same, so, U star is equal to, U star is equal to U. In other words, the Lagrangian parts remain the same for the two observers. That is what I said, does not change. This also stems from the fact that F star is equal to Q F or in other words, the reference configurations are the same for both the observers. Hence Q star is equal to U.

What happens to the other quantities? Say for example, how does sigma transform? Can you work it out, how does sigma transform?

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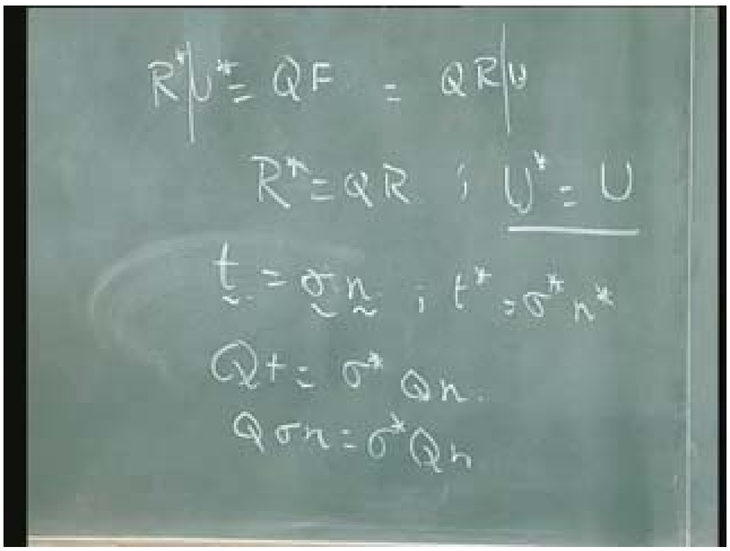

The clue is that sigma is equal to, can be written as, sorry, t is equal to sigma n; t is equal to sigma n, because sigma transpose is equal to sigma is what I take. So, t is equal to sigma n. Let us see how we work out for sigma. Note that, in other words, you can write t star is equal to sigma star n star and the fact that t and n being vectors and that they transform according to the vector laws and hence you can say that Q t is equal to sigma star Q n. In other words, Q sigma n is equal to sigma star Q n.

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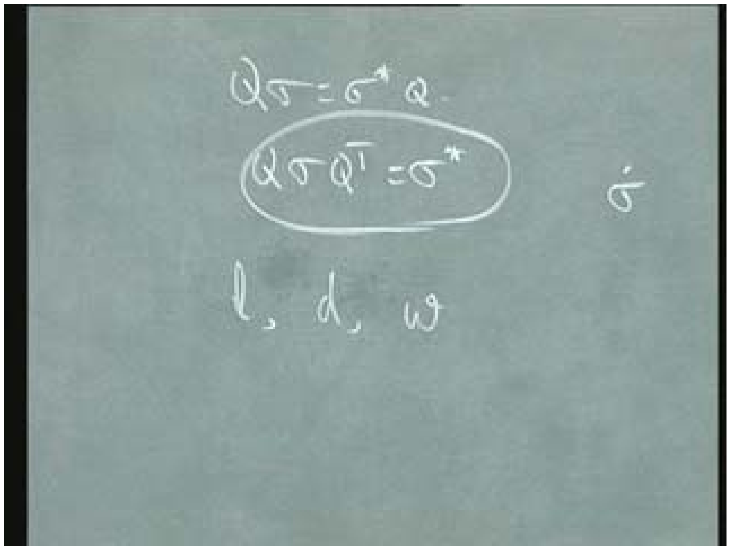

Comparing the left hand side and the right hand side, I can say that Q sigma is equal to sigma star Q. I am noting the fact that Q is an orthogonal tensor, so, Q inverse is equal to Q transpose; Q sigma Q transpose is equal to sigma star. What does this say? This says that, what does it say? That Cauchy stress is objective, correct, is objective. That is a very important thing, so, Cauchy stress, that is the reason why we use Cauchy stress in the constitutive equation without problems, because Cauchy stress is objective. Fine, so, what about two other quantities which are of interest to us?

Say, what would happen to, say for example, we had two more quantities l d and of course w. So, what happens to l d and w? Are they material frame indifferent? In other words, can I use l or what should I use in the constitutive equations and more importantly, even before we go here we had established a simple fact that the rate of stress, sigma dot is one of the quantities that is extensively used in the constitutive equations, sigma dot. That is time derivative of sigma. These are what are called as, there are what are called as rate type constitutive equations; very much useful in many circumstances, including plasticity. So, can I use sigma dot?

You remember that in plasticity we simply divided it by dt and said d sigma by dt, which is sigma dot can be used as, you know, very easily instead of d sigma as well; remember that we did that for lambda dot and so on. So, sigma dot is one of the quantities which are extensively, I will not say used, because we are not, we are going to say that it is not going to be used, is a quantity which is very useful, if it is possible to be used in the constitutive equations. So, let us see whether you can use it. Can you just check that and tell me? Check whether sigma dot, take that equation, first equation, check that and tell me whether it is possible to use it and why it is not possible to use it.

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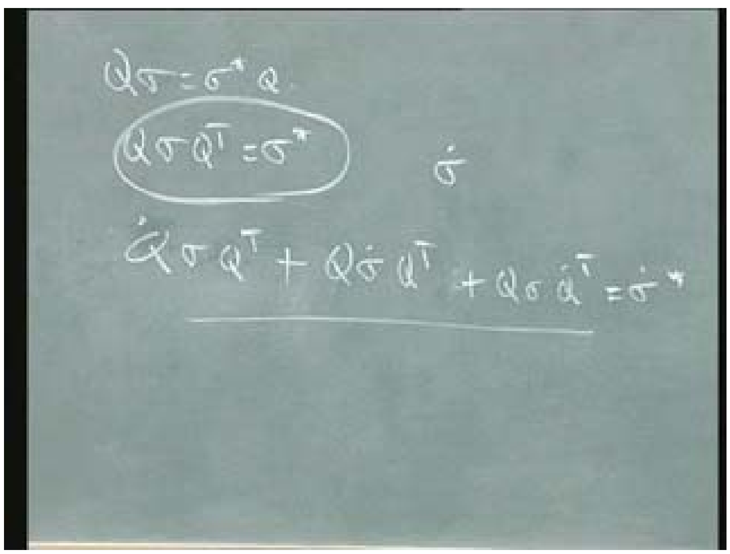

Very simple, known, it is nothing that we need to do much. See, what we have to do is only to differentiate this equation with respect to time, so that you will get Q dot sigma Q transpose plus Q sigma dot Q transpose plus Q sigma Q dot transpose is equal to sigma dot star. Obviously, this equation tells us that because of the fact that Q is a function of time, sigma dot is not an objective quantity and we have to do something else in order to make sure that sigma is an objective quantity. So, sigma is sigma dot rather, is not or cannot be used in the constitutive equations. Is that clear? Yes?

Yes, are we using the same yield criteria? Yield criterion does not come into the picture here, because we are only saying what you are going to use in the constitutive equation. Yield criteria is different; see, what we are trying to say is what is the quantities which can be related by means of an equation which we call as constitutive equation? Can I relate, say, epsilon dot and sigma dot or e dot or yeah, epsilon dot or d and sigma dot? Can I relate d and sigma dot, this sigma dot here in a constitutive equation, when rate of loading becomes important, say for example or where and many of the non-linear equations are written in a rate form? Rate form means that the derivative with respect to time goes into the constitutive equations.

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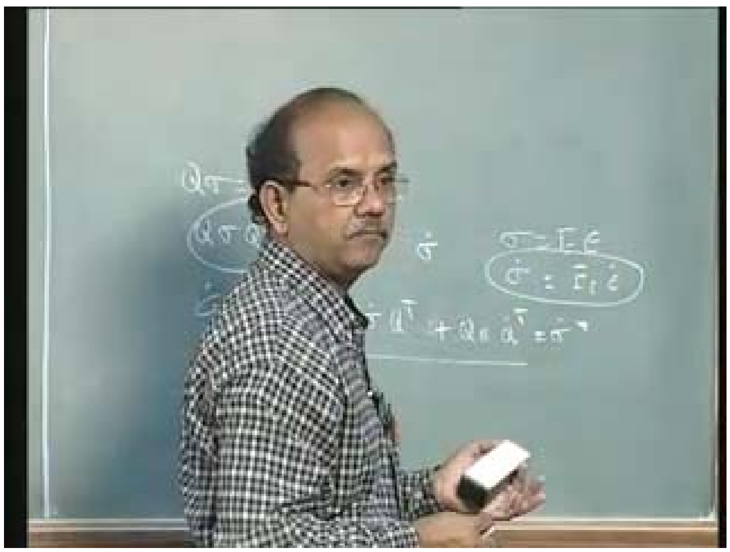

Constitutive equation is, for example, is not written as sigma is equal to E epsilon. It is not written like this, but is written sometimes, most of the times as sigma dot is equal to E t epsilon dot. This is how it is written. If you write it like this, is that equation correct?

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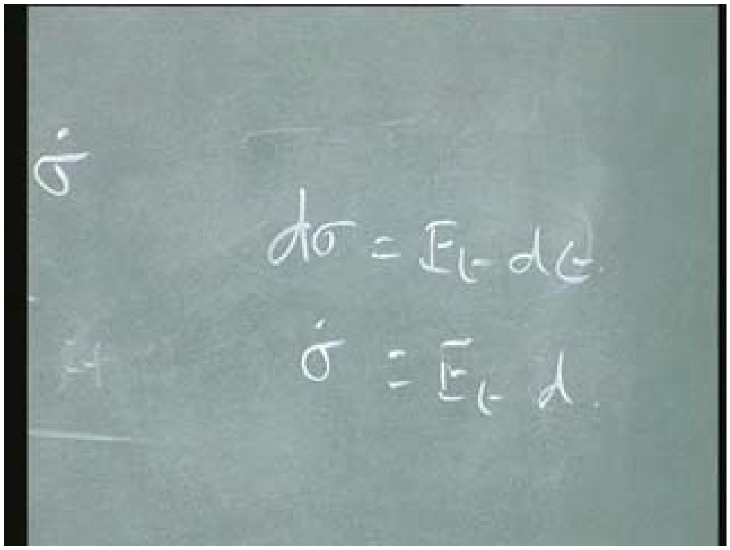

For example, you remember that we all the time wrote d sigma is equal to E t into d epsilon. Can I just convert this into a time derivative and write that as sigma dot is equal to E t epsilon dot or d dot and go ahead, or sorry, d rather and go ahead. No; this cannot be done, because sigma dot cannot be used in the constitutive equation. See, as long as we are talking about very small deformations, where the reference configuration and the current configuration are almost the same which you would have followed in what is called as small deformation analysis, all these issues do not come into picture at all. We will not be looking at objectivity and all these things. These things though, note this, though you are not absolutely correct, but does not matter, because all of them are infinitesimal deformations. But in these circumstances, we are talking about finite deformation cases and hence all these precautions have to be taken, so that the results are consistent; what you get is consistent, not only the theory, but also with experiments.

So, let us move to l or velocity gradient tensor and d and w. They are going to have some role to play, which I am going to tell you shortly. What is or how does l transform? Let us see. Can you work it out, how does l transform? You know what is l?

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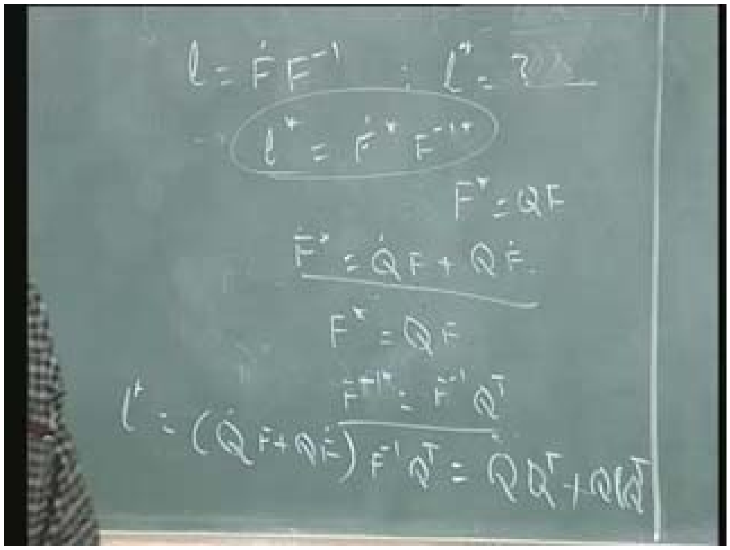

l is equal to F dot F inverse. What is the relationship between l and l star? l star is equal to what in terms of l? This is what we are looking for. In other words, l star can also be written as F dot star F inverse star. We also have the relationship that F star is equal to Q F. We also have that relationship that F star is equal to Q F, so that F dot star is equal to Q dot F plus Q F dot and note that, note this carefully that F star is equal to Q F. I can take an inverse, so that F star or F inverse star is equal to F inverse Q transpose, Q transpose by making use of the fact that A B inverse is equal to B inverse A inverse and that Q inverse is equal to Q transpose, I can, so, F inverse star is equal to this as well.

Now, substituting this on to this expression, sorry, this expression here, this as well as this equation to this expression, let us see what you get and then, of course, ultimately writing in terms of l, let us see what you get out of it? So, l star is equal to, first term is here, Q dot F plus Q F dot; the second term is here that is F inverse Q transpose, so, let us see. The first term is of course, Q dot Q transpose, yes, Q dot Q transpose plus Q l Q transpose, beautiful, so, Q l Q transpose. Note that then l star is equal to Q dot Q transpose plus Q l Q transpose which means that these velocity quantity of course, because it is a velocity quantity, they are again not frame indifferent, they are not frame indifferent.

Now, let us look at w. Let us see what happens to w and d. What is a w? In fact, we had used the symbol w for the skew symmetric part of l.

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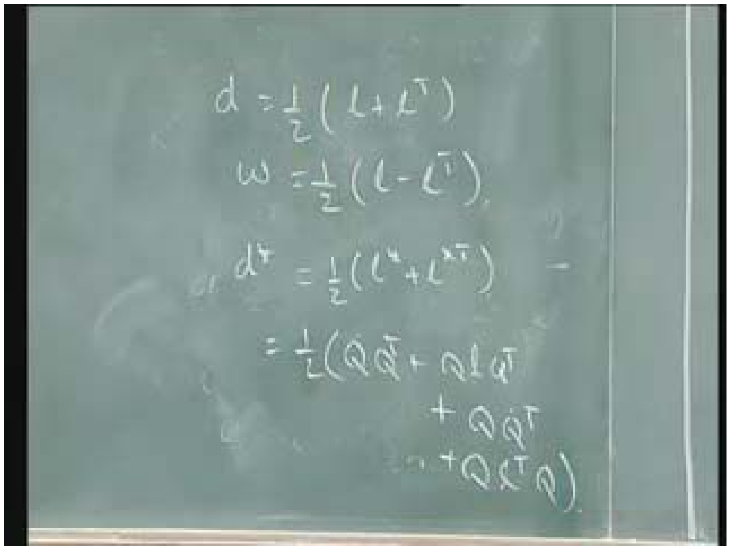

That is d we had used, if I remember right half into l plus l transpose; w is equal to half into l minus l transpose. Now, let us see what happens to d first. If you want you can do this d and then do the second one also. So, d is equal to, so what is that we want to do? I want to know what is d star? d star is equal to half of l transpose, sorry, l star plus l star transpose. Now, l star I know, how it, I mean, transforms. You can write that down for example, that is half into Q dot Q transpose plus the second term here Q l Q transpose, Q l Q transpose plus Q, so, transpose of this, so, Q Q dot transpose plus it will be the same Q l, Q l transpose Q.

Let us see what happens to this. Let us look at these two terms Q dot Q transpose plus Q Q dot transpose. What happens to these two terms? No, no; very simple, nothing very difficult, do not, just take what is Q Q transpose. See, this is a trick which you have to play in every problem, most of the problems; trick is very simple.

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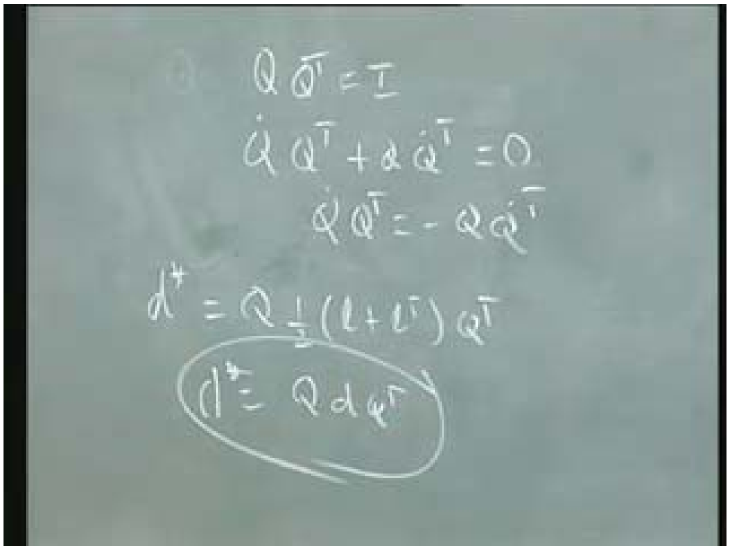

Q Q transpose is equal to I. Note this trick. Nothing great, Q dot plus Q Q dot transpose, so that Q dot Q transpose is equal to minus of Q Q dot transpose. That means that those two guys are out. So, you can take this out, these two guys are here are out, so that d star can be written as d star is equal to Q half of l plus l transpose Q transpose that is Q d Q transpose. Having done that, please do the w part of it, do w. How does it transform? I have done d for you, just do w here. What you have to do is exactly the same thing. Now, just substitute for l l star and let us see what your result is.

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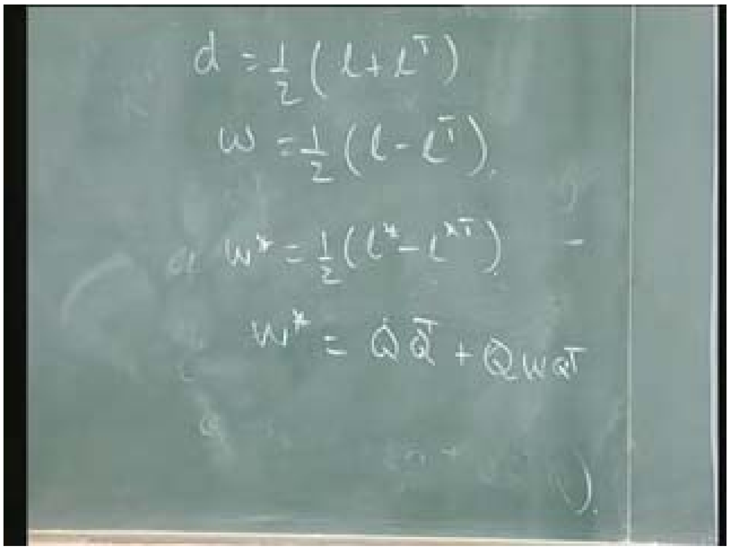

In other words, w star, now keep substituting it, look at that and substitute. Yeah, two minutes, let us see. So, what do you get? w star is equal to Q, very good, Q dot Q transpose plus Q w Q transpose. You see that w again is not a material frame indifferent quantity. Is that clear? We have almost seen most of the things. You can see that e is equal to e star, you can just prove that. We know what the types of quantities are that we have to use, but we are stuck at one place where we showed sigma dot is not a material frame indifferent quantity. We are stuck because, we want some rate quantity which can be used in the constitutive equations; we do not seem to have it. Can someone suggest what can be done? I will give you a clue. One is that we can now have combinations. It is not that, it is not that every independent quantity has to be looked at whether it is objective or not. It is also possible to have a combination of quantities and you would be surprised to see that a combination of quantities would be what I would call as objective. Say, let us take a vector say, u. So, u dot minus w u.

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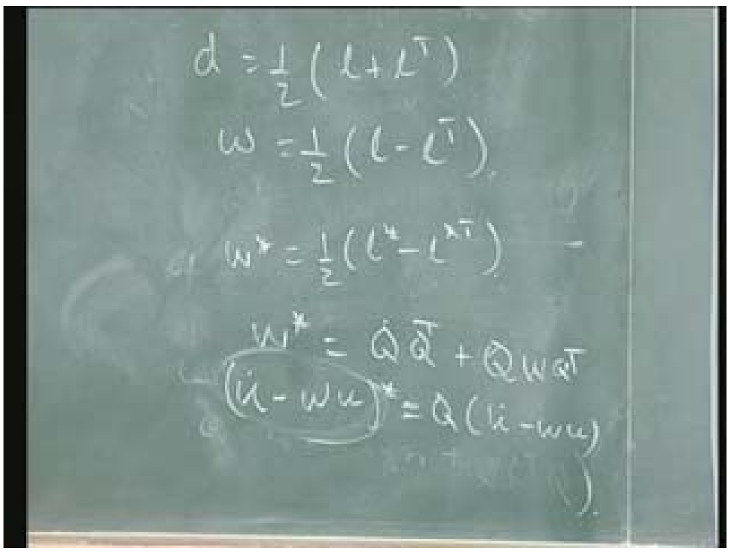

Suppose you take a quantity like u dot minus w u, take a quantity like that; you will see that this quantity becomes objective. So, it is not very difficult to show that or in other words, what we mean to say is u dot of minus w u star is equal to Q into u dot minus w u. Let us, let us check whether that is true. In a similar fashion, I mean, you can check that, but in a similar fashion you can also have a combination of sigma's to show that it is possible to get a combination of sigma to be or sigma including sigma dot to be material frame indifferent quantity. So, these combinations are now used as objective quantities. They are given a very special name as well. Check that, you check that, but meanwhile let me finish this part. Let us take now, as I told you, a combination of certain quantities.

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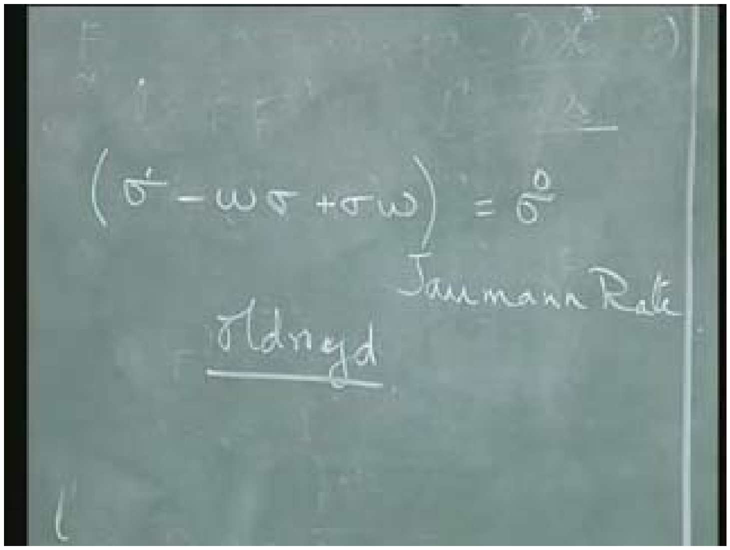

Let us take for example sigma dot minus w sigma plus sigma w; let us take a quantity like this. It is not very difficult to check whether this is a frame indifferent quantity or not, not very difficult to check that. So, this quantity, this quantity is used in the constitutive equations as a rate quantity instead of sigma dot and is usually denoted by sigma with a circle on top. Yeah, that is what I said. You check that. I am going to allow you to check that, it is very simple to check. What you have to do is to only substitute all those for w, what it is. But, before I complete it, let me, before I go to that, let me introduce this.

This is called as the Jaumann rate, this is called Jaumann rate. Jaumann rate is extensively used in large deformation plasticity. You can see that being mentioned in for example packages like Abacus, Abacus manuals you will see Jaumann rate. What does Jaumann rate mean? Jaumann rate simply means that since sigma dot is not objective, a basket of sigma's and dots are taken; this whole thing is taken together and that is used as the rate quantity, so that the quantity becomes objective. There are a, in this sense, there are a number of objective quantities, rate objective quantities. These quantities are used as one of the or one of the members of the constitutive equation. You can check this, but even before we do that, to give a physical picture of these rate quantities, let us look at what is called as Oldroyd rates.

Oldroyd rate is a lie derivative, lie derivative of sigma. What is lie derivative, we had already seen this. Pull back, do a d by dt, then push forward. That is called Oldroyd rate. In order to interpret it, it is very simple to interpret Oldroyd rate. What does it mean? We had already seen this. What is lie derivatives interpretation? That means that the, yeah, remember what it is, which means that we are now moving along with the velocity, all the twists and turns of the velocity field and looking at how the sigma changes with time. We are in motion along the velocity field and we are then looking at the change in sigma. Say for example, Oldroyd rate, why I mentioned this is because it is very simple to understand that way.

In other words, these rates take into account such motions due to Q dot, such motions they take into account or physically what it means is that you are also moving with that fields like that, you know, the Q dot fields and then you are looking at the derivative or in other words, the sigma dot which was not a, why was that not an objective quantity, because there was a Q dot there, Q dot there. So, that Q dot is removed in this fashion; by grouping them together that Q dot quantity is removed. Physically also you can look at it in a slightly different fashion.

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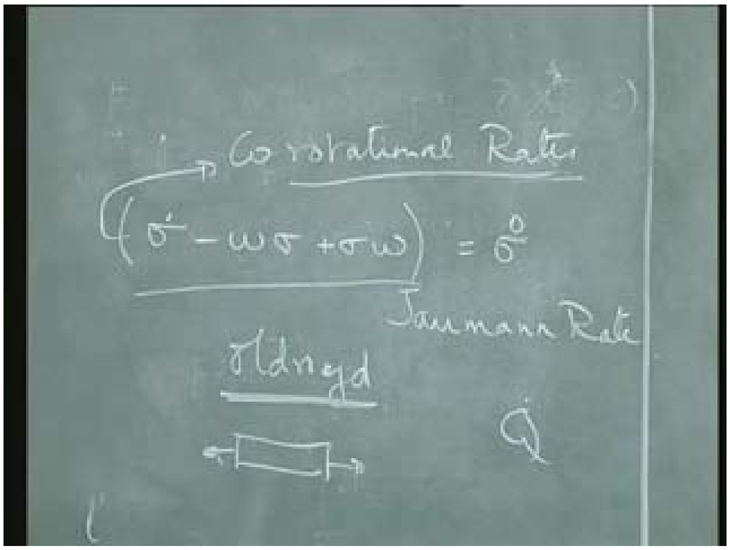

Suppose I have a bar here and the bar starts to rotate. Instead of you rotating, the bar starts to rotate or the observer rotates. You can also imagine that the bar starts to rotate along with the stress field. Let this rotation be defined by Q of t. What my sigma dot relationship says is that the sigma dot in this case is a function of Q of t, the rotation which is a function of time, the rate of rotation or Q dot of t rather. So, that is the reason why sigma dot, or in other words, what it means is, suppose I start rotating it at different, you know, Q t varies and Q dot, Q dot takes different values, then my sigma dot will take different values. That is what it means physically. Suppose you have, you are in a situation where you sit with the field or with this and then start rotating in the same fashion and start measuring it and compensate for that Q dot, then the ill effects of Q dot are out, are out and the measurements that you would do are exactly the same. Is that clear? That is what is achieved by this kind of what we call as the objective rates. Sometime we call these rates as co-rotational rates, because that is that rotation effects that is co-rotation; you are with that rotation, co-rotational rates, co-rotational rates, because these rates that is this rate is called a co- rotational rate, because these rates remove the ill effects of Q dot.

What does it mean, simply, I mean pre-empting things, you may look at later.

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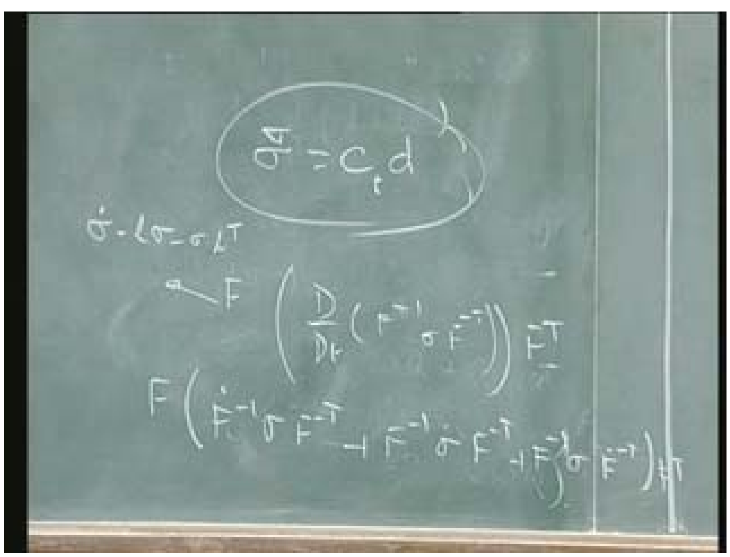

It simply means that when I put down a constitutive equation, I use for example, I can put the constitutive equation of this form say, C_t d. Instead of sigma dot entering constitutive equation, you will have this co-rotational rates that enter into constitutive equation. Jaumann rate is not the only co-rotational rates. Oldroyd rate, see is another rate. There are number of rates like this, depends upon what you use. Ultimately what do you mean by co-rotational rates? There are two rates, co-rotational and convicted rates; I will not go into details of it. I just want to pick only the principle, philosophy that it is possible to compensate for this Q dot and get co-rotational rate, which can ultimately be used in the constitutive equations. Is that clear, any question, yes?

Oldroyd rate, see, people call, people write this as old sigma. Oldroyd rate is usually written as old sigma, sigma old. That is how, that is how you use it. Yeah, can you, why not you write down, yeah, what is the expression? Why not you write down, that is one of the good things that you asked. How do you write down Oldroyd rate? Push back d by dt, then pull forward. So, F D by Dt of F inverse, sorry, F inverse sigma F inverse transpose F transpose, so, this is the Oldroyd rate. You expand that, F into F inverse transpose dot sigma F inverse, sorry, F inverse sigma F inverse transpose plus F inverse sigma dot F inverse transpose plus F inverse sigma F inverse transpose dot F transpose.

You will ultimately see that Oldroyd rate is written as sigma dot; that is written as sigma dot minus l sigma minus sigma l transpose. This is the pull back derivative push forward. So, this sigma is pulled back. You go and have a look at our old notes, pull back derivative; then these two are the push forward operations. Note that, stress if you remember, we had two operations for two different quantities. For contra variant quantities and co-variant quantities, we said that the pull back operations are different and push forward operations are different. Operations are different in the sense that the equations are different and sigma being contra variant quantity, the pull back and push forward is different from what you would do in the case of say for example, strains. So, the meaning becomes clear here at this part.

Oldroyd rate is another rate that is used in the constitutive equation. Of course, when you use different rates, then this C quantity is different. They are not the same; you have to be careful with that as well. Of course, these are not the only rates. As I told you there is what is called Truesdell rate, which is again very popular. Truesdell rate operates on the second Piola-Kirchhoff stress, it is also again used; but, the concepts of all these rates are the same. Having completed objectivity, we will now move to the actual constitutive equations. Next 3 to 4 classes I will look at constitutive equations that are used basically for non-linear elasticity. May be next, we will complete it in the next 3 classes and then we will come back and see the finite element formulations for large deformation problems, how we can extend the small strain problems to large deformation problems. Is that clear, any question? So, there are no questions, we will stop here and we will continue in the next class.