Advanced Finite Element Analysis
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Lecture - 18

Phi, a variable, the scalar field was given and that the observer was supposed to move with the velocity v.

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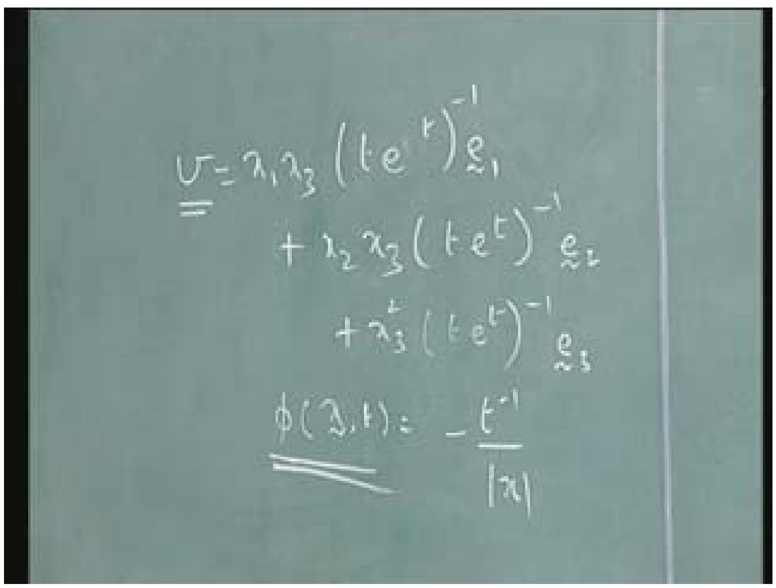

He said it is equivalent to calculating the material derivative. It is a relative term, whether observer moves or the material particle moves, it is all the same.

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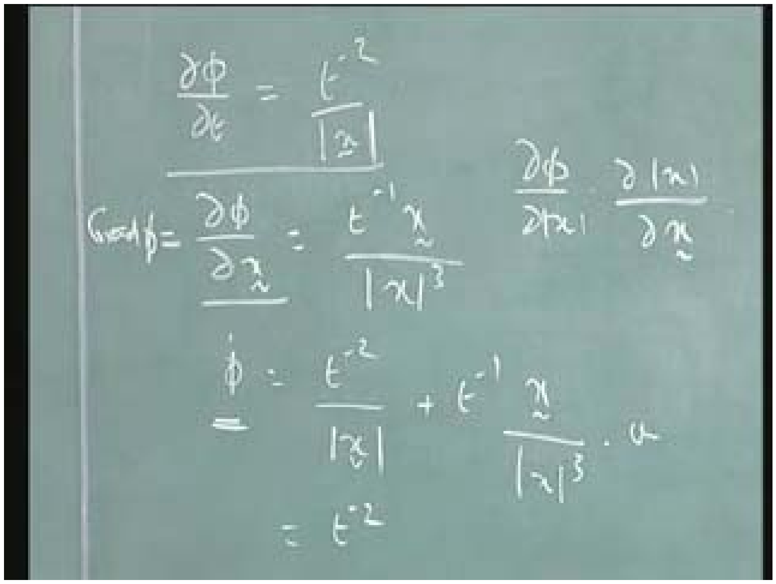

What you do is to calculate dow phi by dow t, then calculate dow phi by dow x or in other words, grad of phi. Then, calculate phi dot by, first term being dow phi by dow t was grad of phi dot v.

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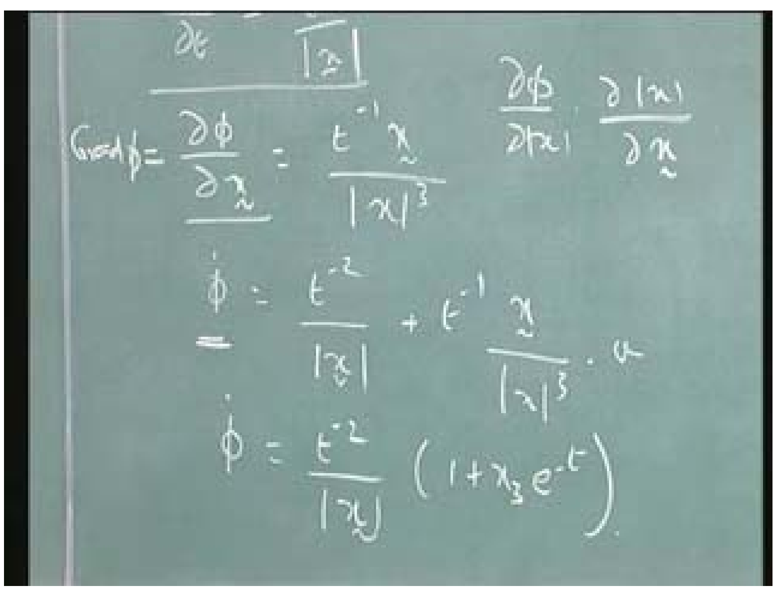

The answer happens to be x into 1 plus, so, that is the material time derivative of phi. Now, let us move to the next problem. Next problem is, same way we will do it.

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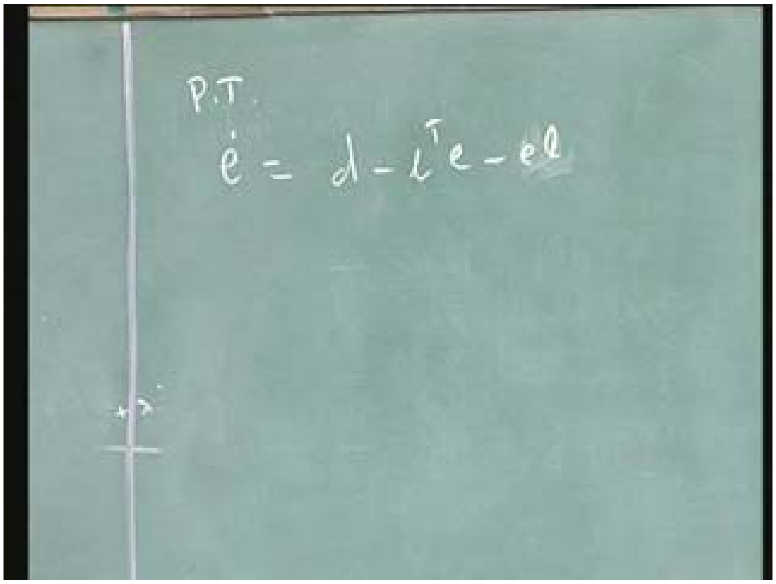

Prove that e dot is equal to d minus l transpose e minus e l. I shall give you couple of minutes, let us see how you work out this problem.

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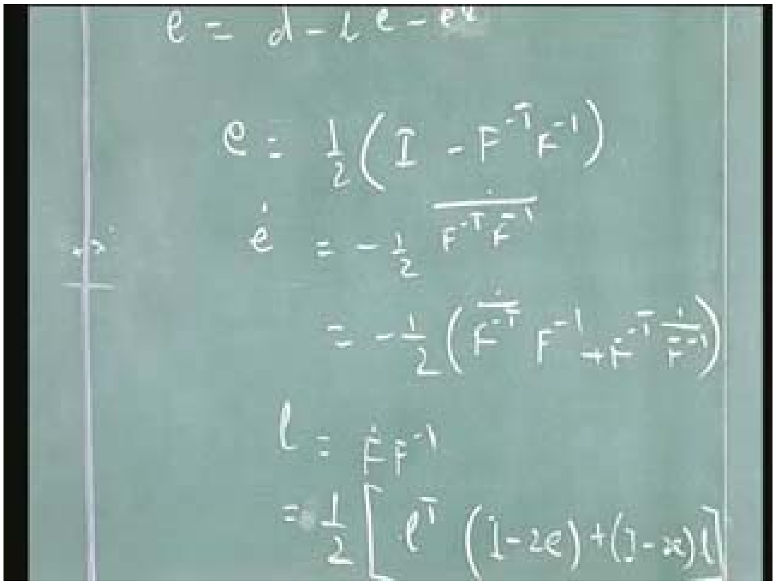

Remember, what was, what was our e? e is the Eulerian; note that it is the Eulerian strain rate or strain rate which is, sorry, strain which is for Almansi strain. That is in other words, it refers to the current coordinate system and it was written as half into I minus F inverse transpose F inverse. I am interested to determine what e dot is. Let us see, continue with that. Obviously, the first term will not be there. So, it will be half of F transpose T F; whole is to be differentiated with respect to time. How do you write that? What is the relationship between l and F and F dot? l is equal to, F dot F inverse keep that in mind; so, substitute that and see what you get. What will happen to the first term? Do that and you will ultimately get l transpose into I minus 2 e plus I minus 2 e l. Note that d is half of l transpose that is l.

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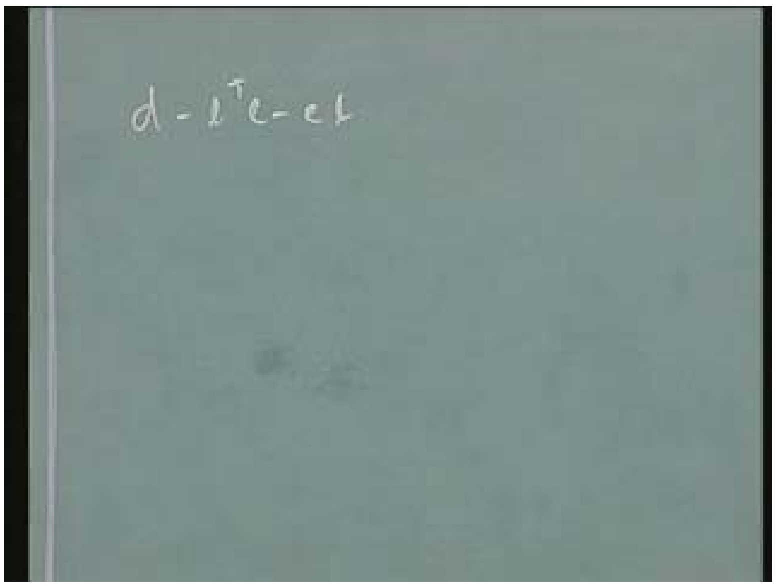

From this, you can write this down as d minus l transpose e minus e l. Fill up the remaining steps, we will move to the next problem; just substitution and then just working it backwards. Both the problems are very simple problems. Only thing is that we have to manipulate a bit. Let us go over to the next problem. Consider a deformation gradient that is F, given by what I am going to write now.

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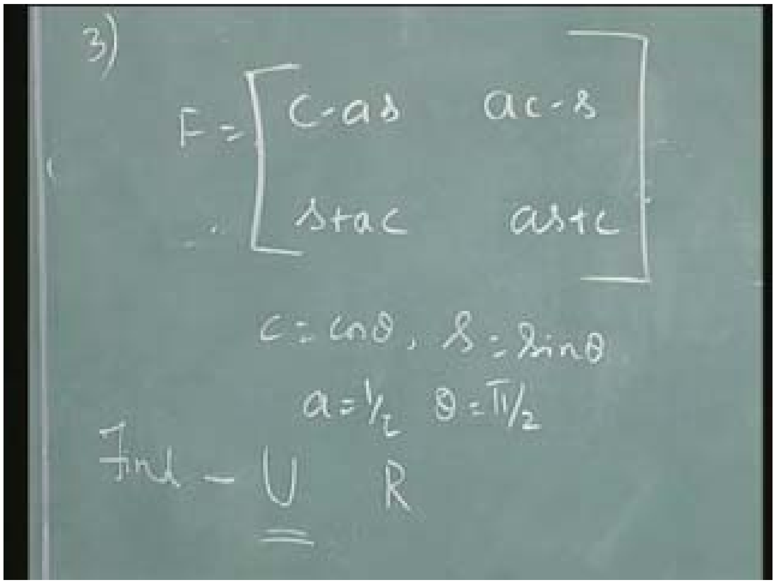

Consider the deformation gradient, F given by C minus as ac minus s s plus ac as plus c. Of course, C is equal to cos theta and s is equal to sin theta. Take for this problem a is equal to half and theta is equal to pi by, pi by two. This is the third problem. Find the stretch tensor; find the stretch tensor or in other words, sometimes U is called as stretch tensor, U and R; U, which represents the pure stretch which is called the stretch tensor. Find U and R. In other words, substituting these things, you can first write down what is F.

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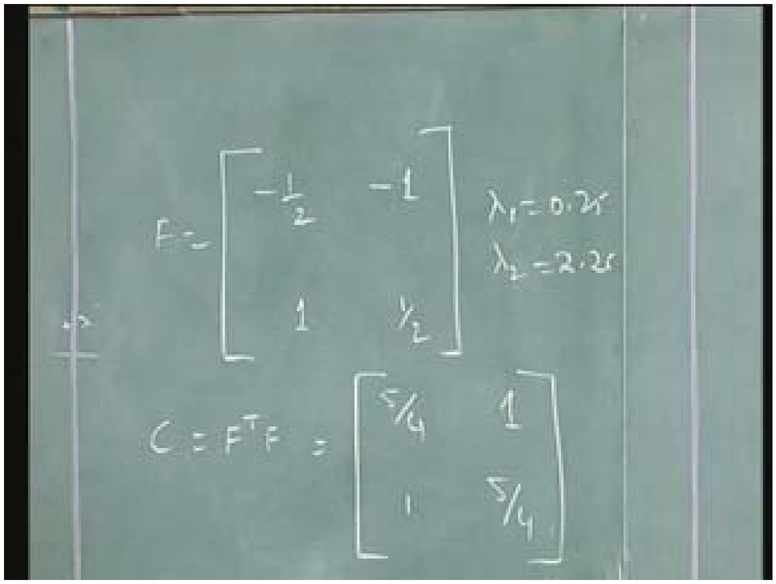

What is F? Cos pi by 2 minus, so, minus half, first term; the second term is a cos theta again that goes to zero, so, minus 1 and then 1, again C goes to zero and half. So, the first step is to find out C. Look at the steps, what I am going to do. C is F transpose F. Determine F transpose F. So, the steps we are going to follow is that we will find out C, solve the Eigen value problem, determine the Eigen values of this and as well as Eigen vectors, write it. Write the Eigen values of the U. What is the Eigen value of U? What is the relationship between that and Eigen value of C transpose C? Root of the values and then determine the Eigen vectors, transform U to the original coordinates and then determine R, because you know F and then, you know U.

So, now determine F transpose F. If you have forgotten something, please refer to your notes and tell me what is F transpose F? 5 by 4 1 1 and 5 by 4. The next step is to find out lambdas for C. Please determine lambda, write down. It is very simple, straight forward, because it is just 2 by 2. Write down the characteristic equation and then find out what lambda is. For the lambdas determine the Eigen vectors as well; determine the Eigen vectors as well. Very good; so, lambda_1 is equal to 0.25 and lambda_2 is equal to 2.25. Find out the corresponding Eigen vectors for lambda_1 and lambda_2. This lambda_1

lambda$_2$ are the Eigen values for this C. We will take the square root for U, but meanwhile, let me determine the Eigen vectors for these two quantities.

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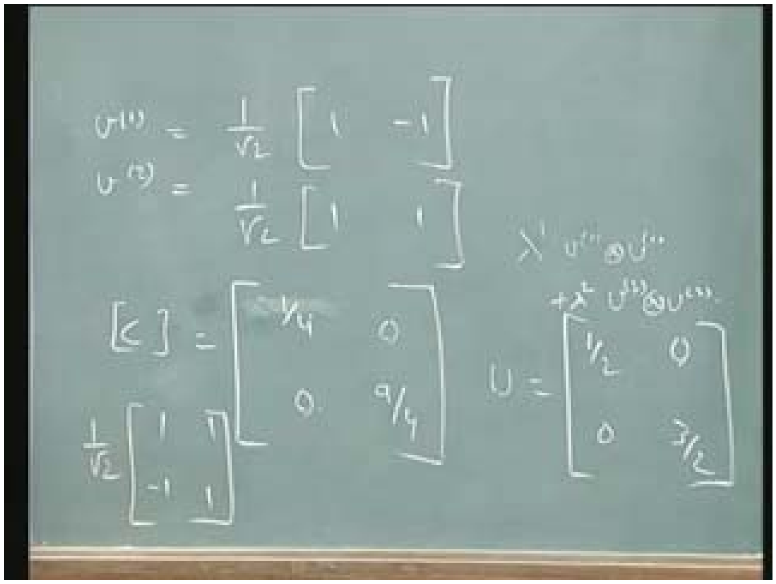

That is in other words, v of 1 and v of 2 corresponding to the Eigen values lambda$_1$ and lambda$_2$. 1 by, you have to normalize it, 1 by root 2, correct, 1 minus 1 1 by root 2 1 1; very good, it is excellent. Look at these steps. You know this is how you calculate stretches and as well as, because these steps or these kind of calculations may become very important when you do finite element analysis, when you implement finite element analysis later. In other words, how do I write the C matrix. So, C matrix is in terms of, this is spectral decomposition; in terms of these lambdas are written as 0.25 or 1 by 4. If you want, you can write it as 1 by 4 and someone said 9 by 4 and the corresponding, what do I mean by basis in this case?

In spectral decomposition remember that we had written for spectral decomposition, lambda 1 v 1 v 1 plus lambda 2 v 2 v 2. So, the basis here are the Eigen vectors, correct. So, those are the Eigen, Eigen I mean, these are the basis. Essentially, what is that we have done? You know, just recapitulate what we know. Essentially what we have done is to get a basis, new basis, in such a fashion that you get diagonal C matrix. This is exactly what you do in, say for example, when you do vibration problems; you find out the Eigen values and the Eigen vectors and then change over the basis. Your original basis is now replaced by the Eigen vectors. What exactly you do? Decouple all those equations, the governing differential equations, you decouple them.

Basically, Eigen vectors are used to change the basis, so that it becomes the matrix, ensuing matrix; in other words, ensuing equation in the other case, …… problems, you know, vibration problems rather, would be much easier to solve. Now, how do I determine U in this basis? Simple; square root of these base here; 2 0 0 3 by 2, but my job is not yet over. What is that I should do? What is that I should do? Now, this is in, this - what we have got here - is in this basis. Now, I have to get it back to my original basis. How do I do that?

Look at this and see, you have to do a transformation. Please look at that. Let us see how much you remember from what we did earlier and then write it down. In other words, the basis actually can be written as 1 by root 2 1 minus 1 1 and 1. This, multiplied by U, multiplied by transpose of this, would give me the U value. That is the transformation that I am going to use, U or the value of U in my old coordinate system. Let us see how we do that.

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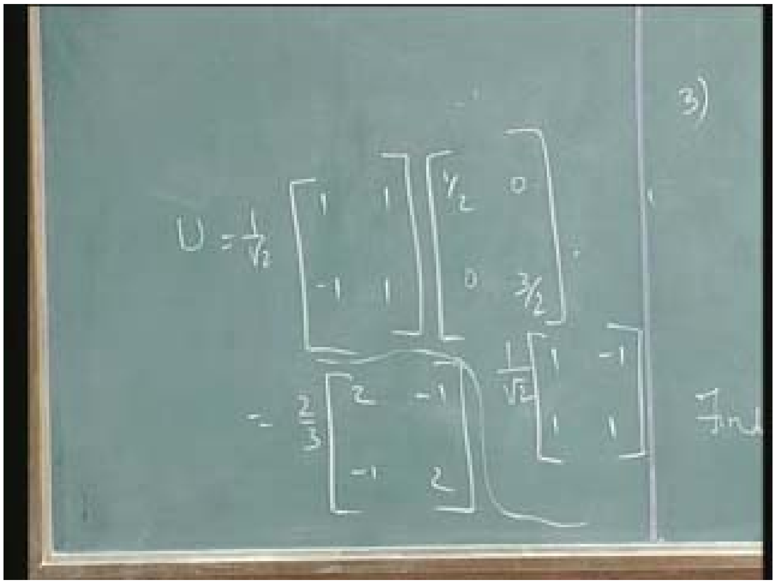

In other words, 1 by root 2 1 minus 1 1 1, half cipher cipher 3 by 2 into, I will write it here; 1 by root 2 1 1 minus 1 1 would be the transformation matrix. What is the result of that? 1 by 2 1 by 2, so, you can take it as half. So, the answer is, if I am correct, 2 by 3 2 minus 1 minus 1 2. Just check this up, see whether you get this. Please check that. Essentially, the steps are very clear, just find out the Eigen value, go to the spectral decomposition mode, standard step; determine the Eigen values. This is a very practical way of decomposing F into the stretch term, U and the rotation term, R. Hope this is correct. Please check this up. You know, I just did that fast, I hope it is okay.

Now, how do you determine R? Yes, that is all. So, invert it, F U inverse, calculate F U inverse and calculate R; please invert that and find out what R is.

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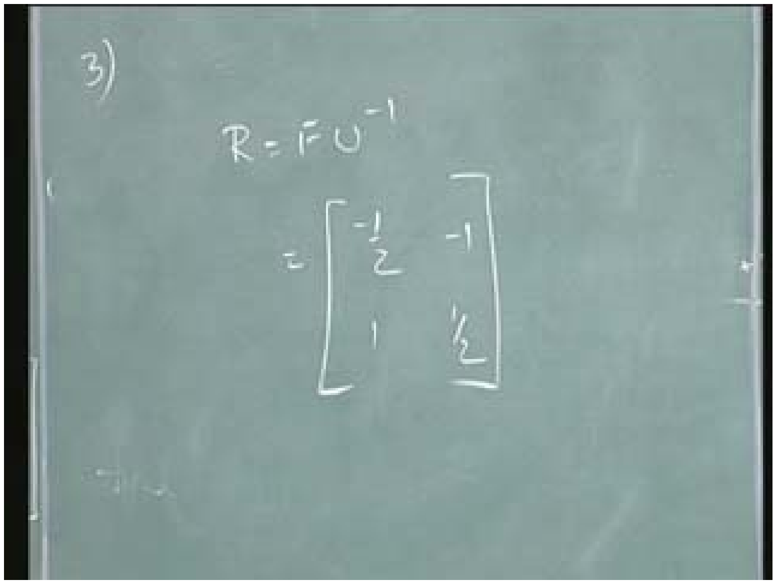

R is equal to F U inverse. F you already know; invert U to get U inverse and then, please attempt the problems, all the problems; as we go along please attempt it.

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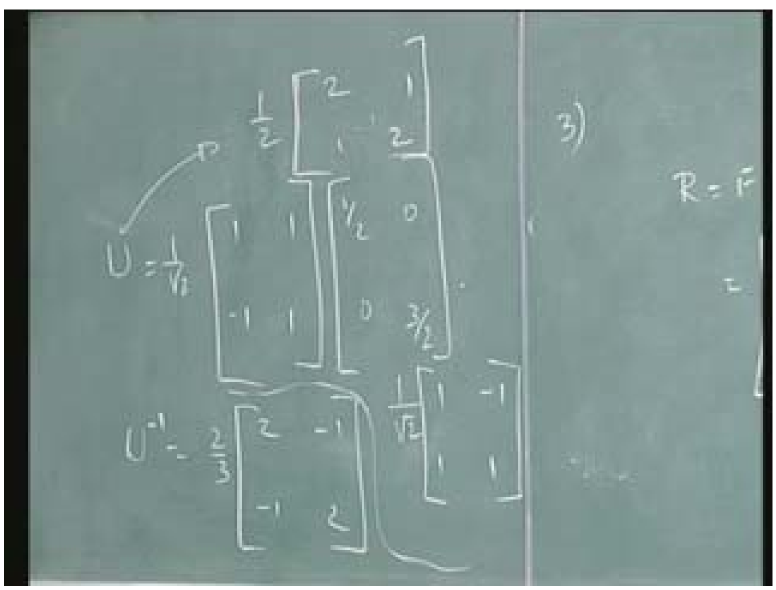

I think, I think this is U inverse, actually. Sorry, this is U inverse; U inverse is this and U is actually half of, yeah, U is half of 2 1 1 2, because 1 by 1 by root 2 1 by root 2 is half of it, then we multiply this 2 1 1 2. This is U and actually this is U inverse. We have already inverted it, so, that is U inverse into 2 by 3.

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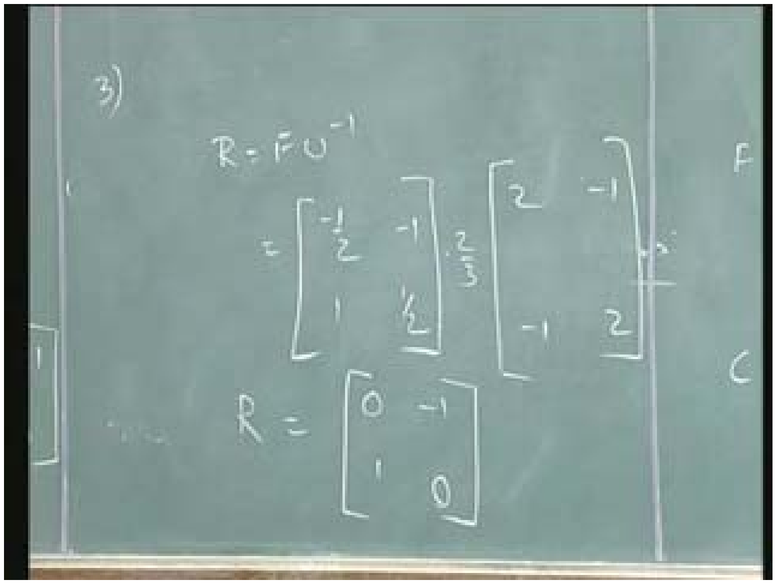

Obviously, we will not get 2 by 3 here, because 1 by root 2 1 by root 2 is here. So, we will get only half. So, U inverse is equal to 2 by 3 2 minus 1 minus 1 2; so, 2 minus 1 minus 1 2. So, R is equal to 0 minus 1 1 0.

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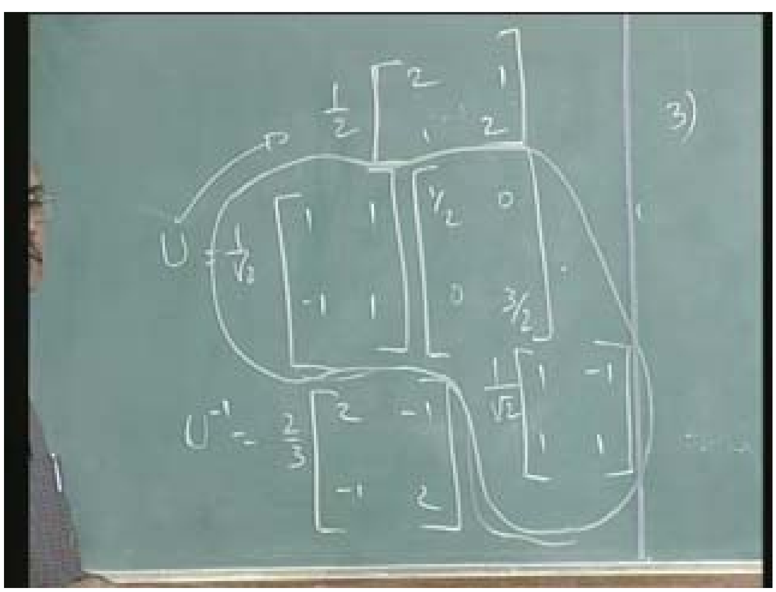

Note that, note that this is U inverse, not U. This U is here, just you have to multiply those terms there; 1 by root 2, 1 by root 2 goes out to half. So, other multiplication gives you this. Inverse of that is given by …., take the determinant, then cofactors; write down the cofactors, it is just 2 by 2 matrix. So, R gives this. You initially note that R R transpose is equal to I. Any question on this problem?

Now, take down the next problem. The uniaxial, we are talking about, now we are going to talk about uniaxial stress and the associated strain. How this condition can be expressed in terms of the first Piola-Kirchhoff stress, the second Piola-Kirchhoff stress and of course, the Cauchy stress and what are the strains? Different strains are capital E, small e and so on.

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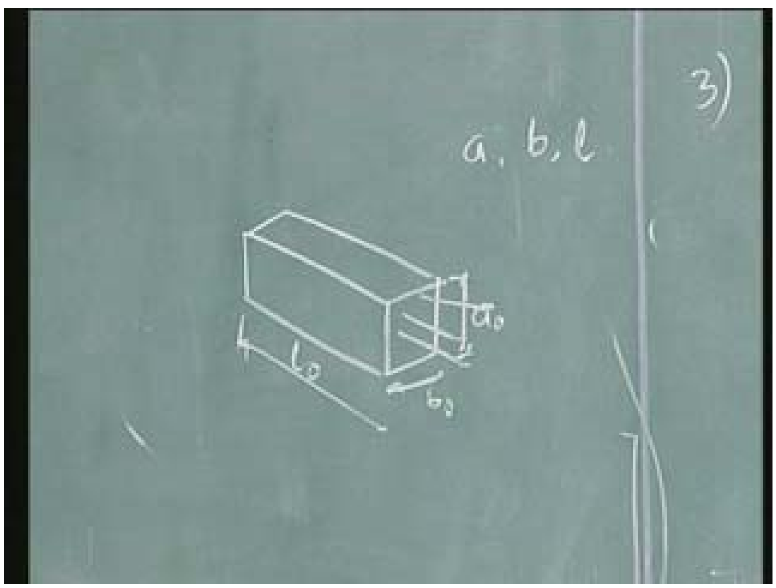

Consider a bar with length l, b and a. Let us say that its original dimensions are l_0, b_0 and a_0 and its final dimensions are say, given by a b and l. Let us say that a uniaxial stress state is what is applied to this bar. Determine the nominal stress, second P-K, first for this bar.

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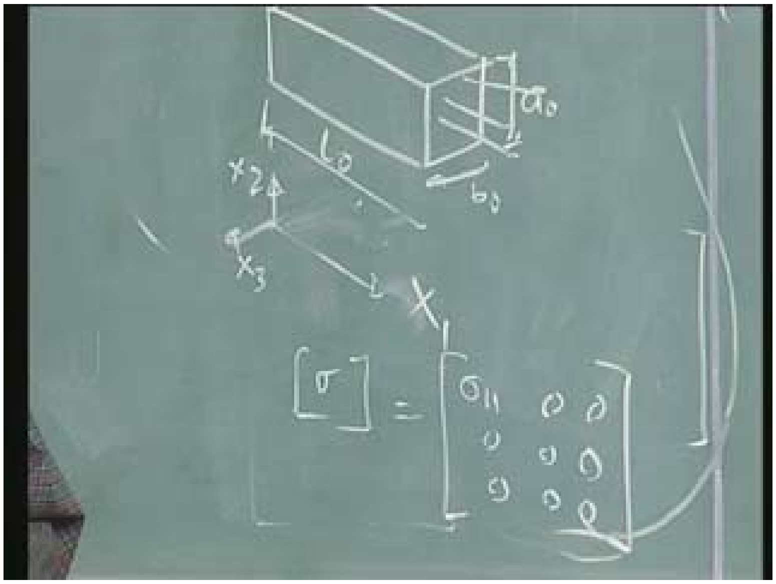

Take the directions to be say, this is the X direction or capital X or capital X_1. If you want, write it as capital X_1 and then that to be capital X_2 or in other words, X_2 is in the or you can X_2 and X_3; to be careful that is X_3. Right handed coordinates …… should be maintained and so, X_1 and X_2 and X_3. Our first job here, what is that we should do? Of course, sigma you know. So, the Cauchy stress is nothing but say, $sigma_{11}$ or sigma X 0 0, 0 0 0, 0 0 0; that is fine. But, in order that we convert this into the first Piola-Kirchhoff stress and second Piola-Kirchhoff stress, what is that we need, the first thing? F, very good; so, we need F.

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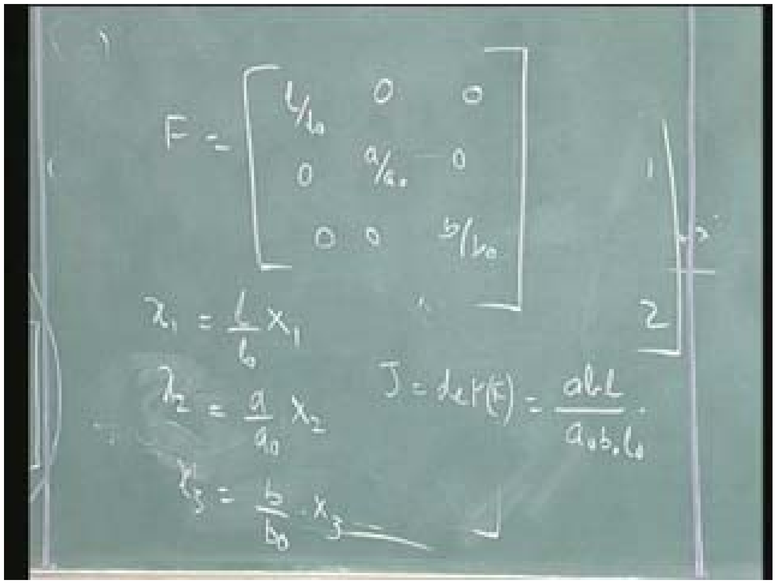

Now, I will give you two minutes. Let us see how many of you come up with what is F for this? No, it is not l minus l, not. Please, first, your first step should be to write down say, what is small x_1 co-ordinate, small x_1 in terms of capital X_1 and then small y_1 and small z_1. How do you write this term? Choose, I mean, I have just given x_1 x_2 x_3. Choose an origin and try to write down x_1 y_1 and z_1. Yes, so, first x_1, write it, write down. So, l by l_0, same what you said, l by l_0 into say, X_1; not x_1, $y_{1;}$ y_1, say, x_2. x_2 is a; so, a by a_0 into capital X_2 and X_3 is equal to b by b_0 into X_3, very simple. When x_3 is equal to b, capital X_3 is equal to b_0. In other words, this is some sort of a, what is that we have followed here; linear interpolation, one end to the other end; that is all.

Now, write down F. Now you have crossed a major thing. Now write down F. That is straight forward dow x_1 by dow capital X_1 that is l by l_0, no other terms there, so, all these terms becomes 0; 0 a by a_0 0, 0 0 b by b_0. Now, what is the determinant or J determinant of F or J, what is it? Yes, so, it is very simple. You know dv by d capital V; dv d small v is abc, you may not even go through this, so, say a b l, rather a b l by a_0 b_0 l_0; that is the determinant of J. Now, we have to calculate F inverse. It is a diagonal matrix.

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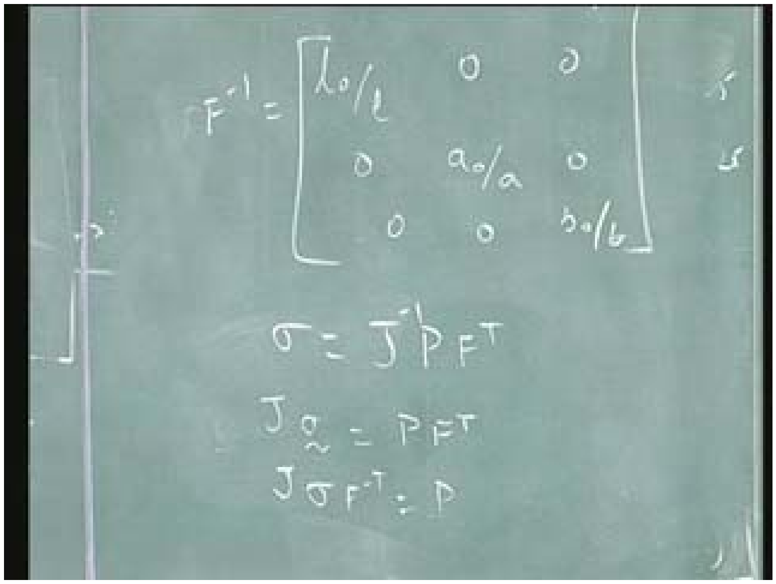

So, F inverse l_0 by l 0 0, 0 0 0 0; very good. Knowing sigma, knowing J, what is that I want to calculate? P, …., nominal, what is the formula? J, no, no, that is for from P. So, J F inverse sigma; yeah, J sigma F inverse transpose, calculate that. Actually, we had sigma is equal to J P F transpose, J inverse P F transpose. So, this will become J sigma is equal to P F transpose. So, J sigma F inverse transpose equal to P.

Pardon? What is sigma? Sigma is sigma. What is that you get for P? a b, let me write it here.

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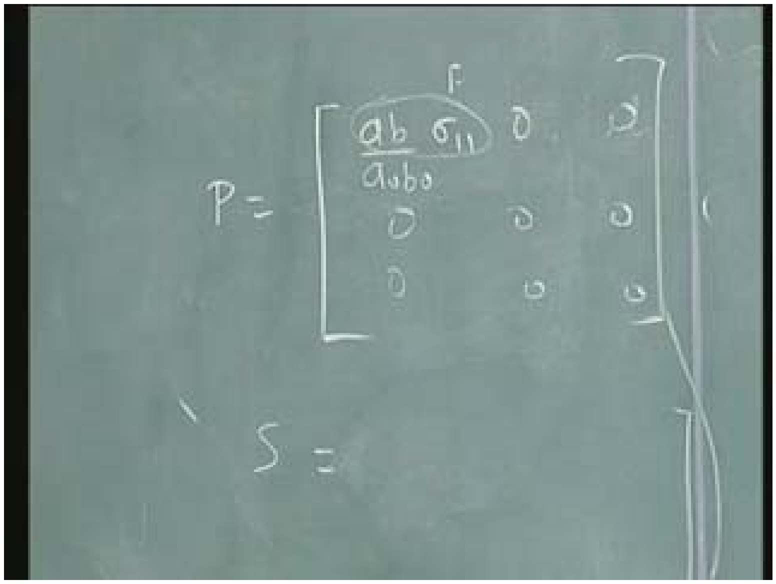

So, P is equal to ab by a_0 b_0 $sigma_{11}$, rest of it are zero. Look at this. What this indicates? ab into $sigma_{11}$, ab is what? The current area, so Cauchy stresses for the current area. So, that area multiplied by $sigma_{11}$, what you get? The force that is acting, that divided by the original area. That is why this is called as a nominal stress. Obviously, I mean you could have guessed it, but nevertheless it is very important that you go through the whole steps, because that will give you the whole thing. This is the force. The force divided by the original area would give you P.

Now, calculate S, the second Piola-Kirchhoff stress. Second Piola-Kirchhoff stress, we already saw that it is the pull back that we have to apply on the Cauchy stress. Rework that here.

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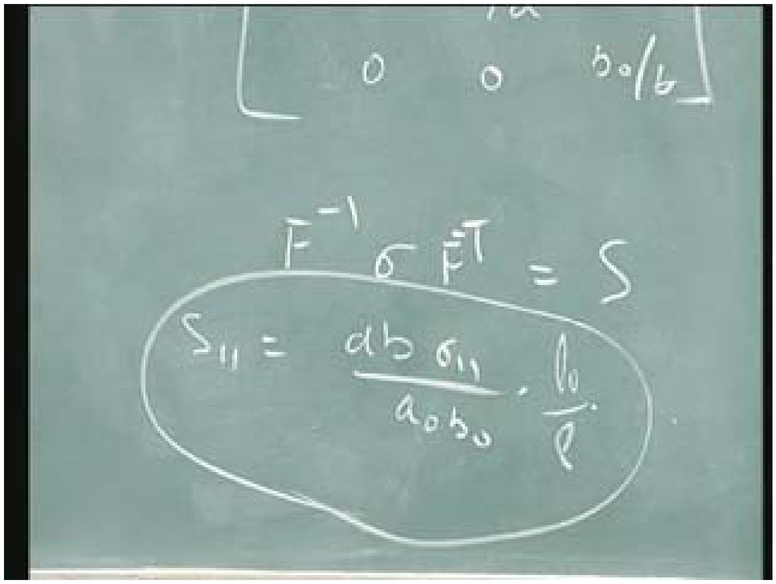

What is the relationship between sigma and S? F inverse sigma F transpose is equal to S. Sigma transpose is equal to sigma, anyway it is the same. So, let us take the S_{11} term, just work it out and tell me what the first term is, the S_{11} term. Yeah, a b sigma_{11} by a_0 b_0 into l_0 by l. Everyone got this answer, into l_0 by l? Yes, I think that is right. Look at that. Here for P, you are able to give a physical meaning for P. But for second Piola-Kirchhoff stress, we are not in a position to give any physical meaning for it. It is just a pullback operation and that is how you should view it. You should not, you know look at Piola-Kirchhoff stress as you know, something which we can physically relate to.

Yeah, this poses problems. This poses problems, especially when you give results and when you go to plasticity, when you deal with second Piola-Kirchhoff stress, since it does not have a physical meaning and many of the quantities that we deal with ultimately, especially when with respect to yielding or failure and so on, they have a physical meaning. So, when you deal with this Piola-Kirchhoff stress, second Piola-Kirchhoff stress, you have a problem. Agreed that? So, we will overcome that by different things and in fact, you would notice that none of the softwares, none of the softwares give results in terms of second P-K. Though internally many of them work with second P-K, they will not give you results on second P-K, basically because it does not have the physical meaning. But, that should not worry you when we apply these things, because you should view it as a mathematical operation of pullback. So, that is the reason, why you will not, apart from theory, you will not see these thing in results. The results are still given in terms of the Cauchy stress. All the results that are still given, in most of the softwares, are in terms of the Cauchy stress.

Yeah, first Piola-Kirchhoff stress, so, you can develop a relationship between P and S. I will leave it to you, it is very straight forward to develop; please develop a relationship between P and S. But, keep this in mind and we will encounter second Piola-Kirchhoff stress, quite often in the implementation levels, but you as a, if you just use it, you will not encounter second P-K or second Piola-Kirchhoff stress. But, there is today a tendency, especially in biomechanics, to actually plot graphs in terms of second P-K itself. There are, I mean, I am not sure whether how much that is the correct way to approach things, because this does not have a physical meaning, but still what people do is to calculate sigma, calculate F and plot the graph; second P-K versus e. There has been recent literatures, which show experimentalist plotting second Piola-Kirchhoff stress. But, personally I feel that that does not take you anywhere, because you do not get a physical picture often.

Let us look at the next problem. I will just give the problem to you. This will be our fourth problem. The deformation of a body is described.

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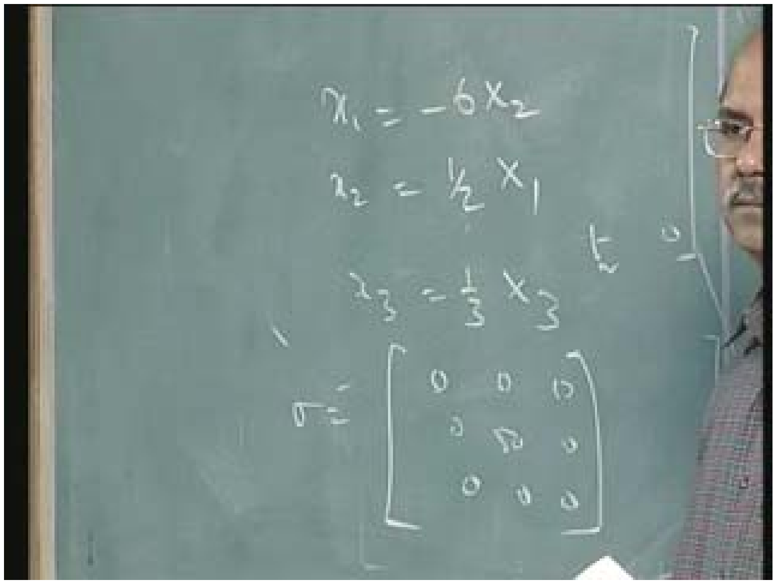

The deformation of the body is described by x_1 is equal to minus $6X_2$, x_2 is equal to half of X_1 and x_3 is equal to 1 by 3 capital X_3. If sigma is given by 0 0 0, 0 50 0, 0 0 0, sigma is given by this, determine Cauchy stress vector t, Cauchy stress vector t and the first Piola-Kirchhoff stress vector. Note the difference between stress tensor and stress vector. We have emphasized that in our first course, we know that stress can be looked at as a vector and as a tensor.

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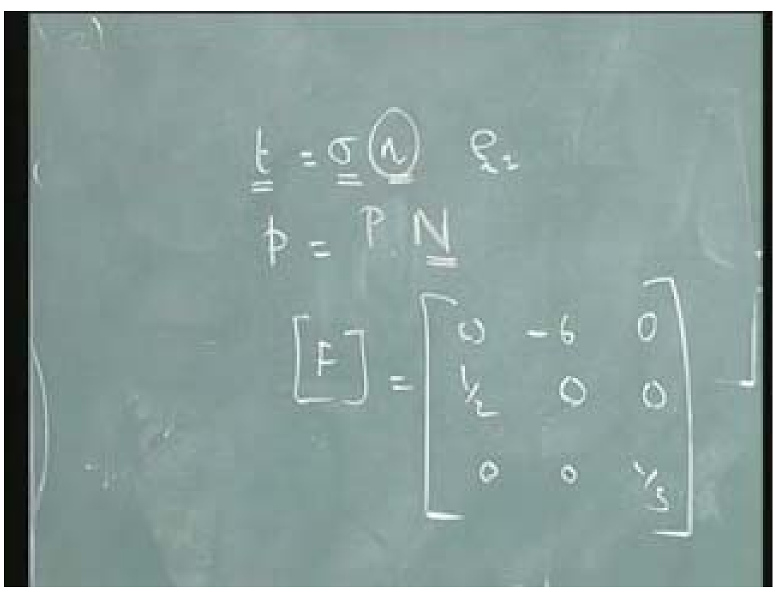

So, sigma n or sigma transpose n is equal to a vector, that is a tensor and calculate the first P-K stress vector taking n to be e_2, taking n to be e_2, the unit vector along the second direction and then calculate what the stress vector is for both Cauchy stress and the Piola-Kirchhoff stress. Our first job, of course, is to calculate from this F. Of course, what you get here, p, what you get here? Same P N; note the difference between small n and capital N. Small n is the normal at the point of interest along a surface ds in the current coordinates and capital N is the same normal in the reference configuration, in the reference coordinates, coordinate; so, note the difference between the two.

Your first job now is to find out F dow x_1 by dow capital X_1 is 0 minus 6 0, half 0 0, 0 0 1 by 3. So, I need F inverse transpose.

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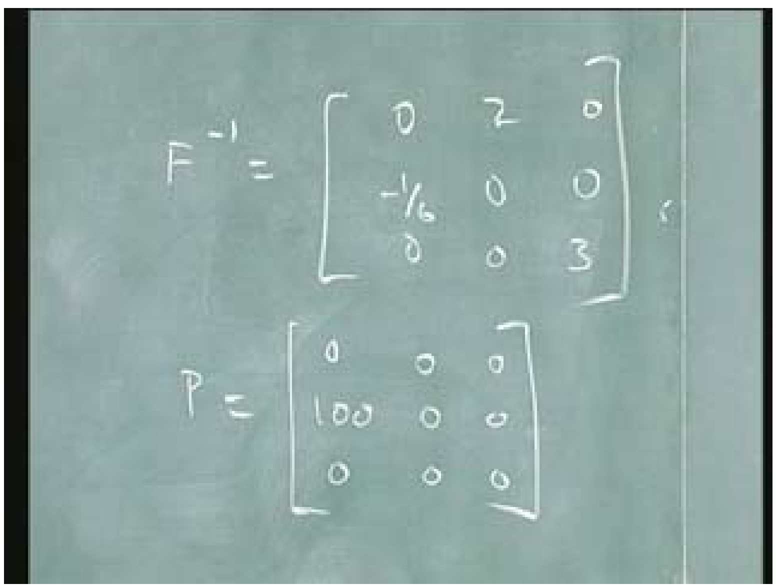

Calculate F inverse, calculate F inverse; 0 2 0, minus 1 by 6 0 0, 0 0 3; that is F inverse. Yeah, F inverse transpose, this is F inverse transpose. Yeah, F inverse that is F inverse. So, you have to calculate F inverse transpose and then calculate P. Of course, determinant J, I think, happens to be 1. Calculate the determinant of F that is determinant of F. This is equal to J is equal to 1. So, calculate P now. P happens to be 0 100 0, 0 0 0, 0 0 0. Now, I have to calculate capital N, the normal. How do I calculate? Nanson's formula.

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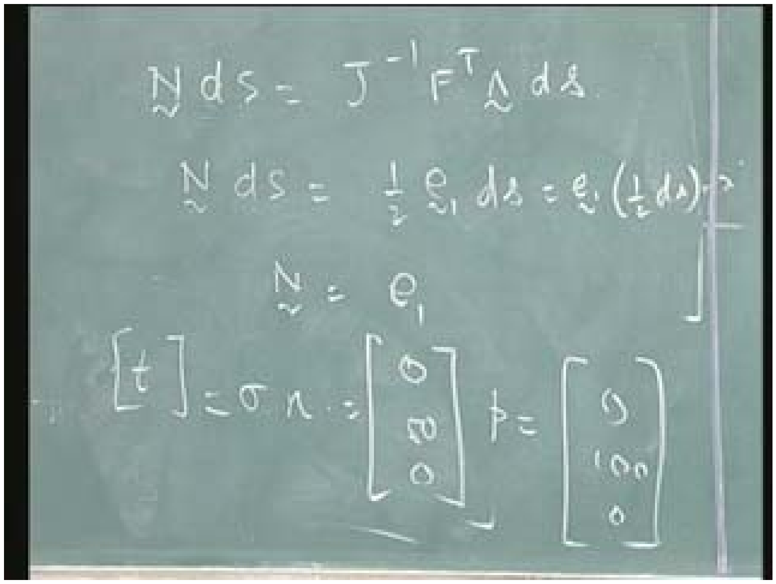

Just calculate what capital N is. So, NdS is equal to J inverse F transpose n ds. What happens to this? Half e_1, yeah, ds. Check that up; is that right? Yes, very good. So, half e_1 ds. What is N? N is actually the, yeah, the so, what is N? It is the unit vector along e_1, which is equal to e_1 itself. So obviously, so, ds is half of ds; that is it. Because, this is a unit vector, e_1 is a unit vector, so, I can rearrange this as e_1 half ds. That is the magnitude. This is a unit vector, this is a unit vector; this is the magnitude. So, ds is equal to half ds. That is straight forward. So, look at that N, which was e_1 has become now small n which is e_2. Yes, so, find out the traction or the stress vector sigma n.

Very good; so, that happens to be 0 50 0. Now, what happens to p? Yeah, 0 100 0; very good, 0 100 0. Obviously, we need not even calculate that, because I see that, I told you already, I gave the clue that ds will be half ds; obviously that is 0 100 0. So, that completes few problems that we were able to do. You now know, how application is done, it is very straight forward. Then, any question? With that, we will close the topic on stress. We will move over to few more fundamentals that we have to learn before we get back to the non-linear finite element analysis.

Yes, the softwares have these things as routines - calculation of U, calculation of R and so on. We will see more about that when we come to the formulations of elements in terms of these quantities later. Yes, of course, there are shape functions that are defined. We will see about them during the formulation. We will close now, see you.