Advanced Finite Element Analysis

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Lecture - 22

Let us now start with our constitutive equations, our next step in continuum mechanics.

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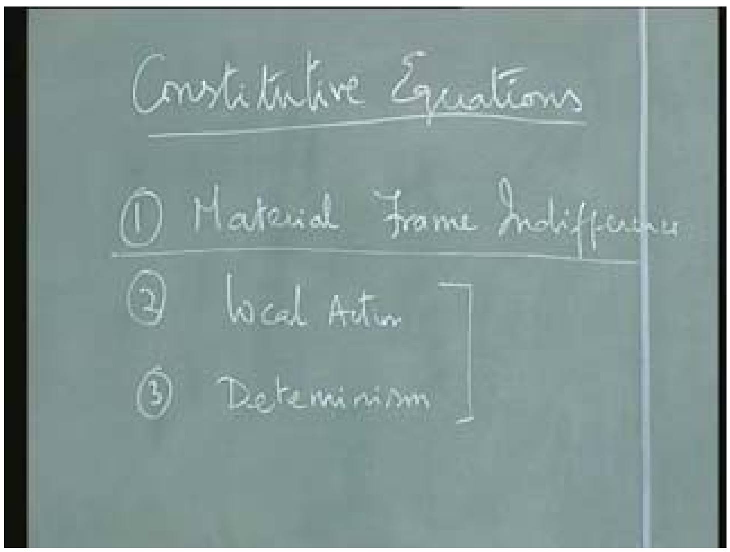

Before we move back to the finite element analysis, let us see what the things are that we are going to look at in continuum, sorry, constitutive equations. As I had already told you that the constitutive equations are ones which gives us the relationship between state variables. Basically we are interested in certain stress measures and strain measures and this will be very useful to us, of course, to further apply finite element analysis to much more complicated problems. So, let us look at certain principles that are necessary for constitutive equations. This will throw up some more theories on stresses and strains. In other words, what we mean to say is that, you cannot just take a stress measure or stress rate measure or strain or strain rate measures and just use it in constitutive equations. Constitutive equations independently have to satisfy certain conditions. Whatever you do with the constitutive equations, they should of course satisfy all our thermodynamic laws as well as the other balance laws we had put in. Your equations should not in any way hamper the other things, the thermodynamic laws.

Apart from that we have two important principles. One of the most important principles is what is called as material frame indifference and the second one is what is called as principle of local action and if you want, you can add one more called principle of determinism. These two are sort of more technical. You just state what it is, but we will concentrate on material frame indifference much more closely. So for example, what this principle of determinism says is that the stress on a body, the current state of stress in a body depends upon the history of motion or deformation of a point.

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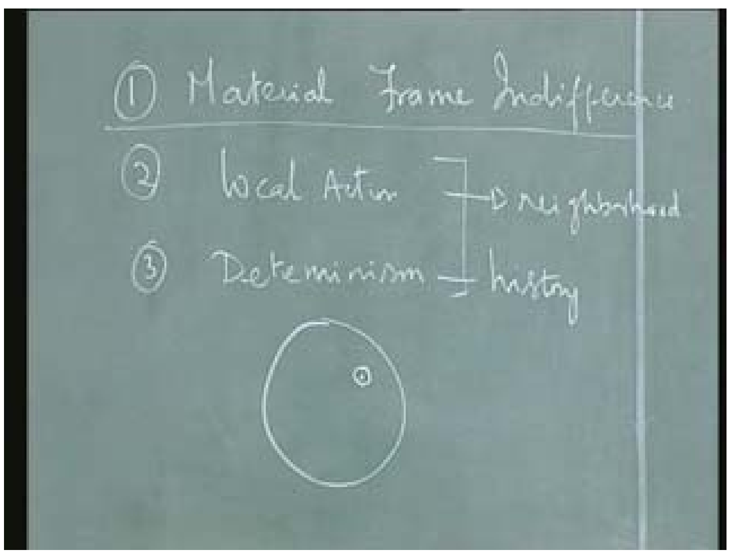

In other words we may put it like this, say, if I am interested to determine what would be the stress at a point in a body, at a point in a body rather, then that stress is determined, of course uniquely, by the history of deformation or motion of this point and that if at all there is interaction of other points on this point, it is restricted to a very, very small neighbourhood or infinitesimal neighbourhood of this point. In other words, determinism talks about the history of deformation or history of motion and the local action talks about the effects of a very, very small neighbourhood, infinitesimal neighbourhood around that point. These are more technical, in the sense that these are things which you can understand. But, more important thing that we have to understand, have to apply very rigorously, is what is called as the material frame indifference or objectivity. What essentially it states, I had said that before, is that when you study any physical quantity, this quantity should not be affected or rather I would put it like this, decisions based on these quantities should not be affected by the observer.

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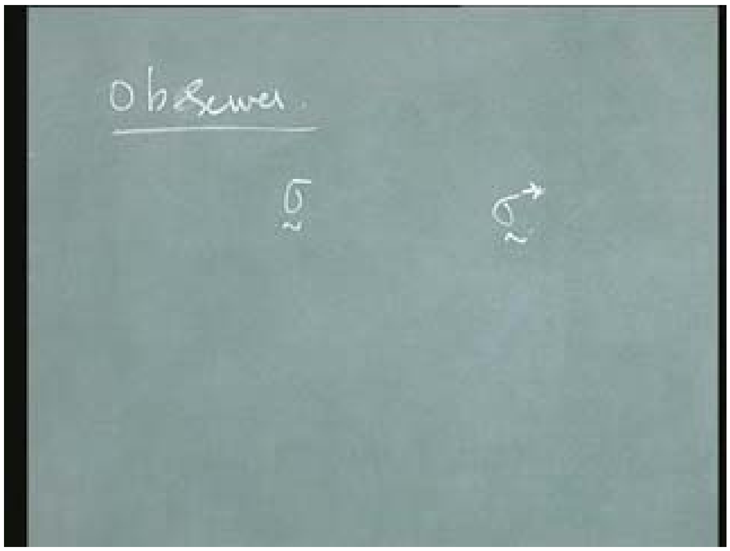

So, the concept of observer is central to the whole of continuum mechanics, so, it should be independent of the observer. We are going to state later, may be towards the end of the class that it is not that the vector quantities which are measured by two observers are exactly the same. That is why I put in a clause there, but they have to transform according to certain rules. Now, let us first look at the concept of two observers and then develop this theory of material frame indifference. Now, it is possible that any event, in this case an event, we had already defined that, is an ordered pair of x and t, that is in other words, it is vector x and the time t can be observed by more than one observer.

Let us say that you and I watch an experiment. It is not necessary that both of us are at the same place, same place with same co-ordinate system. You may have, you may be at a different position that is number one and you may have a different co-ordinate system that is number two and number three is you may not even be stationary. You can move as you observe or you can rotate or whatever it is or in other words, the relationship between you and me may even be time dependent. If this is the case, how does a quantity behave? How a quantity does behave under our observations, become important.

To put it in a very simple fashion, suppose I am observing stress and you are also observing stress; let me say that you are a starred observer, say star is what I put for you. Both of us are observing stress in a body, same body, undergoing the same type of deformation; not same type, same deformation rather, both of us are observing this. You may be just in a trolley and moving out or you may even rotate. But the point is that ultimately when I say that there is going to be yielding, you also have to agree with me that there is going to be yielding. When I say that there is going to be failure or fracture, you will agree with me that there is going to be fracture. It is not that, it is not that when you observe stress, you can sit in a chair, a revolving chair and then just revolve or rotate your chair and say that the stresses are now increasing or in other words, whatever you do should not have any effect on the body, what you are observing on the body.

This is in essence the concept of material frame indifference or frame indifference. The quantities that you are observing are indifferent to the frame that you choose in order to observe.

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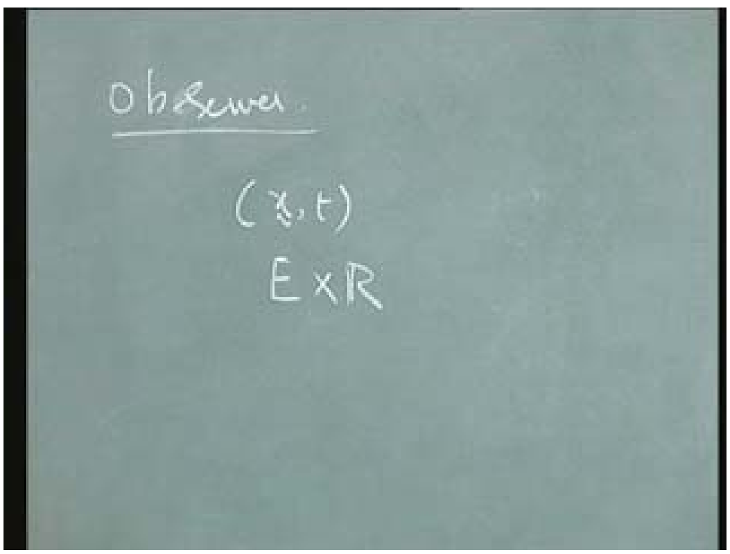

An event, say for example, the unstarred observer can be say, let us write this down as say, x comma t. This is what we called as event, x chosen from the Euclidian space and t chosen from real space and usually it is written as E cross R, but we need not worry about that.

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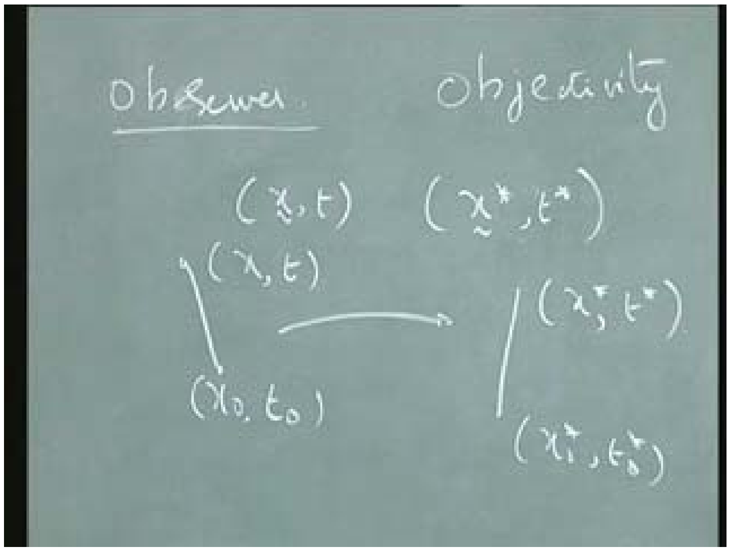

Let us say that, as another observer, let me call you as the starred observer and let the starred observer observe this, the same thing as x star comma t star. Let me clarify, let me point it out again that it is not necessary that x should be equal to x star. It is not that x is equal to x star; t also need not be equal to t star. We are not talking about that; not the same, but there has to be an allowance for we two being different. What is that allowance, how do we get this allowance? That is the question or in other words, if I tell you what is x, closing your eyes, can you say what should be your x star? Can you predict it?

Can I develop a relationship between them and can this relationship stay and will that relationship ensure that there is objectivity and in fact, material frame indifference is also called as a principle of objectivity; it is objective, it is not subjective. What you observe is objective. So, it is also called as a principle of objectivity. Now, immediately two things come to our mind, when I say that I have to be objective. One is that all the observations which are of interest to us are ones which involve a change in length. If you had followed all the things that we had done, one of the key things here is that we are interested in change in length. So, the change in length that you observe or I observe should be the same. In other words, the transformations that I have to apply when I want to predict what you observe should be length preserving transformations, should be length preserving transformations.

Say for example, I observe two events. Say let me call these events as x_0 comma t_0 and let me say that this is say, x comma t and you observe that same thing and you record these events happening at two different times and with two different x's as say x_0 star comma say, t_0 star and x star comma t star. Suppose you observe in this fashion and the, sorry, you observe like this and I observe like this. What do we mean by length preserving transformation?

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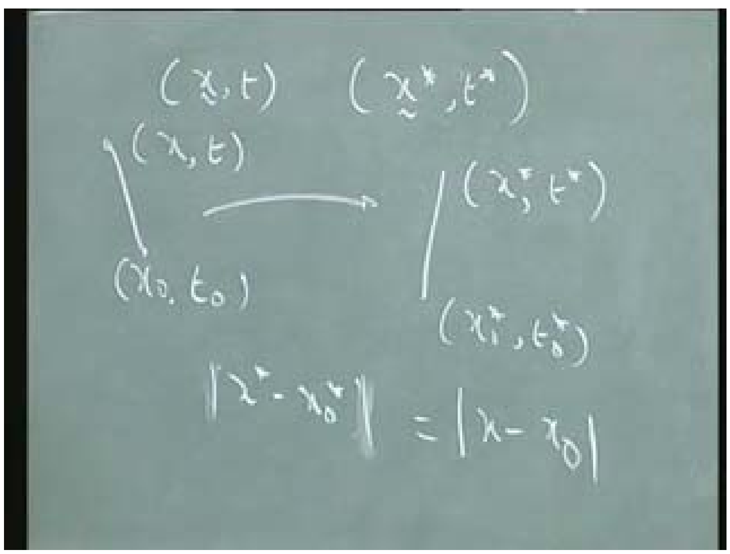

The lengths are given by say, x star minus x_0 star or rather the norm of it should be the same as x minus x_0 or in other words, that is the first condition. I will come to time in a minute.

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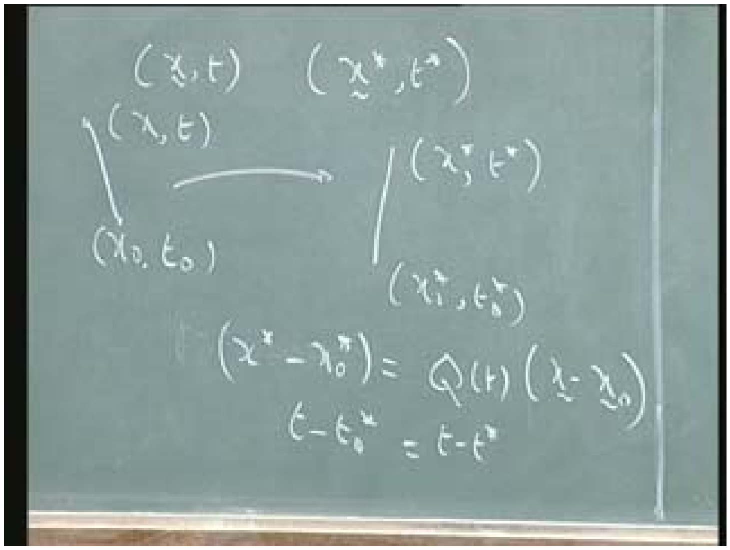

In other words, x star minus x_0 star should be equal to Q t where Q is, remember that we had already done this, length preserving transformations or transformations which are operated by an orthogonal tensor into x minus x_0. If the transformations are like this, then obviously, what is it? Obviously, length is preserved and that that is number one. Number two, if I look at the difference in times which is say, t minus t_0 star, the difference in times should be the same or in other words, this is equal to t minus t star, so that t minus t_0 is equal to t star minus t_0 star. So, the time differences are preserved. You can write it any way you want, you can bring this side and then bring it to the other side. So, the two things one is that time differences are preserved and the length distance, lengths are preserved. If these two are preserved, then whatever conclusions you and I come to ultimately will be the same. Yes?

Yeah, that is a good question. Why am I taking two different times, t_0 and t? What we are saying is that they are two different events. They are two different events that are taking place at two different times; they are two different events that are taking place at two different times. It is not, it is not restricted to a body that is say, being deformed, even then deformed with respect to time is a factor, is a factor.

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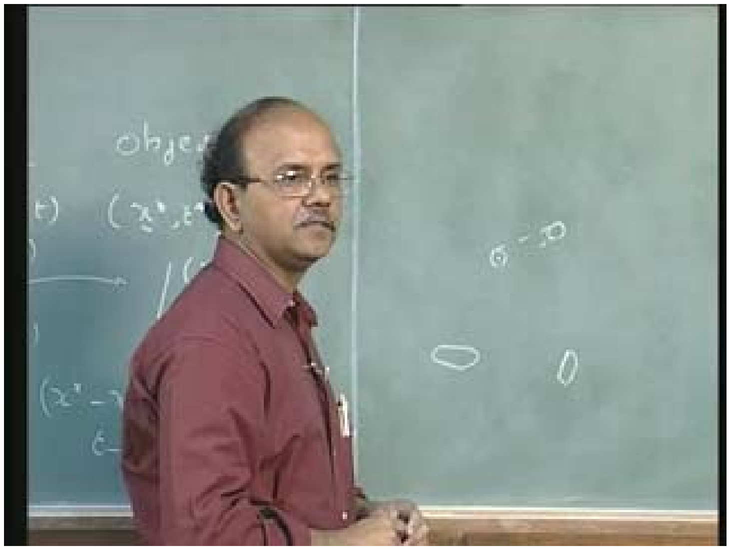

I mean to put it more you know, more lucid fashion, what is being done in some of the continuum mechanics books is that suppose you are observing some say, you are in a in a ship and your friend who is in another ship and both of you are observing say, a target. So, you say for example, they are warships and some other, you know, an enemy ship is here. Now you are looking at the position of this from here and you radio the guy that, right now I am seeing this guy at a particular position. He also say, starts looking at it. This starts moving, so, after some time say, this is the position. The movement of the ship, here the enemy ship, is independent of your observation. So, ultimately whether you observe it or your friend observes it, the distance moved by this target ship should be the same for both of you and the time that is required for this motion should, time as it moves from here to here should, also be the same. There has to be a relationship between your co-ordinates and my co-ordinates that is all. That is what I said.

So, having understood this that same thing, yeah, either way that is why I said you can bring that this side and then write.

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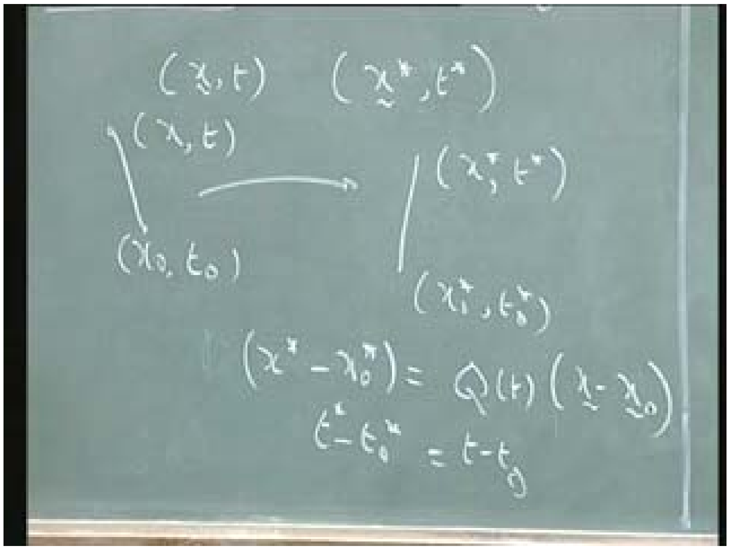

If I bring this this side, I will write this as t star and you can write that as t minus t_0; either way you can, t minus t naught star is equal to t minus t naught. Oh, I had put, if I had not put t_0, yeah, this is t naught, obviously. So, either way you can write. That is the example which I gave you as a very naïve example, but it is nice to understand it; but more importantly here we are talking about deformations following say, points and over a period of times. So, the time should be the same, the distances. These are more mathematical representation of an event consisting of one from the Euclidian space and one from the real space R. You can rewrite this in fact and let us see what happens.

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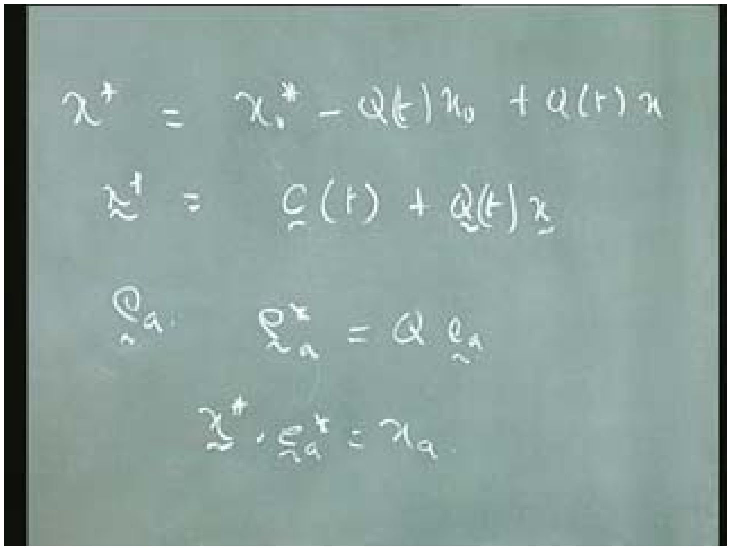

If I rewrite this part, I can write that as say, x star is equal to, look at that, x star is equal to, this I am going to take it to the other side, is equal to x_0 star minus Q t x_0 plus Q t x. Q t, what is Q t, in the sense that Q, the most important thing here is that, this Q is a function of time. The most important thing is that this Q is a function of time, so that this ultimately can be written as say x star is equal to C t, this whole thing can be written as C t, x_0 is a constant there plus Q t into x. In other words, the length preserving transformation can also be represented in this fashion.

What does it mean? It means that if you have an x star and if I have x, what you have measured and what I have measured, if these two are related by this kind of equation what we have observed is very objective. That is what it means. Look at this, what is this? This can be looked at something like a translation term and this can be looked at something like a rotation term. Now, let us go further and say that let us say that you have a co-ordinate system. Obviously, all of us observe these events using a co-ordinate system. Let us say that I have a co-ordinate system. Let us say that it is written by e_a. That is e_1 e_2 and e_3 and you have a co-ordinate system written as say e_a star and obviously, your co-ordinate system and my co-ordinate system are such that they are related by this Q e_a. I am just removing t, so you can just write it as Q, Q e_a. Let us see what this does for our actual co-ordinates that we measure. Now, we know that very simple to find out that x star, of course all of them are vectors here, tensor, x star dot e_a star is equal to the co-ordinate e x a. In other words, this is a very straight forward thing, which we have done again and again in the last course. What we do is that we take this dot product with e a star on this.

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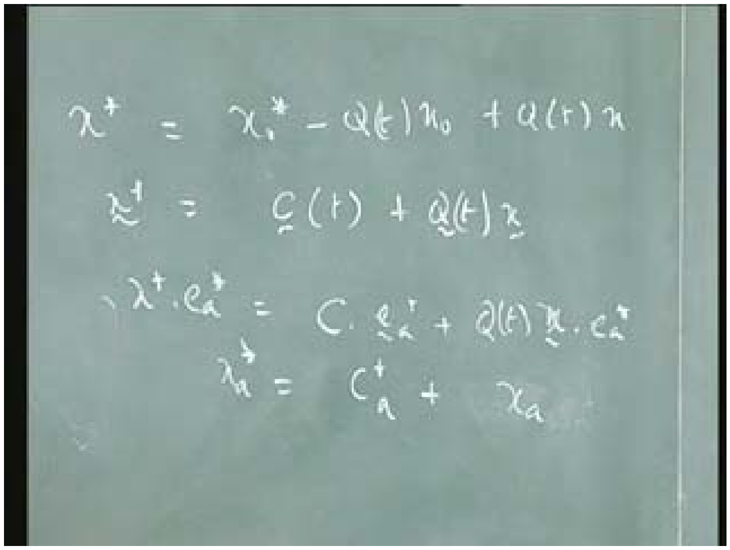

Let us call this as x star dot e a star is equal to C dot e_a star plus Q t x dot e_a star. In other words, this is x_a, the coordinate, of course, a takes the value 1 2 and 3. If I use e_1 star, which is the basis, e_1 basis, of course, there are three basises e_1 e_2 and e_3, so, a takes the values from 1 2 and 3. If I put 1 here, what I get is the co-ordinate x_1. That is what we mean. So, what does it give? This shows that, ultimately if I write that as x_a star is equal to C dot e_a star is the, when I write C in starred co-ordinate and this can be written as C star or C_a star plus how do you write this?

You can write that as x dot Q transpose e_a star and Q transpose e_a star is what? From our previous expression, e_a star and e_a, this becomes e_a, so x dot e_a and that can be replaced by x_a. In other words, if we are observing an event which is supposed to be objective, then the co-ordinates what you observe and what I observe are only removed by, are only removed by, a constant C here. They are removed by constant C_a star. Is that clear? Now, there is a, there is a subtle difference between this kind of approach and rigid body rotation. Can someone tell what would be the difference? Is it that just a rigid body rotation? Is it that, because that is also a length preserving transformation? Both of them are very similar; there is no doubt about it. But, technically they are slightly different in the sense that the co-ordinates in this case do not change, since they change only by this; but, whereas there the co-ordinates themselves change, because co-ordinates are given by, position vector by, Q t, so, the co-ordinates change. Your observer is the same and observers, either you can look at it as observer is the same and the body is rotating. So, the co-ordinates of the body are changing as the body rotates. This is, yeah, more technical both of them, but are length preserving transformations. In fact, you can check this as length preserving transformation.

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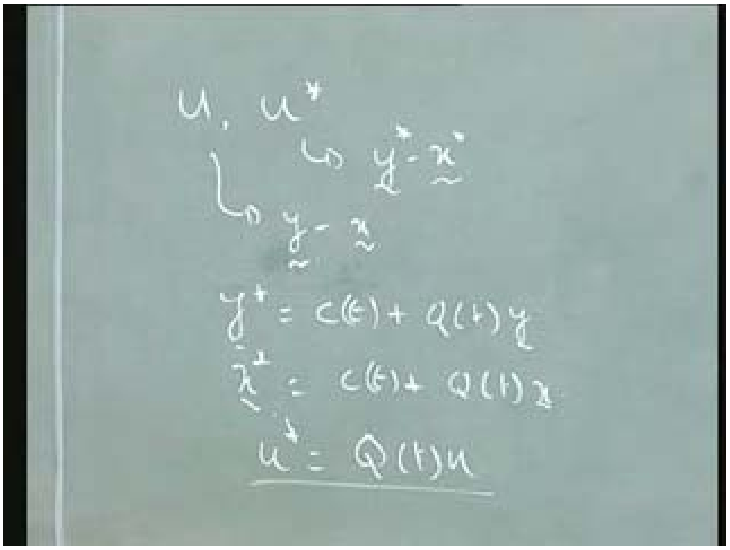

For example, you can take say, a vector u and u star. Please do that. u star is written say, y minus y star minus x star and u can be written as y minus x. This is a vector, y minus x and say, y star minus x star. You can check that how y star, y star of course, happens to change like this. y star is equal to C t plus Q t y and x star changes like this - plus Q t x and from these two, you can see that y minus or y star minus x star is equal to Q t into y minus x or in other words, what it means is that, any vector u star can be written as Q t u and if a vector transforms, I am not saying that every vector would transform like that; if a vector in fact transforms according to this transformation rule, then the vector is objective; vector would result in length preserving transformations.

So, point number 1, if I want a vector to be objective, it does not change or I would say, I should not say it does not change, but its meaning does not change whether you observe it or I observe it. Then, that vector has to change according to this fashion or in other words, any vector which changes according to this equation is termed as objective, is termed as objective. In order to make things clear, let us look at a vector like velocity, vector like velocity. So, let us see what happens when I choose velocity. It is very simple, just work it out and you will get very interesting results. Please look at that, how you can define velocity.

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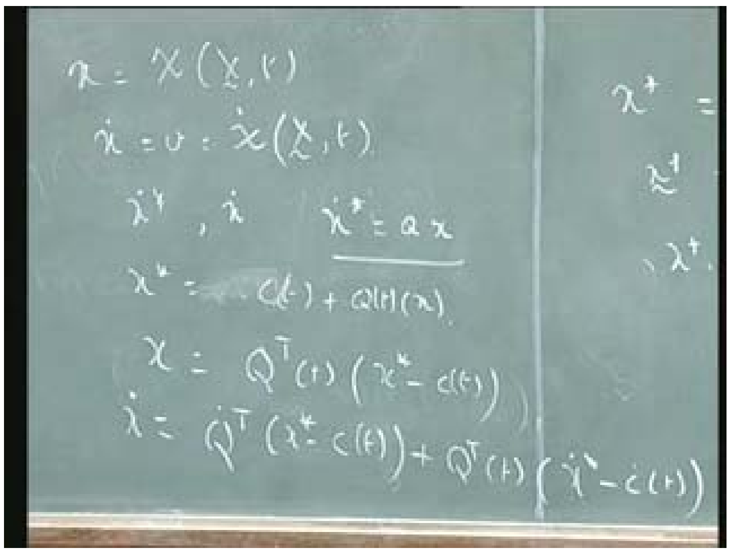

Say for example, x, you know, how it is defined. x is equal to deformation, so x dot which is v is chi dot x comma t. Let us see how this transforms. Now I have x dot star and x dot. This is what I want to find out, what is the relationship between the two? If the relationship between the two are said to be objective, then they should transform according to this rule, Q of course function of t; should transform according to this rule. If the transformation rule is not like this, then that quantity is not objective. So, any decision taken based on a quantity which is not objective depends upon your actions, depends upon your actions. That is what I said that when I talk about stress, the stress has to be or the stress measure which I am looking at should be objective. If it is not, if it is not, then that quantity will be affected by your motions or your actions. So, let us see what what happens to velocity.

Simple; x star is equal to Q t into sorry, C t plus Q t into x. C t is a translation; you can view it as a translation term and Q t is a rotation term. So, from this you can also write down x in another fashion as x is equal to Q transpose into x star minus C t. Now, find out what is the relationship between x dot and x dot star. So, x dot is equal to x dot is equal to Q dot transpose, Q dot, look at that, because Q is a function of time. This is one of the major factors which distinguishes it from just a small rotation that you give to the body or to the co-ordinate system into x minus C t plus Q transpose, sorry, yeah, Q transpose t into x dot star minus C dot t.

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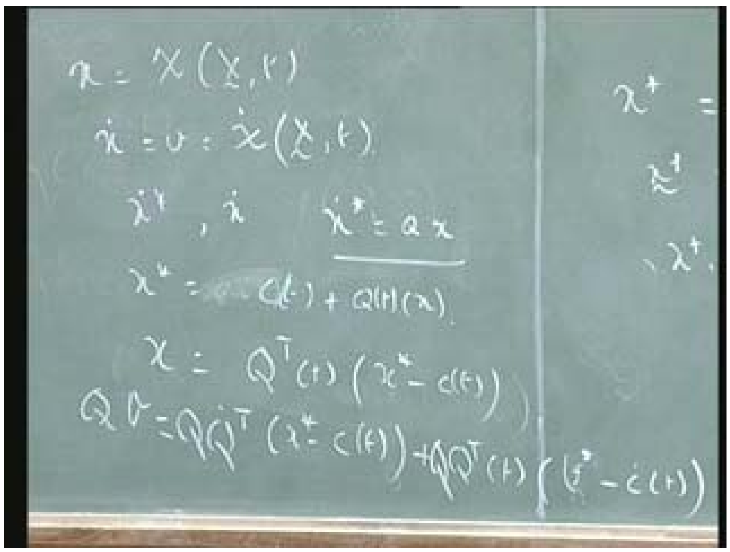

So, just replace that x dot by v, so that v is equal to and v star. Just multiply throughout by Q, multiply by, so that Q v, so this becomes Q; that becomes Q. We know that Q Q transpose is equal to I, so that ultimately you can express v star, look at the way I am going to express v star.

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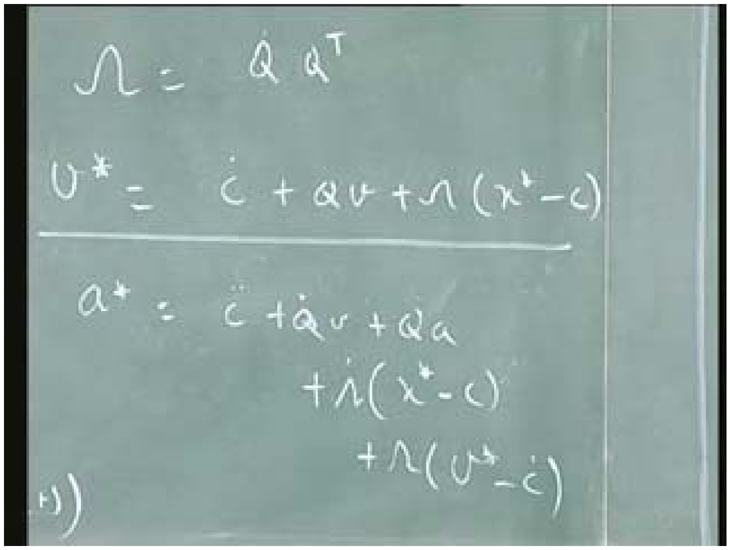

You can do that, it is the next step. So, v star and and let me define one more quantity, let me define omega. Define I am saying, please note that I am defining omega to be Q dot Q transpose, Q dot Q transpose and of course omega is equal to minus omega transpose. Using the principle of or Q transpose is equal to Q I, sorry, Q inverse, so, I can write down ultimately v star to be, very simple, just look at this equation; you can write that as v star is equal to C dot plus Q v plus omega into x star minus C. Now, look at this expression. Now, I have two terms here, C dot term and this term, which are additional terms; Q dot term and C dot term which are additional terms which defines the relationship between v star and v. In other words, my relationship should have been just v star is equal to Q v; my relationship should have been v star is equal to Q v. But, the relationship happens to be now v star is equal to C dot plus Q v plus omega into x star minus C.

In other words, velocity is not an objective quantity; velocity is not an objective quantity. It depends upon your motion, the way you are going to rotate, obviously. Now, you know it is, everything you know dawns on you - the way you are going to move C dot, the way you are going to rotate, will change the velocity that you observe on a particle. We know this. So, velocity is not an objective quantity. Look at acceleration, just let us see. Can you please derive from here? Differentiate it once more and please write down what would be the relationship between a star and a. Differentiate this with respect to time and write down what is the relationship between a star and a. It is quite simple. This when I differentiate, it becomes a star is equal to C double dot plus Q dot v plus Q a plus omega dot x star minus C plus omega v star minus C dot. Right? I think I am right; I have not missed out any terms. Now, I am not going to spend time in simplifying it, please simplify that.

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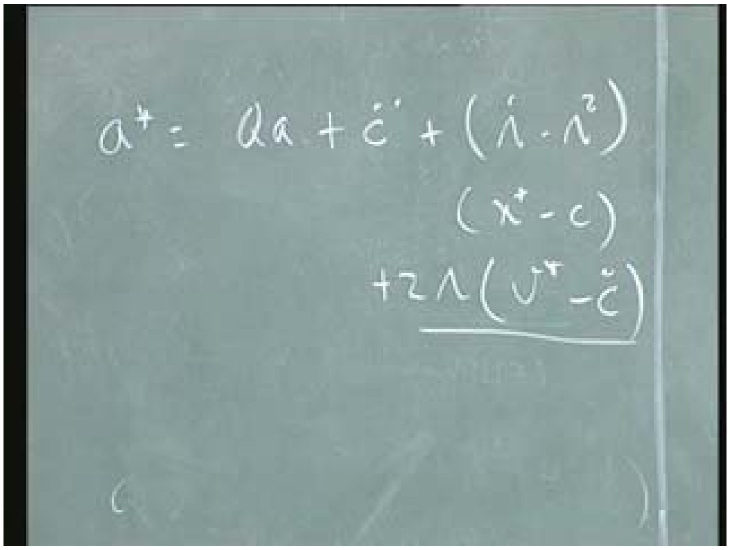

I will write down just the final result with whatever we had defined; omega, please substitute that, I mean very straight forward substitution, so that I can write a star, a star, is equal to Q a, the first term, that is this term here plus C double dot plus omega dot minus omega squared. Omega squared is omega omega, see, you can find out easily, substitute what it is, into x star minus C plus into v star minus C dot. Some of you have done multi body dynamics may recognise this or if you are going to do it later, you will recognise what these are. The last term here is what is called as the Coriolis component. So, last term there is what is called as the Coriolis component.

The omega squared term is the centrifugal acceleration and the omega dot term there is what is called as Euler acceleration. So, they are different accelerations that exists, because of this kind of relative motion. So, again acceleration is not an objective quantity. What does it mean? It simply means that if I use velocities and accelerations as a part of my constitutive equation, then I am in trouble, I am in trouble. I cannot, I cannot definitely say that, in a very naïve sense that look if the velocity of a particle reaches a particular value, then there is going to be failure. If I make a statement like that, what does it mean?

You can start moving, you know, sit in a revolving chair or may be in a trolley; just start moving, get whatever velocity you want and look at that specimen and then it should break. It is exactly what it means by saying that your quantities should be objective. So, I cannot use velocity, I cannot use acceleration. From that sense, you will immediately recognise that say, Newton's second law of motion is not frame indifferent. Your motions are going to have an effect on the second law. So, second law is not objective or not frame indifferent. Does not mean that we have to, we need to throw it out, but we have to keep this in mind very carefully. In fact, many of the laws put certain restrictions, certain restrictions on the way you look at the constitutive equations. You have to understand that very carefully, but most of the laws that we encounter in constitutive equations are the ones where you would see that they are frame indifferent.

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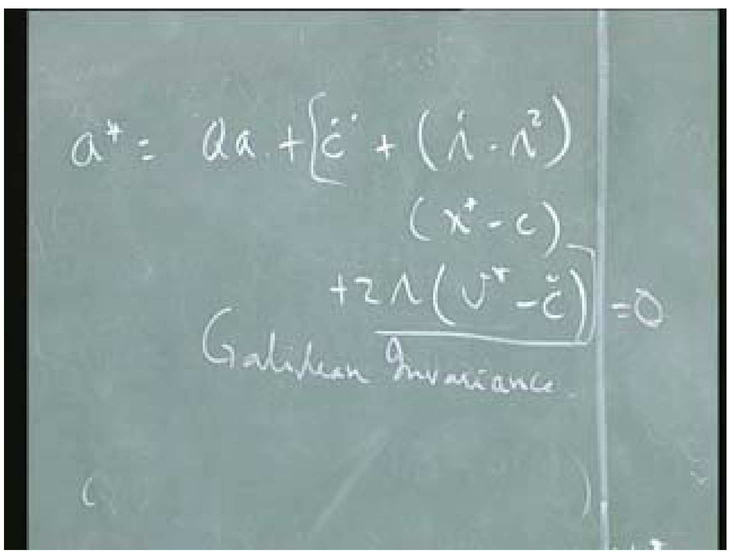

Suppose this quantity from here to here, if this quantity happens to be zero, then a star is equal to Q a and the acceleration happens to be objective, then we call this transformation or this invariance to be Galilean invariance. We call that as Galilean invariance. So, that is fine, this is for a vector, but we are interested in tensors. How does say, a second order tensor transform? Let us call a second order tensor to be say, A. How does it transform?

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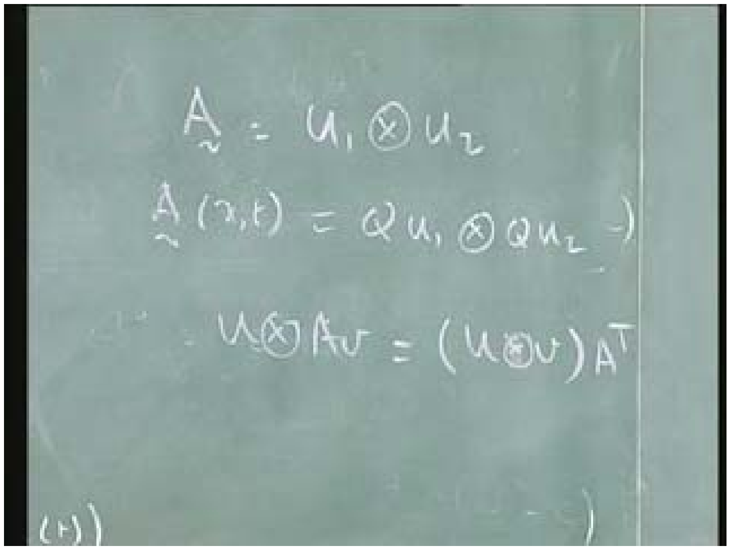

So, second order tensor can be written as dyadic of two vectors, say for example, u_1 dyadic u_2. Of course, the second order tensor can be or should be written as x comma t, x comma t. Now, how does this transform? This means that this is equal to Q u_1 dyadic Q u_2 and you can verify this very easily. We have u dyadic A v, can be written as u dyadic v A transpose. This you can, no, dyadic, sorry. You can verify it using indicial notation.

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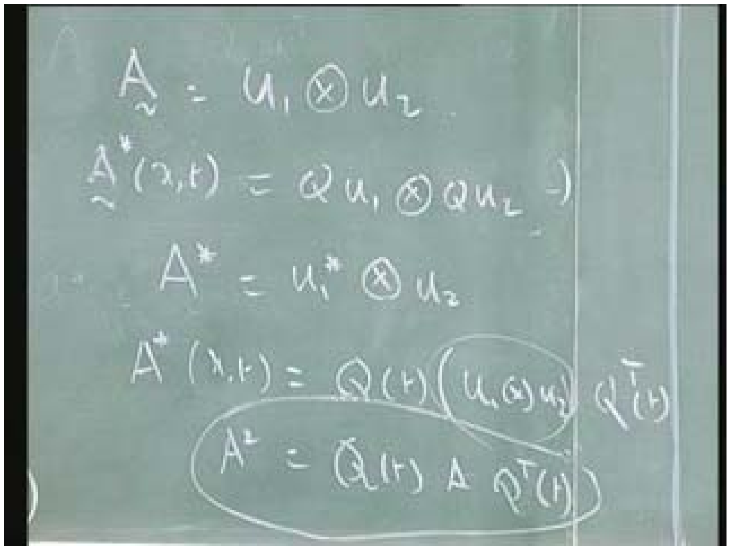

Hence the previous equation, let us see how it transforms for starred quantity, sorry A star I should put here. A star is equal to Q u_1 dyadic Q u_2, where A star of course is equal to u_1 star dyadic u_2, so that A star x comma t can be written as Q t u_1 dyadic u_2 Q transpose, so that A star is equal to Q t A, because that is A Q transpose. In other words, in other words, we call, when we call as the vector to be objective, then the transformation rule should be like that and if I call a second order tensor to be objective, then my transformation rule should be like this and what happens if it is a scalar?

Suppose we are observing temperatures, what would happen to it? Both of us are observing temperatures, common sense.

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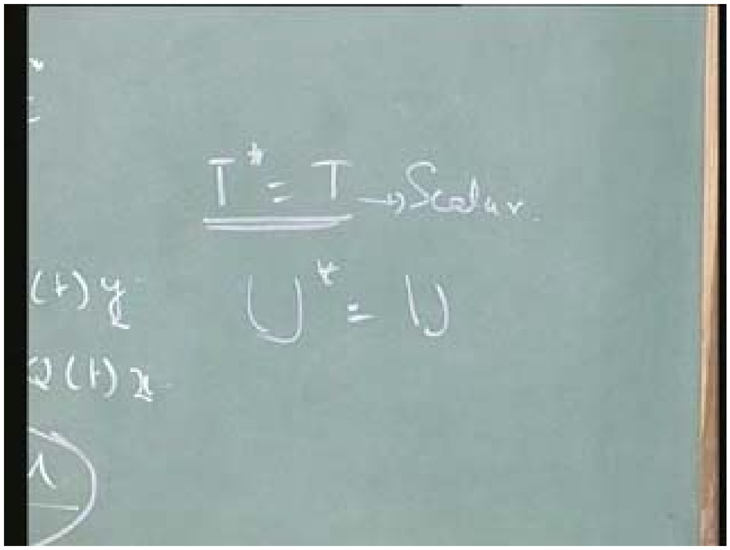

Same, so, if there are scalars like say temperatures, then both T star should be equal to T; both of them should be the same. That is for a scalar. You can extend that, but for time being this is enough for us, scalar and that is for a vector and lastly this is the one which talks about the second order tensor to be objective. You have already seen two quantities velocity and accelerations which are not objective. We will see that many more quantities, which you have come across, are either going to be objective or are not going to be objective.

Now, one of the interesting things that you have to notice is that, I had all the time put some x, small x comma t, which means that what we are considering are spatial quantities. In fact people like Ogden, you know, ….. an excellent book on Non-linear elasticity, he calls these quantities to be Eulerian objective, Eulerian objective, because these quantities are observed in the current co-ordinate system and hence they are called as Eulerian objective. But, if I have to transform this to the original co-ordinates or reference configuration or if you are looking at Lagrangian quantities, then Lagrangian quantities are, we are going to see that the references, a reference configuration is going to be the same for both the observers. That is the assumption we make that the reference configuration is the same for both the observers.

So, if you look at say for example a quantity like U, which results from our decomposition, F is equal to R U, then since it is referred to as Lagrangian and U star should be equal to U, because both of them belong to the reference configuration, so all those quantities, Lagrangian quantities which refer to the reference configuration should be the same or U star should be equal to U. Is that clear? We will stop here and we will continue with certain other important aspects of objectivity in the next class.