Advanced Finite Element Analysis
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Lecture - 19

Yeah, before we go further, I think I should clarify one of the doubt which was asked in the last class, at the end of the class, though it does not concern what we are doing now, just to do with plasticity.

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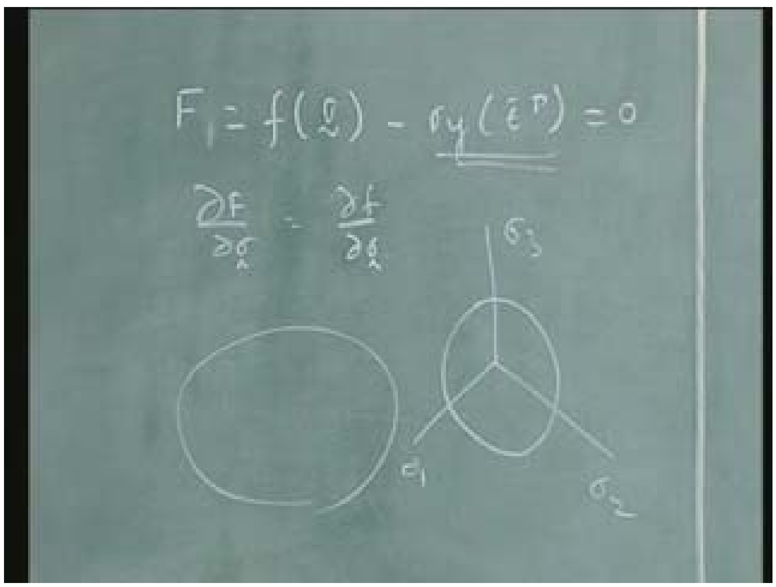

What is the difference between capital F and small f? I think there is a lot of confusion on this. If you notice that, we had said F is equal to small f the function of sigma minus, if you had noticed we had put, $sigma_y$ epsilon bar p and that is what we had put as zero. Note that this capital F is what we had put as zero and not small f. I would like that to be noted. Dow F, if I write dow F by dow sigma, then that is the same as the small dow f by dow sigma, because this guy here does not have any sigma. So, both of them will be the same, but note that small f is not equal to zero. It is the capital F, which is actually the yield surface that is equal to zero.

What is this yield surface? People still are bit confused. Please note that it is just a geometric representation, just a geometric representation of this equation that is all. You cannot obviously imagine what is there in six dimensional space, but if you want to imagine this, you can imagine very well in the three dimensional space or two dimensional projection of it. For example, in a $sigma_1$ $sigma_2$ $sigma_3$ axis, we can imagine that it is a cylinder and perpendicular to the axis of the cylinder, this would appear as a circle like this. Of course, it is not necessary for you to imagine this. It is only, it is of notional value or just to make it easier for you to understand we have been talking about the yield surface to be like this and so on. It may be very complicated and it is not necessary, but of course, this can help you a bit to understand what it is. So, it is just that this mathematical representation is converted into a geometric representation and that is what we call as the yield surface. So, we will move on now.

We have seen few of the stress measures. Before we go to some of the balance loss, which we will do may be in the next class, we have to look at a very important topic, usually confusing to many students, though it is a very fundamental aspect of calculus and a part of which we have already seen, which is called as linearization in, I can call it as a mechanics or solid mechanics.

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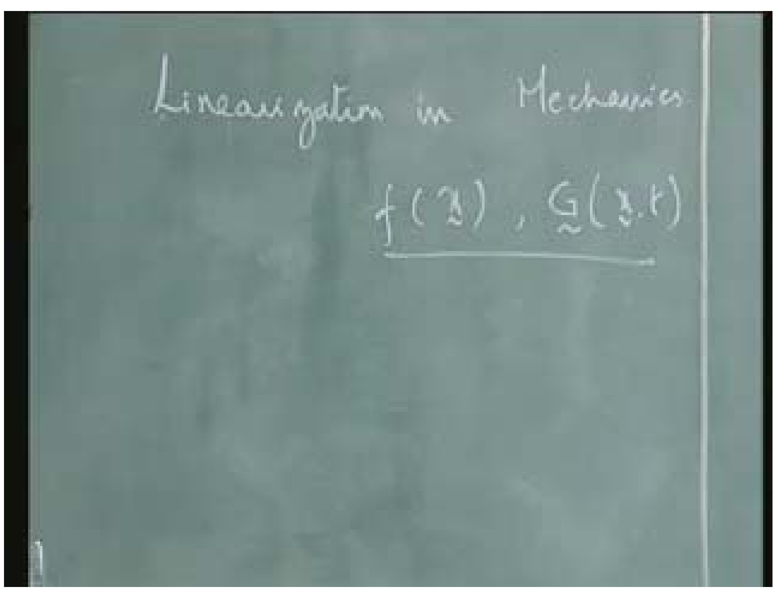

If you bluntly ask me, linearization is nothing but a Taylor series approximation, but the way linearization is written is slightly more technical than what is written or what we can look at as a Taylor series. In other words, there are a few more things which we have to learn and one of the reasons why I am going to talk about linearization in mechanics and the ensuing say, symbols and other things that are used here is basically because, most of the papers that appear today, if you want to look at research papers and some of the non-linear finite element books, talk about linearization and have a very particular way of writing it. So, that is the reason why I am going to talk about linearization in mechanics.

As the name indicates, what it means is that any quantity whatever, whether it is scalar of a vector value, function of a vector that is in other words, the function which is a scalar which depends upon vector x or a vector or whatever is the corresponding mapping that you get, it can be in any dimensional space, we call this as a, as this belonging to a particular function space, whatever be the function space. Then, this can be, say for example, function of x and t and so on, this can be linearized in the sense that you can determine the value of these functions at a neighbourhood, a small neighbourhood, around a point by Taylor series approximation.

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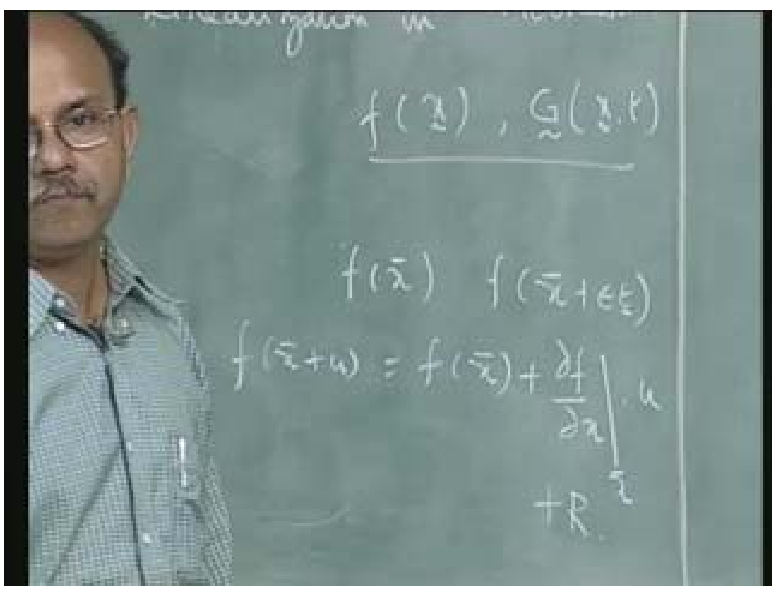

For example, given this function at say, x bar, I can get the function at a small neighbourhood of x bar, for example, at a point where epsilon tends to zero or along a line t. Now, before we go further, let us write down our first statement on linearization which comes from Taylor series approximation, so that you can write say for example, f x bar plus say, u is written as f of x bar plus, look at this symbol what I am going to write; of course you can write that as, let me write that as, first say, similar way as we write it, x bar u plus the residues or higher order terms. Linearization means simply that we are going to restrict our attention to this first term; we are not going to further go ahead with this term.

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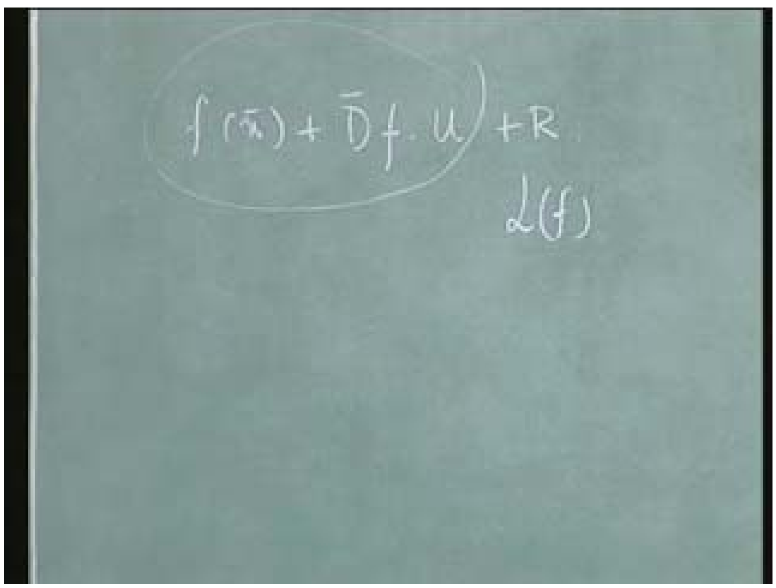

This is also written as, this term is also written as f of x bar plus D f dot u plus the higher order terms and this is called as the, look at the symbol L of f which means that linearized part of f. Look at this operator. This is nothing but, dow f by dow x dot u. This D which is dow f by dow x at x bar which is written as D bar, can be looked at as an operator which operates on f; it can be looked at as an operator which operates on f. In other words, this can be, this is, if you have x to be a vector, then it can be written as grad of f and so on.

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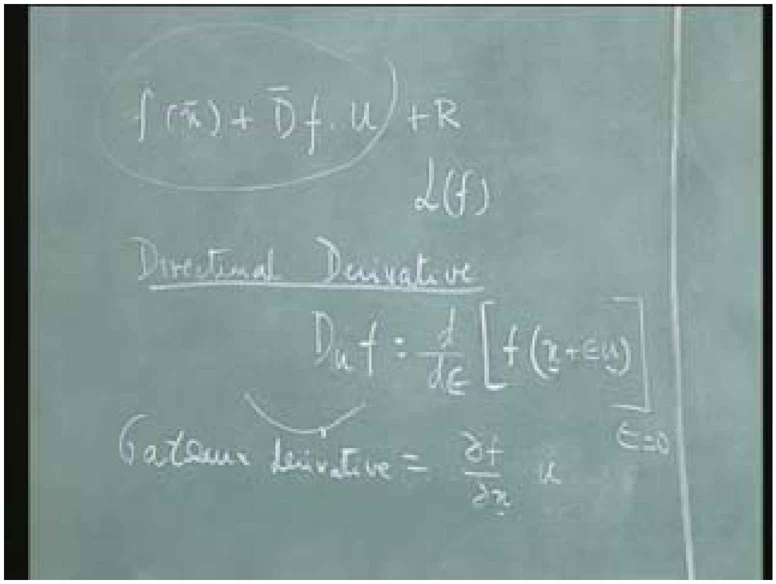

Let us define what is called as the directional derivative of a function f along say, u; same u we will put, u of f to be d of d epsilon into function, that function for which we want the directional derivative, x plus epsilon u at epsilon equal to zero. Of course this can be, we will see the connection in a minute between the two, but look at this and see what it really means. What does it mean? It means that, suppose I have a function. This function is defined say with respect to x at a point, at many several points and that I want to find out how actually a function varies. See, this is just 2D; this can be say, 3D. This function can be in terms of x_1 x_2 x_3 and so on. So, it can be something like, whatever it is, whatever shape it takes. What we are trying to do here is that to find out how this function varies along a certain direction at a point x. This direction along which this function varies, it is in 2D, it does not matter; but if it is n dimension, then it matters. Then, the direction along which it varies is given by u.

Look at this term very closely. What we mean by this? We have put an epsilon u as epsilon tends to zero, which means that we are calculating this change of this function, of this function, at a point along a particular direction. That is why it is called as directional derivative. Yeah, it is a fundamental calculus, but nevertheless it is good to know this more clearly, what is meant by directional derivative. Now, if you look at this term, then I can just chain rule this, so, I can write that as dow f by dow x dot d epsilon into du by or sorry u by, I mean, d epsilon of epsilon u which happens to be u; that is the, just chain ruling this. So, you will see that dow f by dow x dot u which is the same as that of what we had seen as the increase in delta f in our previous expressions on Taylor series. d by d epsilon there is no t or theta, d by d epsilon of epsilon u which is u at, we are calculating that at epsilon is equal to zero. This is not t; there is no t here. This is epsilon. Is that clear?

The most famous of them is what is called as a directional derivative and we are interested in directional derivative, which basically is one which gives us the linearization procedure and is useful to us in many instances. We will see where these things are useful to us, but before that let me give another name to this. This is also called as Gateaux derivative G a t e a u x, Gateaux derivative. Gateaux derivative is the word which most mathematicians use it, especially if you are studying variational formulations or variational calculus, then, for example, Gateaux derivative is the one which people use. We will see why this is important in variational calculus later, may be after we give some amount of introduction to you. So, Gateaux derivative, directional derivative and the increase along the direction of u, it all means the same thing. We are going to introduce some more things to it, you know, some more concepts to it called Lie derivative. We will do that in a minute, but before that let us see how we calculate the directional derivative of say, quantities which are referred to the current configuration.

How do you calculate directional derivative for quantities which are referred to current configuration? The usual procedure is say, what is a typical quantity which is referred to a current configuration? Say for example e, Almansi strain.

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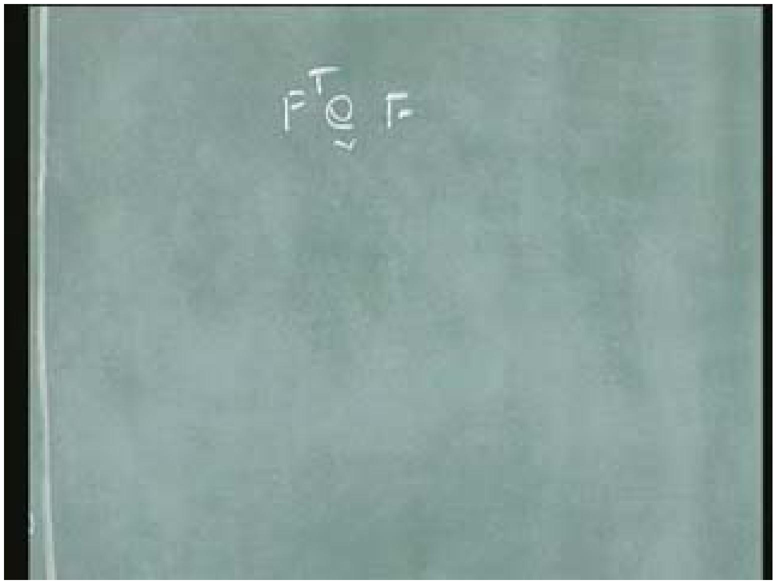

If I want to calculate the directional derivative of a quantity like e, the procedure is to do a pull back operation on e, calculate the derivative and then follow this up by a push forward operation; a pull back operation, calculate the derivative followed by a push forward operation. So, what is the pull back operation for this? So, that F transpose e F, just check that up, so that is the pull back operation. Yes, we will see that in a minute why we do not do that directly, why? What is the importance of the directional derivative like this? This is how it is defined. We are defining the directional derivative of quantities which are referred to the current configuration.

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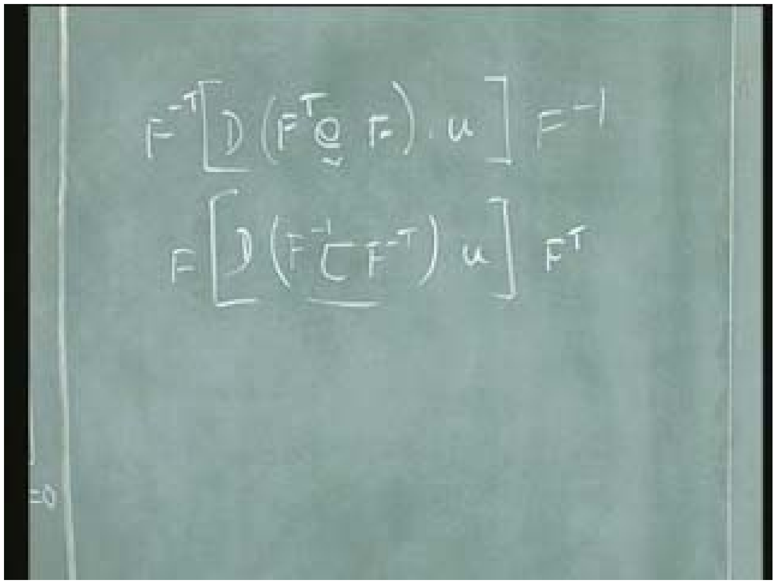

Look at the way I am going to write that. D of this dot u is the quantity which we write, directional derivative after we do a pull back operation and then we follow this up by push forward. What is that that I do for this quantity, for push forward? It is just F. What is it? F inverse, correct and then what is it? This is the push forward operation, push forward operation. Please refer to it and tell me what this operation is? F inverse, correct, F inverse transpose, this quantity, F inverse; so, this is the directional derivative of a tensor like strain.

Now, if you look at, please write this down. What would be the directional derivative of a quantity like Kirchhoff stress, which is referred to a current coordinate, which is defined as J into sigma, what is the directional derivative? Same, you do the same thing; you do a pull back operation and then do the directional derivative, calculate it and then do the push forward. What is the pull back operation for this? F inverse tau, then F inverse transpose D u and then what is the push forward operation? Just the opposite, F F transpose; so, that is the, that is how you calculate the directional derivative of these two quantities which are referred to the current configuration.

F transpose that is; that is called F inverse transpose. Yeah, that is what we had written here. This is F. There is no J there. What is the tau? F inverse tau F inverse transpose, yeah; F inverse is written, F inverse of transpose is usually written as F inverse transpose. Yeah, this operation comes from what we had studied in the previous, yeah, that is this is the pull back operation and then followed by the derivative and then the push forward operation. You look at, you please note that the way we had defined the pull back and push forward operation for two different vectors, vector like strain and stress are different. We said that this is due to one being covariant and other being contravariant. But, if you do not understand covariant and contravariant, it does not matter; just say that, it is basically because they are, one is the conjugate of the other, when we define the say, stress work or work done by stress or stress power, however you view it and in which case, you will understand why they are, two are different, because ultimately my stress work should be a scalar quantity independent of whether I define these quantities in the current coordinate system or in the reference coordinates system.

Yes; what is this u? What is this u? Please note that is why this directional derivative is taken in the pull back operation. That is the thing. So it is defined in the pull back operation and we use it.

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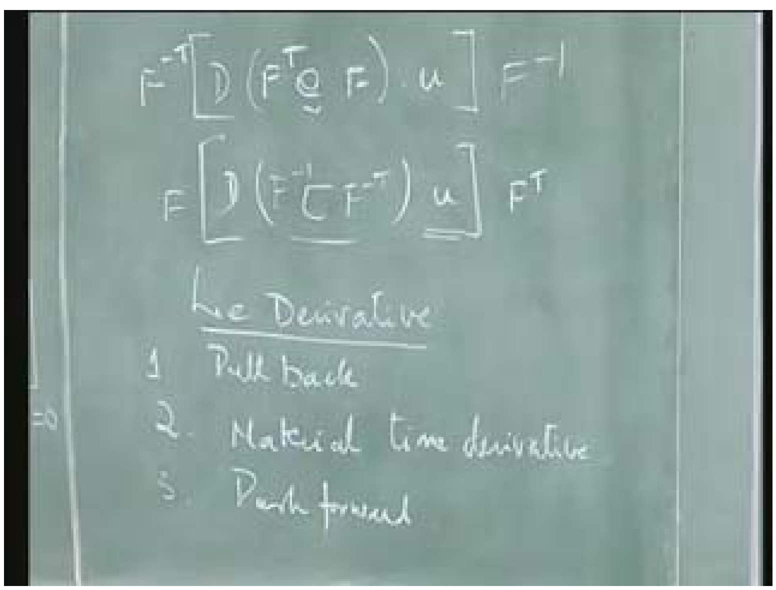

To understand a few more things, let us define what is called as Lie derivative; sometimes Lie derivative is what is called as Lie time derivative, you know, lie derivative and lie time derivative are both of them are the same. No; please note, yes, I understand what this u. Please note that what we are looking for is a small change. That is in other words, these are the, this u defines the direction. So, we are looking at the gradient along a particular direction. So, it is just, it is just, symbolic way of writing that I am moving in this direction. So, it denotes a direction. So, that is why we have defined, look at this here, look at this here. It is not, it is not unit direction, but it is a direction along which you want to calculate. So, that is why we had written it as epsilon u, where epsilon tends to zero. That means that it is just, it is something like calculating the tangent.

Yeah, of course, of course, when I define it like this, they are in the Lagrangian code. Now, note this, note this carefully and then see what they are and you can see it yourself, what they are? Now, lie derivative or lie time derivative again has a very important role in continuum mechanics. We will see in a minute why it is so? Lie time derivative is carried out like this. First, you do a pull back operation, pull back operation of the quantity of interest to you, whatever be the quantity. My next step is to calculate the material time derivative of that pull back quantity, followed by the push forward operation, followed by the push forward operation.

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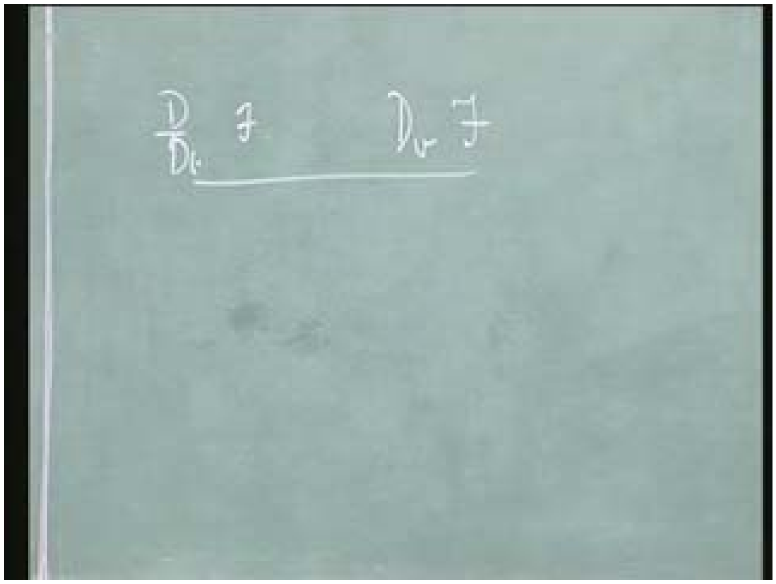

The material time derivative is say, we had written this in terms of D by D_t of f. This is what we defined as the material time derivative. You can notice that this D by D_t of f can be written, I am going to leave that as a small exercise now, can be looked at as the derivative of F along the, along say, a velocity vector v. So D_v of F is actually the material time derivative of F. So, what does it mean? First, first of all let us understand this - physically what we mean by this, this operation or what does this mean or why is lie derivative important? We will, we will just show that in a minute, but let us understand the physical significance. What do we, what is the significance of this statement that it is the derivative of this quantity along a particular vector v? So, what is this pull back material derivative and push forward operation? What it really means - the question which you asked.

It simply means that, if I take a quantity and I want to know how this quantity itself varies; say, this F varies as it moves along say, a particular line or particular velocity, then what I should actually do is to travel along with it in that particular velocity and along a particular line and see how this quantity varies. So, lie derivative physically what it means is that, it is a quantity which comes out as I move along with the particle, with the velocity, along the direction in which it moves. So, it removes all the other external things and it gives me only how that quantity varies or in other words, lie derivative gives me an objective quantity, independent of the observer motions. That is the importance of this quantity that whatever be the velocity, whatever be the direction in which it is travelling, it gives me the correct flavour of how this quantity changes.

You will see that, you will see may be in the next class that, lie derivative gives me a quantity which is one, which can be used directly in what are called as constitutive equations. These are quantities which can used. Lie derivative gives me quantities of certain of these derivatives, time derivatives, which can directly be used in the constitutive equations. In order to understand this let us say we will do a small exercise.

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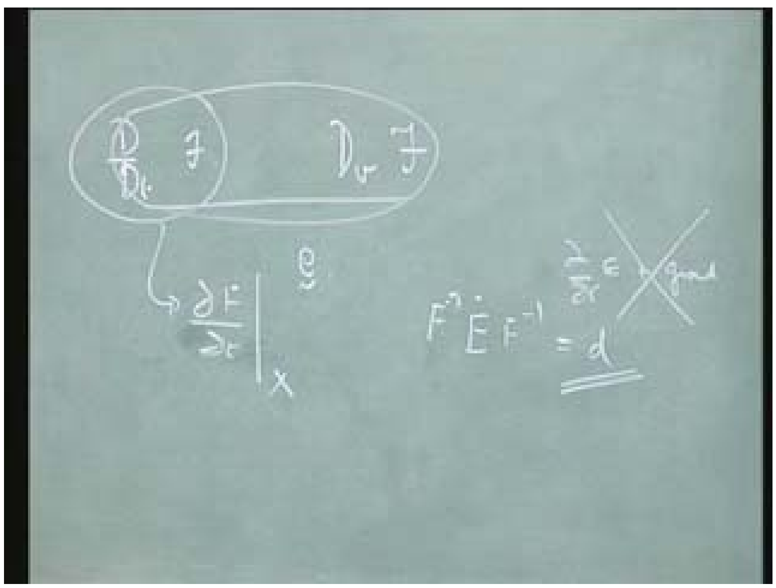

Calculate the lie derivative of Almansi strain, calculate the lie derivative of say, Almansi strains. Look at this definition here and then do that. Let us see how you do this? Just calculate. Yeah; you know all the formula, what all we have done. I want you to refer to it back and then tell me what this is. Lie derivative, note that D f by D_t is nothing but dow F by dow t at a particular X, at a particular X and just you can calculate that. What is the pull back operation of e? What is that you will get? Yes, no, no. What is this quantity, when you pull back small e? Very good, so, that is all. So, the pull back operation gives me capital E. Then, what is the material time derivative. I wontedly gave this and I wanted you to think. What is the material time derivative of E, what is the material? Please note, we had two things - spatial time derivative and material time derivative; the two things that, so, D by, capital D by D_t, so this is what we are saying. This quantity, what is this?

Yes, so, that is D by D_t here D of capital D by D_t; this is not equal to dow by dow t of e plus grad, if you remember grad e dot t, it is not equal to this. That is the reason why I gave this, because this is only for a spatial quantities material time derivative. E itself is a material variable. It is defined in terms of the Lagrangian in a coordinate. So, it is in the reference coordinates, so obviously, the material derivative of a quantity like E is E dot. Yeah, E dot, straight away it is E dot. Because I have pulled back, I have got E, capital E which is E dot.

Now, what is the next operation that I do? I give a push forward operation, a push forward operation of this quantity. Fantastic, that is a good answer. So, what you get is small d, small d. So, the lie derivative of small e is small d. The material derivative, E dot; see, E is a material quantity. That means that it is defined with respect to capital X, all this F transpose F and e is equal to half of F transpose F minus I and so on. We said that there are two quantities for strain. This is the Lagrangian quantity and the other one we defined as the Eulerian quantity, small e, so, we also define that e. When you pull back, we get capital E. Now, when I want to take the material derivative, there is no need for me to do velocities and other things to go into my calculations, because it is a material quantity. So, I can just say it is E dot. Then, I do a push forward operation. So, F inverse transpose E dot F inverse which is equal to d.

The lie derivative of small strain e is equal to d, which is the, what is this? The symmetric part of the velocity gradient tensor or we call this as rate of deformation tensor; we call this as the rate of deformation tensor. So, this rate of deformation tensor is going to be very useful to us and it participates in many of the constitutive relationships, because basically this quantity is free from the observer's movement. We are going to see what it is or objectivity, in the next class. But before that, we understand this as well. Is that clear? I leave this as an exercise; just calculate this, what D_v is in terms of velocity. If you do not understand, we will come back to that later.

Now, where is this? We will talk about lie derivative again. Once we have defined it, we have seen physically what it is we will come back to it when we talk about constitutive equations. But now, let us understand directional derivatives and where we use directional derivatives or in other words, Gateaux derivative. Directional derivatives have two roles for us. One is, most important role is, in our minimisation of functionals, minimisation of functionals. In fact, the minimisation of functional is defined through, actually through, the directional derivative or Gateaux derivative. We had sort of did not look at it so very carefully when we did the first course, but actually we have to look at that more carefully now.

Why gateaux derivative become important when we do a minimization problem is basically because if I have a function, look at that.

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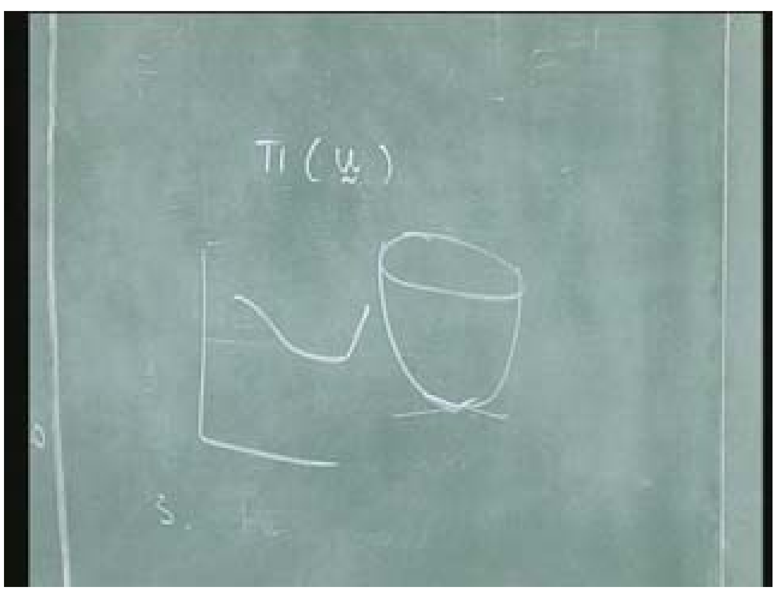

If I have a function or a functional or a function of say, u whatever be that u and u is a vector quantity; u may be a function of X and so on, then I have to calculate this pi to be a minimum, whatever be the direction. Say for example, just to say that say, may be say, this functional may have, it may be something like a say, a vessel like this. So, in whatever direction I move, perpendicular, …. plane or so on, this particular function has to be minimum or the directional derivative, whatever be the direction, should be equal to zero; hence the importance of the directional derivative or gateaux derivative. Is that clear?

So, geometrically you can see that this derivative gives us all the aspects that you would see in just a function whose minimization be carried out, for example, by just differentiating it with respect to X, because we had only say for example, some y is equal to some function of X. Then, we do not have this kind of problems, because we can state dy by dx to be equal to zero, because we just have a nice graph like that and then you want to calculate what the minimum is for this kind of thing, then it is simple. But now, when you define a function with u which can vary or which can vary in n dimensions or it can have n dimensions, then we do this minimization problem by defining a directional derivative or a Gateaux derivative.

Before we go further, let us look at certain other quantities, which is of interest to us. Now let us see that. I mean, before we go into the details of minimization problem, we have to look at certain other aspects.

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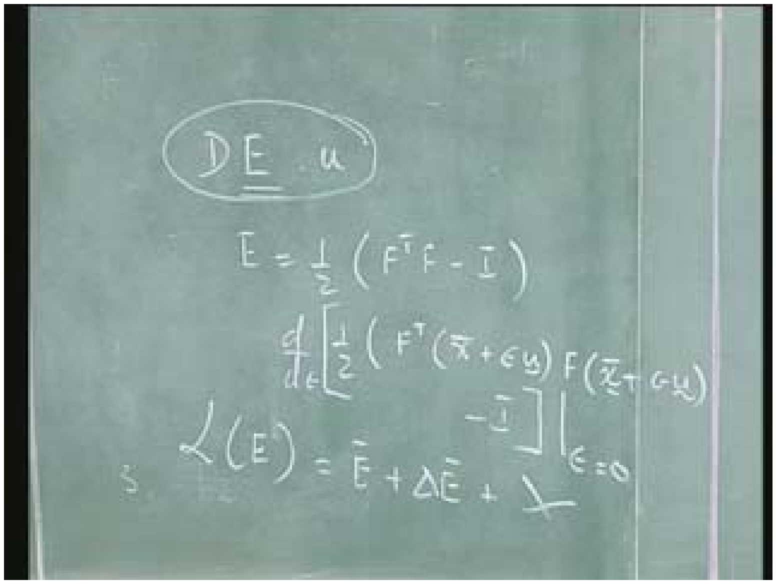

Now, let us take a quantity like E. We had calculated the material time derivative of it; that is fine. But, what is this DE or how do I linearize a quantity, a kinematical quantity like E, because this is going to be important for us later to define this in the, either in the potential energy theorem or later the virtual work principle in the next step. What is the linearization of E or along a direction u, how does E vary? We know that E is equal to half of F transpose F minus I. What we want to find out is half of F transpose defined for a particular configuration plus say, epsilon u along a certain direction and, first I think I have to define d by dt, F of chi plus epsilon u minus I, the whole thing taken at epsilon is equal to zero.

What I have essentially done is to just say that we have defined the directional derivative or in other words, what we are trying to say is what is the linearized part of E; linearization of E. You will notice later that say for example, this can be defined at E bar at a point or at a configuration rather plus say, delta E bar plus higher order terms which we are neglecting and just bear with me for two minutes, then you will understand the whole thing. This delta E bar is nothing but DE dot u. In order to do that, I have to know, I have to calculate this, the linearized part of F which is deformation gradient tensor.

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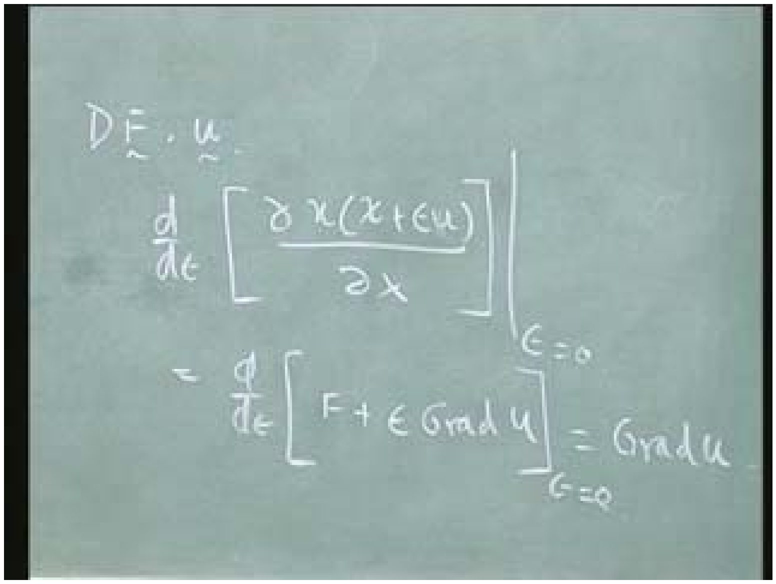

How do I do the linearized part of the deformation gradient tensor or in other words, what is the D F dot say, u? How do I do that at a particular configuration chi. This is given by d by d epsilon of dow of epsilon v at a particular configuration dow X at epsilon is equal to zero, which can be written as d by d epsilon with respect to small chi or small x, we can do that; so, F plus epsilon into Grad sorry, not v, u, Grad u at epsilon is equal to zero, which gives me Grad u.

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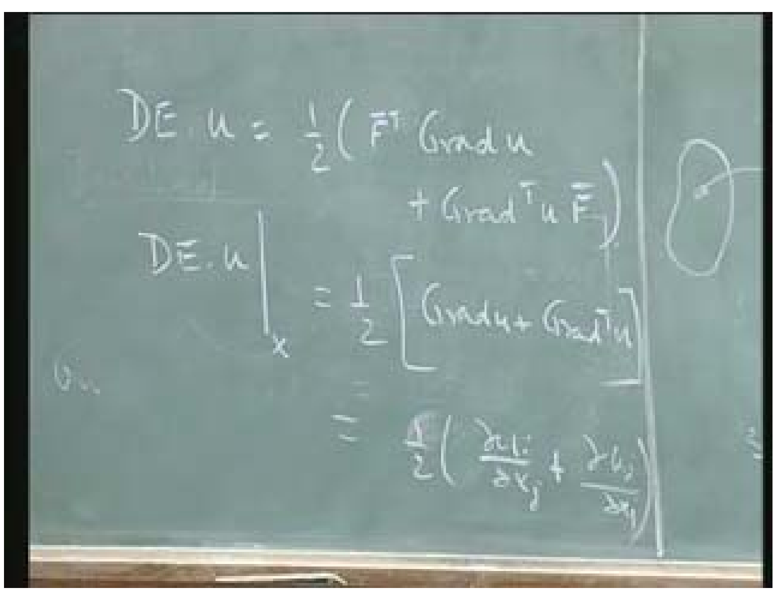

In other words, this expression can now be written as, look at this expression and look at that, what we have got now and so, substituting that you can say that the linearization of E with respect to u can be written as half of F bar transpose, F bar indicates the calculation of F for a configuration given by chi bar, say, what we mean is if this is the reference configuration, we are linearising E with respect to a configuration given by chi bar into the other part is given by here, what we had done, so, it is given by Grad u plus Grad transpose u F bar. What we have essentially done is to replace, see this, when I differentiate it with respect to epsilon for this, this is constant, differentiate this. Then again, of course, I is going to be zero. Then, again differentiate this and that is say, d of u v. That is all; that is what we have done. So, this gives me what we call as a directional derivative of E along u.

To understand that, that is why I said just wait for a minute, suppose I take the reference configuration as the configuration and do a linearization of E, let us see what you get. So, at reference configuration, this chi bar is equal to or small x equal to capital X, which means that F is equal to 1 or I which means that the linearized part at the reference configuration x of this will give me half of Grad u plus Grad transpose u, because it is at the reference configuration. That means that this fellow coincides with this which means that this F bar calculated at the reference configuration itself that means there is no deformation or no change. So, it is I. So, when you substitute that you will see that linearized part of E is equal to half of Grad u plus Grad transpose of u.

What does this mean? What is this? It is what is this? Very simple; no, it is not L. No, there is no E dot. u is just, just a displacement, because in this case it happens to be a displacement. Epsilon, small strain; this is nothing but half of grad u is dow u by dow x I plus epsilon. What does this give you? What does this statement in indicial notation means? Indicial notation simply means that this is equal to half of dow u_i by dowX_j plus transpose of this dow u_j by dow X_i and this is what is our definition for small strain, so, this happens to be just the small strain.

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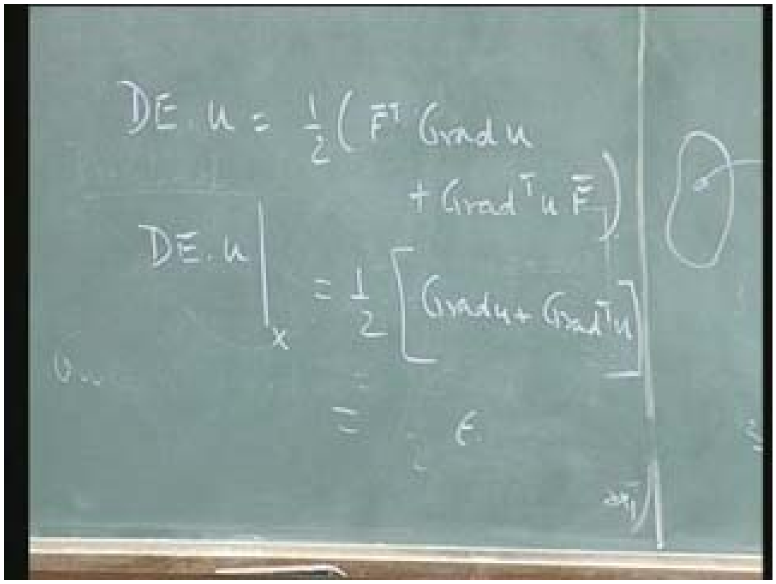

In other words, what this means is that the linearization of our large strain actually leads to small strain. Is that clear? Linearization of our large strain really leads to small strain. In order to understand this, I know it will take some time to sink in; we will do that again in the next class, but in order to understand this let me give you a small problem. Why not you do a small problem?

Assume a scalar field. We will get used to this directional derivative, may be some of the things I will repeat it again, but let us do this problem and get more familiarized with how to calculate it and when we apply it, so that it will become easier.

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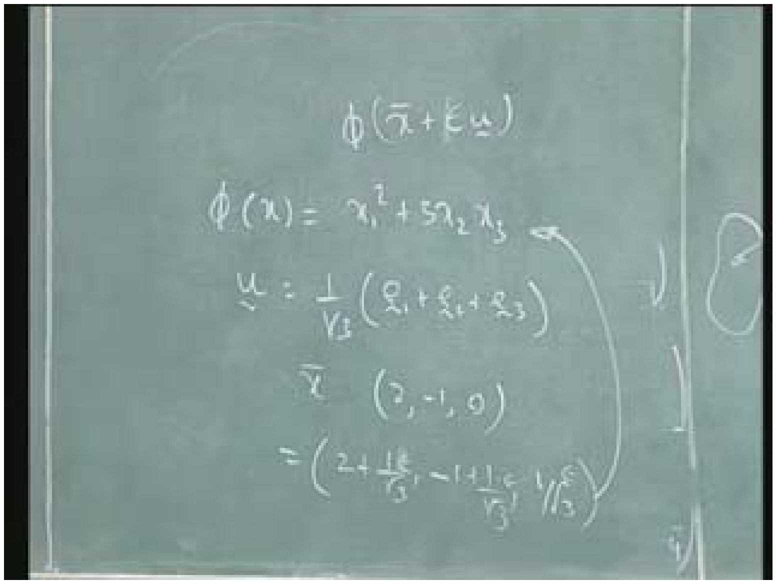

Assume a scalar field to be defined by x_1 square plus $3x_2$ $3x_2$ x_3, assume the scalar field. Let it describe some physical quantity in space. Compute the directional derivative, compute the directional derivative of this quantity phi along the direction u. See, look at that, I am giving you a direction; so, along the direction u say, which is given by 1 by root 3 of e_1 plus e_2 plus e_3 at the point x given by 2 comma minus 1 comma zero. The first, is that clear, at the point x?

This is what we call as x bar at the point, when I say at the point x, that is what we mean by x bar. The first step for me is to calculate x bar plus say t into some or epsilon into u and calculate phi along that line. So, what is this x bar plus epsilon u in this case? This happens to be 2 plus 1 by root 3 minus 1 plus 1 by root 3 comma 1 by root 3; all of them has of course, that is epsilon, yeah, epsilon in it, into epsilon, into epsilon, into epsilon. So, 2 plus 1 by root 3 into epsilon that is epsilon by root 3 minus 1 plus epsilon by root 3 comma epsilon by root 3. This is what is my x bar plus epsilon u. What do I do? I now substitute this into that expression. That is my next step. So, substitute that into this expression and then calculate d by d epsilon, d by d epsilon of this.

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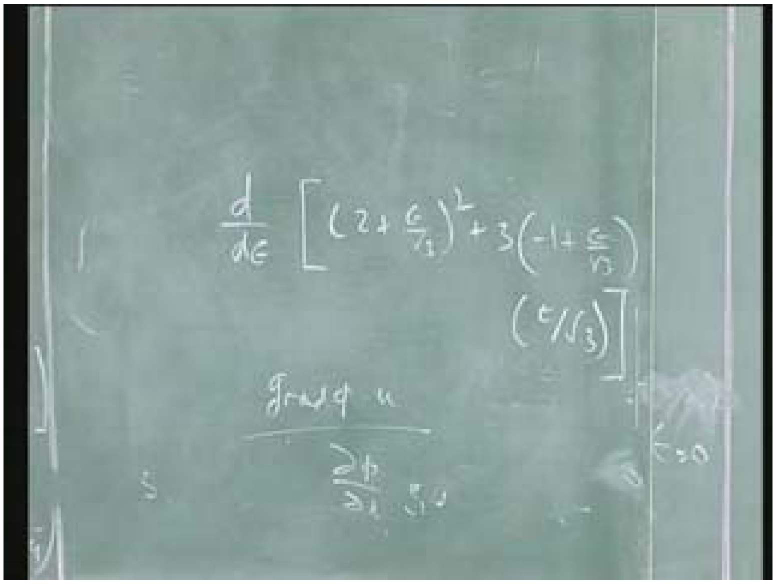

What is x_1? So, 2 plus epsilon by root 3 whole squared that is x_1 squared plus 3 into x_2 x_3; x_2 x_3 is minus 1 plus epsilon by root 3 minus 1 plus epsilon by root 3 into x_3. What is x_3? x_3 is epsilon by root 3, epsilon by root 3 at epsilon is equal to zero. This is what we mean by the directional derivative of phi. That is what its increase is along the direction of u. Why do we do this? This is derivative along u when we take epsilon equal to zero. That means just at that point that is what we mean by that, by that particular derivative.

What is the, please calculate this. You will see that this is also equal to grad phi, grad phi. Just look at that answer and also you will see that this is equal to grad phi dot u and this is what we saw also as the directional derivative. What it means? Grad phi is the derivative along what is grad phi? This is nothing but dow phi by dow X_1 into that is e_1 plus dow phi by dow X_2 e_2 and so on. So, it is the derivative of this along dot u. That means that it is the derivative of this along the direction of u; you would see both of them are the same. So, directional derivative, in other words for this case, you can also calculate. You can say that the directional derivative along any say, vector u can be calculated from grad phi that is derivative along the basis e_1 e_2 and e_3 and then, take the dot product of it along u. You would see that this forms the basis for our minimization problem also, minimization problem also for our potential energy theorem.

We will stop this; we will continue this topic in the next class, may be zero in on certain things which we are supposed to know on potential energy theorem and virtual work principle.