Advanced Finite Element Analysis
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Lecture - 17

Yeah, in the last class we had looked at two stress measures.

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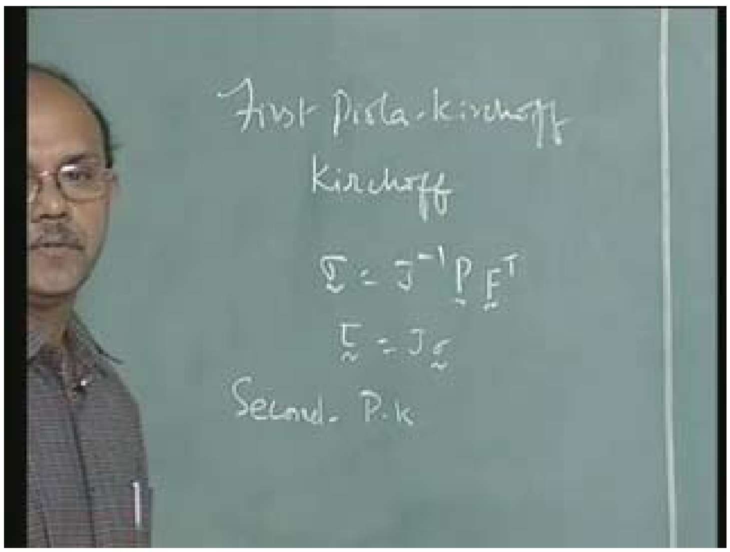

We called the first stress measure as the first Piola-Kirchhoff stress measure; of course this is apart from the Cauchy stress you already know. The second is the Kirchhoff stress measure. We remember that we developed a relationship between the Cauchy stress, the first Piola-Kirchhoff stress which can be written as sigma is equal to J inverse P F transpose and this P, which we call it as the first Piola-Kirchhoff stress and we said the Kirchhoff stress can be written as J sigma and we also observed that the first Piola-Kirchhoff stress is not symmetric.

There is another stress measure called the second Piola-Kirchhoff stress, second Piola-Kirchhoff stress, but before we go to the second Piola-Kirchhoff stress, we have to understand a very important operation called pull back and push forward operations.

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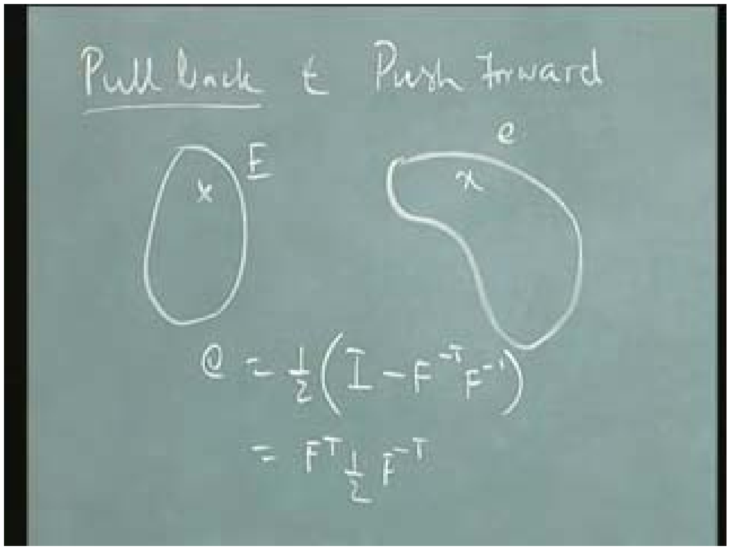

Though the mathematics behind pull back and push forward operation is slightly involved, because this comes from differential geometry, we will understand what we mean by this pull back and push forward operation, because that concept will help us also to understand what we mean by this second Piola-Kirchhoff stress. So, we will physically understand what it means. This pull back and push forward, it is very simple to understand that and mathematically we will develop not so very rigorously, but you may have to take some of the statements which I say, because it may be difficult for me to prove all the things right now, because the back ground what you have may not be sufficient to understand few of the facts that I am going to say.

Let us look at what is pull back and push forward, what you understand by the term say, pull back. The setting is something like this. I have now the body at a reference coordinate or reference configuration and that body is deformed. Let me really deform the body; so, that body might have really deformed. There is definitely now a pull back and push forward that is possible between these two configurations - the X belong to this, small x belong to this and so on. If you remember, we had two quantities and we called this is a material, the spatial descriptions and we had two quantities.

For example if you look at strain, we had a quantity e here and we had a quantity capital E here, defined with respect to this and small e defined with respect to the deformed configuration. Let us now look at the relationship between the two - the small e and capital E. Now, what is small e? If you remember right, half into, no, that is small e, I minus F inverse transpose F inverse is what we had seen as the, very good; so, all of you are familiar now with the quantities. Now, that I can write it for example, as F transpose half of F inverse transpose; just multiplying by or rather I will do the other way, so, that is easier.

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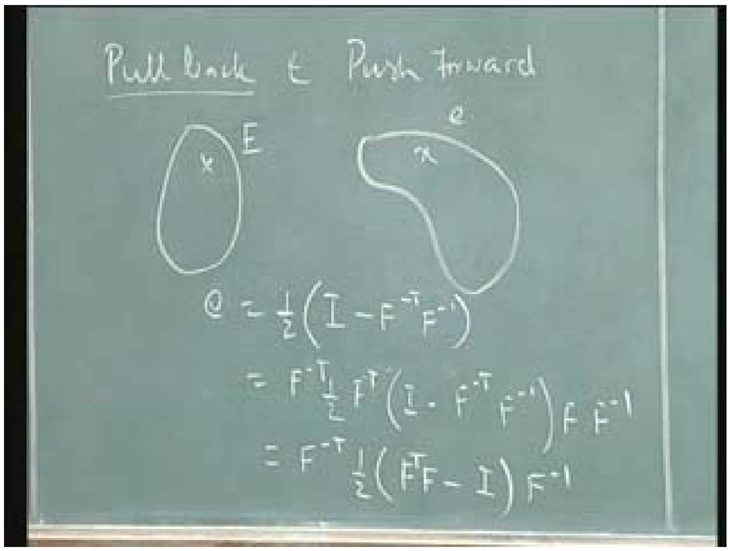

F inverse transpose, because ultimately I want, you will just see why I want this. I want e there, into I minus F inverse transpose F inverse and I want F there. So, I will put F here and F inverse here. So, what I have done essentially? F transpose F inverse transpose F transpose which is I, this quantity, and then look at this quantity. So, this can be written as F inverse transpose half of F transpose; that F, I am bringing it inside, F minus, when I multiply that, that will become I F inverse.

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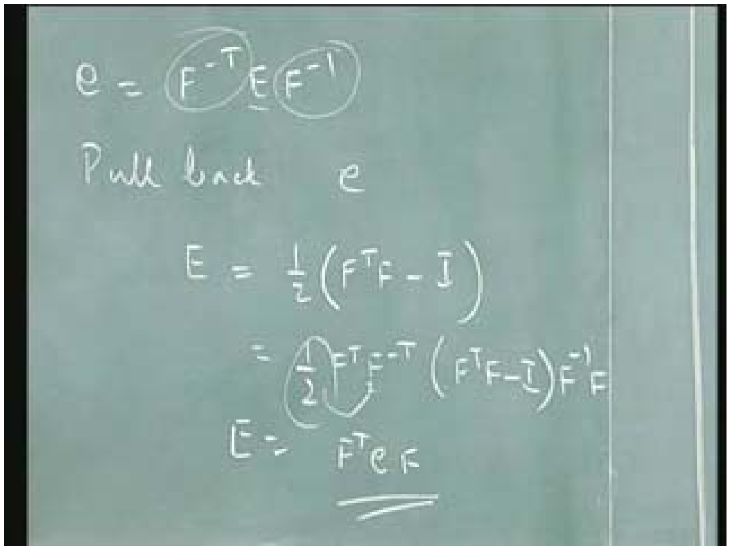

This, you can recognise that as e, so that, capital E rather, so that small e can be written as, look at this here, F inverse transpose E F inverse. Look at that expression there. What does it do? What does that operation does, what does this operation do? It pulls back E. This operation, operation in the sense that F inverse post E and F inverse transpose pre E; what does it do? It pulls back this capital E or the green strain or rather pushes forward, I am sorry, pushes forward the capital E, which belongs to the reference configuration, to small e which belongs to the current configuration; in other words, material configuration and spatial configuration; in other words, again Lagrangian, Eulerian. Now, these are synonymous terms. So, this is a kind of an operation which pushes forward E. This is called as the push forward operation.

Pull back is the inverse of this operation, pull back is the inverse of the operation. In other words, the question is what should I do like this in order that I can pull back e, which means that what is that quantity, capital E? How do I get that quantity capital E by pulling back small e, because after all both of them have a relationship? Both of them are strains - one is referred in one co-ordinate another is referred in another co-ordinate. So, how do I now pull back small e, so that I will get capital E, like what I did here?

Let us see how you do that? Do this in a similar fashion. What I do require in the left hand side is capital E. So, capital E is equal to half into F transpose F minus I. So, I need F inverse transpose F inverse. So, what do I do? I multiply by F transpose F inverse transpose, so that that will be half into F transpose F inverse transpose multiplied by F transpose F minus I into F inverse F. Rearranging this, the same fashion as we did, bringing that 2 inside, rearranging that I will get F transpose e F. This is called as the pull back operation; pull back of e. Yes, that is exactly you could have got it. It does not matter; just you could have got from here also. F inverse transpose, you multiply by F transpose, this will go off, F; we will get this equation. So, this is what is called as the pull back.

Pull back and push forward operations are mathematical operations for physical things, what you have seen. I had talked about in words, like we are referring to the reference co-ordinate; we are referring to current co-ordinate and so on. This is a mathematical expression, but unfortunately these mathematical expressions are not the same for every tensor. In other words, it is not that if I give a quantity in the current coordinates, I can get a quantity, an equivalent quantity in the reference co-ordinate by applying say, the pull back operation. Say for example, we know sigma; we know sigma belongs to the current co-ordinates, because it is a Cauchy stress. So, it is not that if I take Cauchy stress and just apply this, I will get an equivalent stress, an equivalent stress.

There is a subtle difference between the way quantities like stress behaves and the quantities like strains behave; most, many of the strains, rather, one or two things, let us not worry about that, but, more popular strains. Why is that? Do you have any idea why the stress does not behave in the same fashion, in the same fashion why is it that the stress do not behave. The answer is very simple, because when I go to stress work, stress work, say for example, you would have seen in your earlier classes on say, virtual work; virtual work. In the virtual work, internal work is what we call as stress work. People call this with different names. What is it? Delta epsilon transpose sigma. You have two terms there. What are the two terms? Stress term and a strain term. When you have these two terms, the stress term and the strain term, one is the conjugate of the other; one is the conjugate of the other.

Ultimately when I calculate the stress power, whether I do that in the current co-ordinate or whether I do that in the reference coordinates, I should get the same stress power. If I have, for example, the same transformations for the stress, then when I multiply that by sigma, in other words, delta epsilon transpose sigma, then I will not get, then F will enter into my F term or stress power term and I will not get, I will not get the same value on two reference configurations. In other words, stress is that is what I called as the conjugate of the strain. This is similar to covariant and contra variant basis. We have not covered covariant and contravariant basis. These things behave like a covariant tensor and the stresses which are conjugate of this, is called as the contravariant tensor, but we will not go into these details. But, let us now look at how quantity like stress behaves when we do a push forward or a pullback operation. The concept is very simple as I told you that we will look at stress power.

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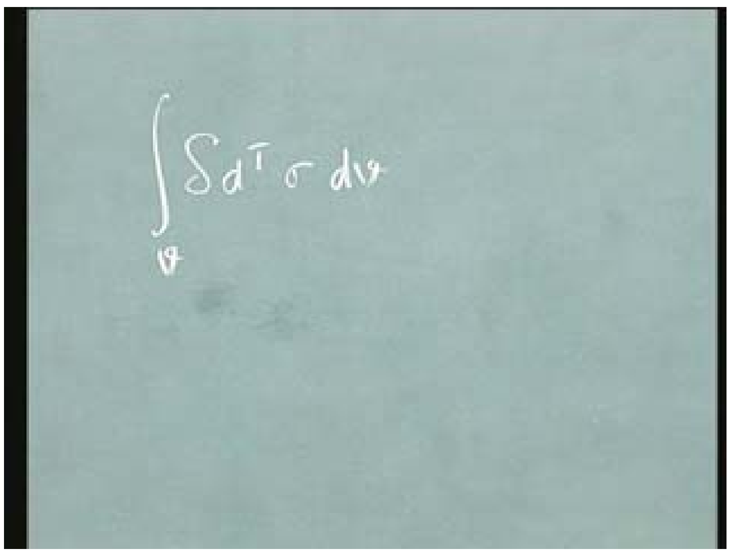

When I say stress power, actually we replace that epsilon by epsilon dot or in other words, what we say delta d transpose sigma dv, the first term, remember that this is the first term, actually this is small v because now it is in the deformed. This is the first term which you had seen in one of your earlier classes on virtual work principle. Now, this is what we have to look at and see how this transformation takes place when I now convert this guy here to capital V. Is that clear? The ensuing transformation will result in the transformation for stress. When I convert that to d capital V, then that will be the pull back transformation, such that you will see now that the stress power is not affected by varying the co-ordinate system or the reference co-ordinate or whether it is the current co-ordinate system.

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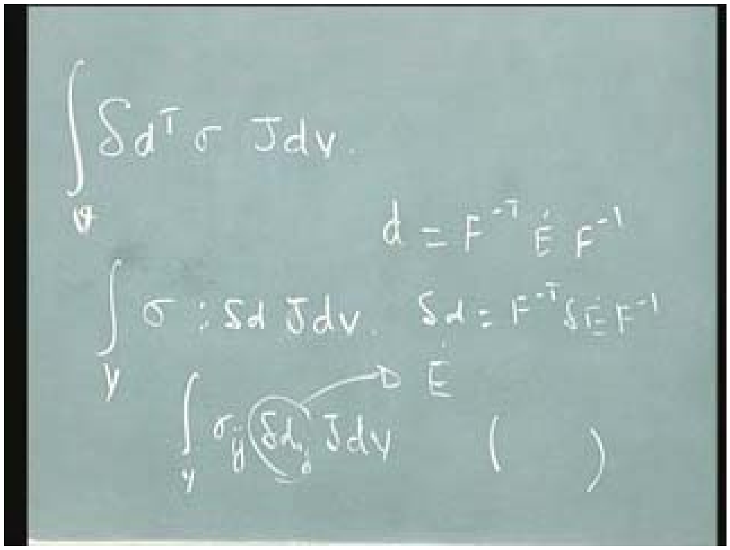

So, the first step, obviously, is to replace the d small v by J d capital V. This, in a vector notation, is also written as sigma double dot delta d J dV. What is this? Sigma$_{ij}$; in fact sigma being symmetric, sigma$_{ij}$ delta d$_{ij}$, so if you want, write this sigma$_{ij}$ delta d$_{ij}$ J dV. Actually, I am still not correct. I have not, I have done only the first step. What is the second step I have to do? I have to replace this quantity. What is this quantity d? I thought we had already seen. No, d, yesterday we saw that. I used a symbol d only, if I remember right. Please have a look at that. Yes, rate of deformation tensor, what is that? That is the, what is the rate of deformation tensor? That is the symmetric part of l. So, rate of deformation tensor being …, actually you can easily guess it because it is in small d, is the one which comes from small l and small l has grad v, small g grad v. That means that this quantity refers to the current co-ordinate system. So, it is not enough if I just replace that by J d capital V. I have to replace as a next step, this as well to the current co-ordinate, sorry to the reference co-ordinate system. In other words, I have to express this in terms of what? E dot.

What is the relationship between d and E dot. I think we had already seen that; d is equal to, no, d is equal to, so e is equal to E dot, you see, F inverse transpose E dot F inverse. What is delta d now? Delta d obviously it goes to E dot, so, F inverse transpose delta E dot F inverse. Now, let us see you manipulate it and see. I give you two minutes. Let us see whether you can manipulate this and get it. That is going to be a very nice manipulation and let us see how you do that, so that you will get E dot there and another quantity, so, you will get two quantities. In other words, I am giving you in a nut shell, what answer I am looking at. I am looking at an answer where I will get one quantity, which I can call as the pull back operations result and another quantity which will be based on delta E dot. Let us see whether you can do that. Look at that.

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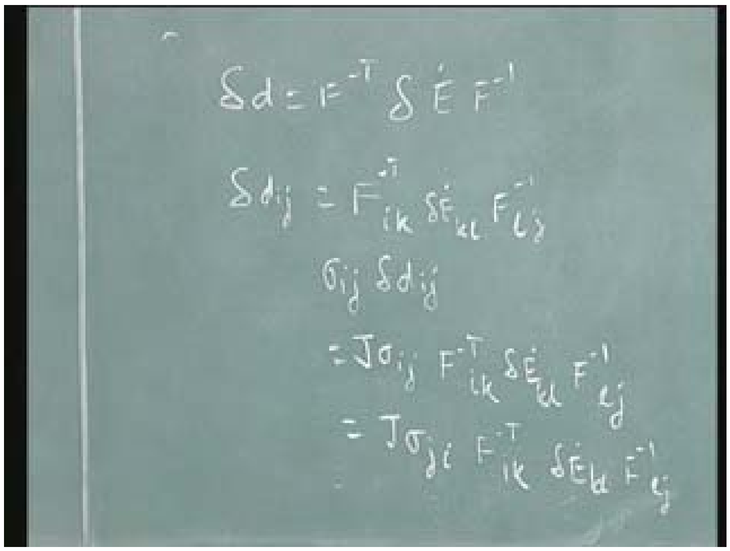

Let me help you out. I want you to do this, because that will be, that will give you lot of confidence to do the things, manipulations, because in non-linear continuum mechanics as well as in finite element lot of things depend upon manipulations. You have to, you have to know how to do manipulation with this kind of tensor quantities. That is the reason why I feel that you should do. In fact, most papers are difficult, are rendered difficult, because many of the simple manipulations you may not be able to derive unless you have some practise. So, write this down, first step; let us write it down in the indicial notation.

Let us see; say, delta d_{ij}, how will you write this down in an indicial notation? Obviously, I should start with i and end with j. In the middle, I can write delta$_{ik}$ delta E dot say, k l F inverse l j. Substituting it in sigma$_{ij}$ delta d_{ij}, of course J is also there. J we will keep, we will reserve it, we will put it in the end. So, this can be written as sigma$_{ij}$ F inverse transpose i k delta F dot k l F inverse l j. Is that clear? Delta F, not F dot sorry E dot. Still you have to manipulate, because I have to get that delta E dot outside. I should write delta E dot outside with another quantity here, then only I will be able to get the stress.

Let us see, let us do the next manipulation. Look at this. This can be written as of course, J, if you want I will add that J. Since sigma_{ij} is equal to sigma_{ji}, I can replace that ij by ji. Then I can write that as J sigma_{ji} F inverse transpose ik delta E dot kl F inverse lj. That is the first step. In fact, since I put j and i, then I can write that as j and that to be i or I can keep it as it is. How do you further do that? Just try that out.

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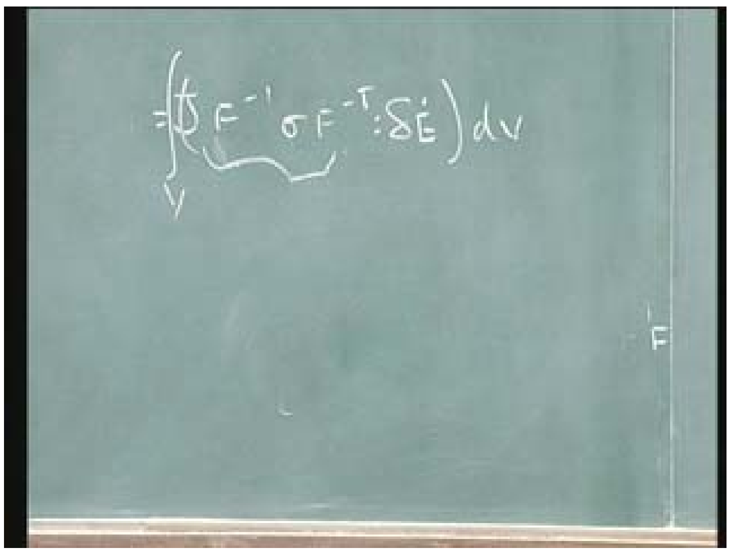

Ultimately, you can write that quantity to be J F inverse, you can bring sigma inside; F inverse sigma F inverse transpose delta colon delta E dot; by juggling this i and j, you can keep juggling this, so that ultimately you will get that. That will be the stress power integral of this V over dV. Now, all of them are dummy here. That is all. What I made use of is that all of them are dummy, because the result is a scalar. i and j, j and i all of them are dummy. I can keep shifting that up and down three steps. Then, I will get that F inverse sigma J sigma F inverse transpose delta E dot dV.

Now, look at this quantity. That quantity is the one which goes with delta E dot or the conjugate of delta E dot in order to give me stress power. So, that quantity can be called as a stress. What is the physical meaning? There is no physical meaning to it.

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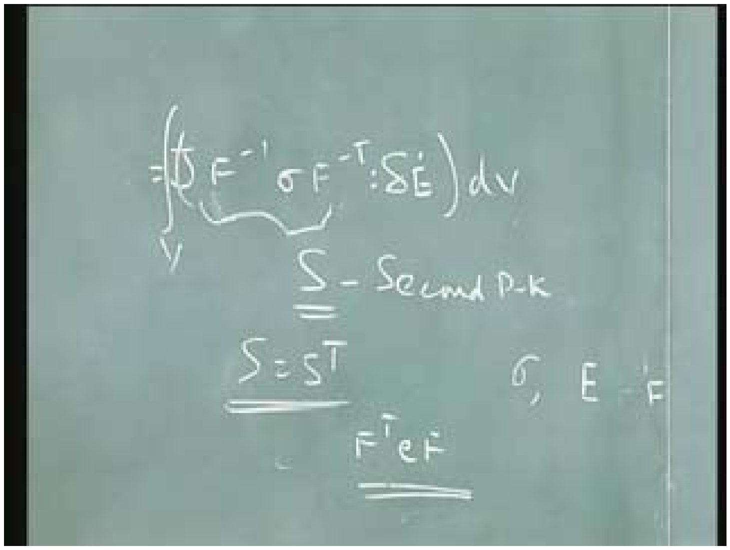

That quantity can be called as a stress and that is called as the second Piola-Kirchhoff stress, S. That is called as the second Piola-Kirchhoff stress, S. Essentially, what is that you have done? You have pulled back sigma, sigma to the reference co-ordinate. You have pulled back sigma to the reference co-ordinate. So, the pull back operation is slightly different now, than what it was in the previous case and the resulting stress is the second P-K, Piola-Kirchhoff stress.

The advantage of this is that look at this. Suppose I take the transpose of S. Obviously, then F inverse transpose becomes F inverse; sigma being equal to sigma transpose, so, F inverse transpose, F inverse rather becomes F inverse transpose. So, S is equal to S transpose and hence, second Piola-Kirchhoff stress is a symmetric stress tensor. It is a very important conclusion that the second Piola-Kirchhoff stress is a symmetric tensor. Note this very carefully, another important point. Now, I have defined casually three measures, stress measures. In fact, I can go on defining another two more, if need be I will do that later also. But, note that it is not possible for me to pick one stress from the basket of stresses and another strain or strain rates from the basket of strain rates and then say that, for my problem I will use say, second Piola-Kirchhoff stress and I will use say e, small e as the strain measure. It is not possible to do that and you will see that there are conjugates for every stress or every strain.

For example, if you see that, if you take P, it has its conjugate say, F dot. In other words, what is conjugate? What we mean to say is that whatever be the stress measure and the strain measure we use, there should not be any change in the scalar; there should not be any change in the scalar. In fact, we arrived at second Piola-Kirchhoff stress from that concept. So, we are totally wrong if I say that I will take sigma and I will take E and do my problem. This is not correct, they are not conjugate measures. Is that clear? Though you have a number of stress measures, number of strain measures, whenever you do a problem, you have to be careful in selecting one and the other in such a fashion that they go together as far as this is concerned. Is that clear?

Note the difference between this pull back operation, the pull back operation just we defined for E. What was the pull back operation which we defined for E? That was F transpose e F; compare that with this. They are different. So, we have defined the second Piola-Kirchhoff stress measure as well. With that we will stop for a minute and do a couple of problems, so that we become familiar with what all we are doing. We will continue these problems in the next class as well, so that we have some exercise to manipulate and get certain results. Can you note down a problem?

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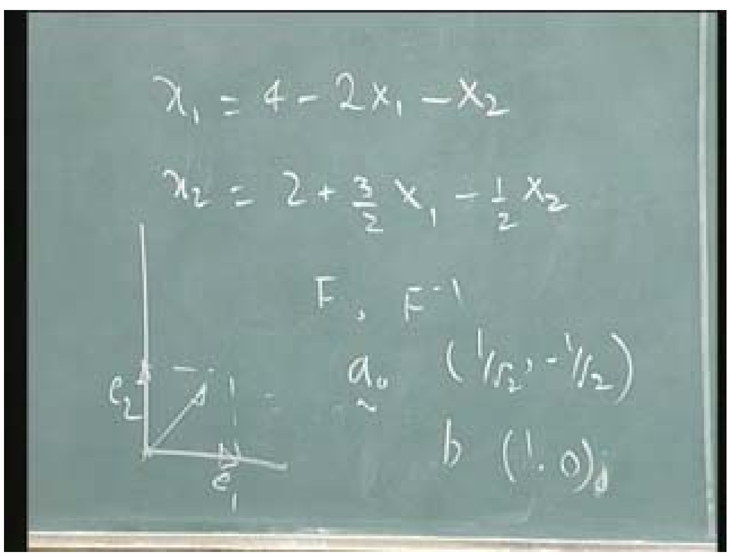

For a two dimensional problem, the deformation for a two dimensional problem, the deformation is given by the explicit equation x_1 is equal to 4 minus $2X_1$ minus X_2 and small x_2 is equal to 2 plus 3 by 2 X_1 minus half X_2. Now, determine the matrix representation of the deformation gradient. That is the first step; determine the matrix representation of the deformation gradient. Also, determine the inverse; also determine the inverse. That means determine F and F inverse. I am sure all of you know how to do F inverse of a matrix and study the deformation of a square, of a square say, formed by say, vectors e_1 and e_2. In other words, tell me what happens to this square under this deformation, which is the deformation that is where you can say that it is at zero. But before you do that, determine the elongation of a material line element, small a_0 given by 1 by root 2 comma minus 1 by root 2. This means that it is in the reference, this reference co-ordinate, so, it is something like that. Also, determine the length of b, original length of b rather; original length of b, where b is given by 1 comma 0. That means that b is in the deformed co-ordinate.

So, your first step is to find out F. Let us see what is F? I will give you two minutes and then, we will write down the result for F. Look at that and then, my first step is to write down F.

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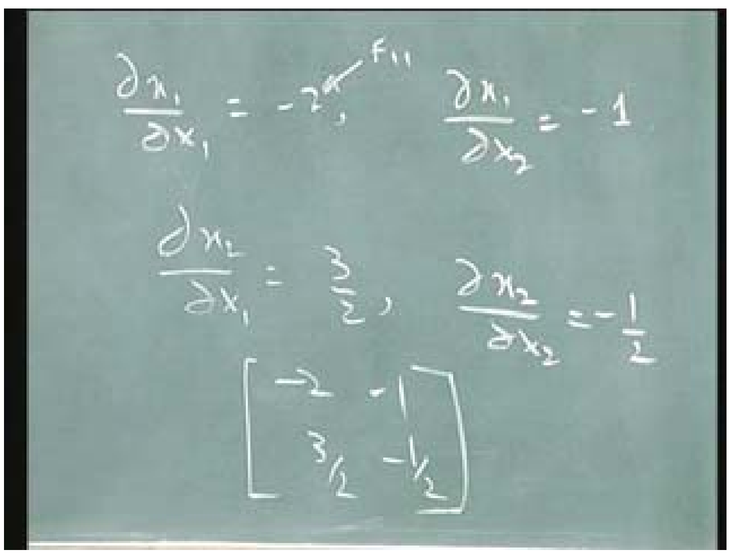

What do I do? Simple; dow x_1 by dow capital X_1 is my F_{11}, because I told you that we have to write it in a matrix form. So, dow x_1 by dow capital X_1 that gives me minus 2. So, that becomes F_{11}. Then, dow x_1 by dow capital X_2, which is going to become F_{12}, so, that becomes minus 1. Then, dow x_2 by dow capital X_1, dow x_2 by dow capital X_1 is equal to 3 by 2 and dow x_2 by dow capital X_2 minus 1 by 2. In a matrix form this can be written as minus 2 minus 1 3 by 2 minus half. The first thing I want you to look at is, it is not symmetric.

Unfortunately, I have seen over the years, many students making the mistake of just like that assuming that F is equal to F transpose. They are so used to symmetric matrices in their earlier mechanic, structural mechanics courses, K is symmetric, sigma is symmetric, epsilon is symmetric, so, they think that every matrix that comes in continuum mechanics or structural mechanics is symmetric. See that; so, you have to be very, very careful in it, because you would see that many formulations in non-linear analysis would give you, will land you up in, I would say not even give you, land you up in unsymmetric matrices. They have its own problems; the storages are going to be high, the time required to solve them is going to be high and so on. So, please note that.

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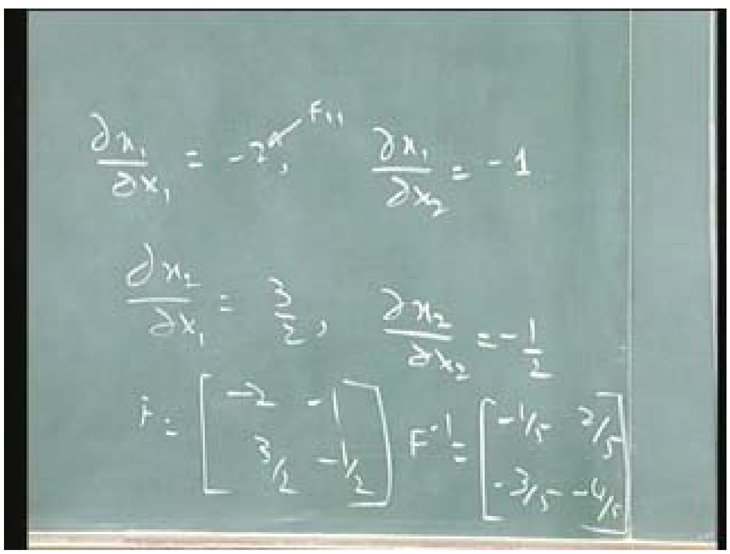

Now calculate the F inverse for this. Quite simple; calculate the determinant and put that. Yes, is it minus 1 by 4, minus 1 by 5? Let us give some time, let us see; minus 1 by 5 2 by 5 minus 3 by 5, very good, minus 4 by 5. Fine; so that done, I have to find out what happens to a_0? How do I find that out? How do I find out what happens to a_0?

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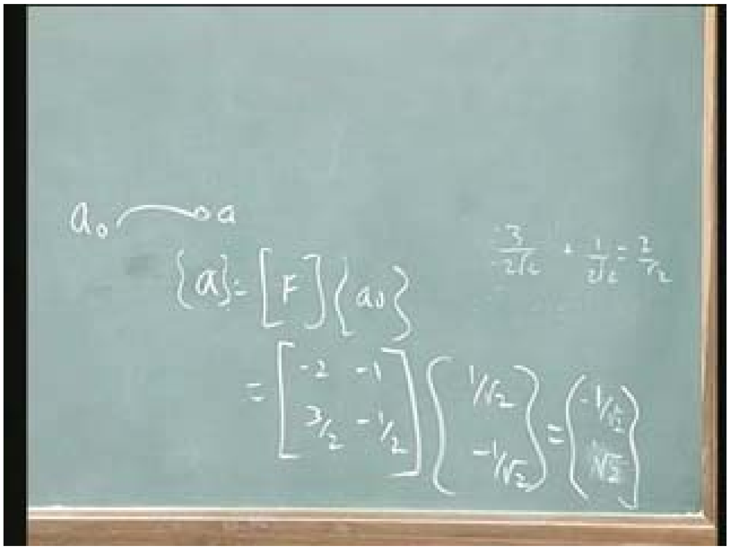

Let us say that the a_0 is now transformed into a. So, how do I write a? a is equal to, correct, so, F, let us see what happens. Ultimately, I want you to find out of course, the length of a_0. a_0, I am sure you know that the length of a_0 is, what? 1, half plus half, 1. So, what is the length of a and see how you can calculate the vector a? This is what is F. That is all, you know, there is nothing great in it. So, that will be minus 3 by root 2 and 1 minus minus 2 minus 2; no, this is minus 2 and plus, plus 1 by root 2. So, this is plus 1 by root 2 minus 2 1 by root 2, so, minus 1 by root 2 and this one 3 by root 2 root 2; pardon? root 2, shall I take your word? 3 into 2 by root 2 plus 1 by 2 into root 2, so 4, 4 by root 2. No, no. This is 4, so 2 by root 2, so, root 2. Are we correct? Everything is fine. Actually my, this vector is not right.

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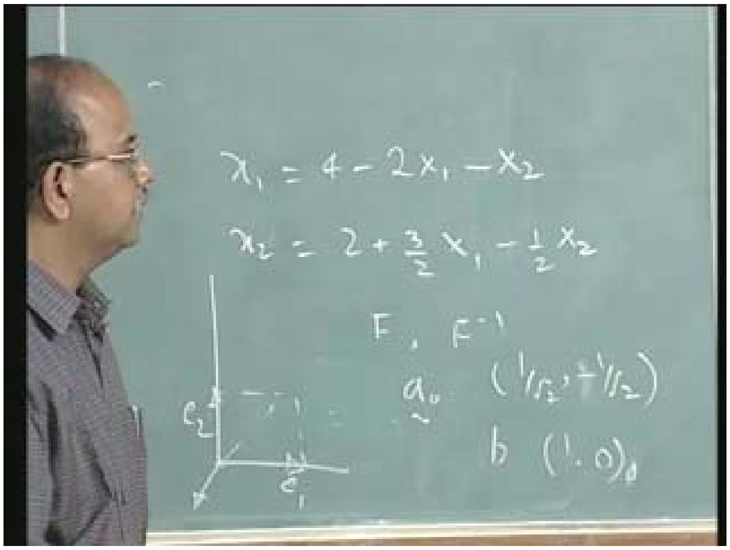

So, if I have to keep this vector, then actually it is both of them are plus. Anyway, it does not matter. So, that is the opposite direction, minus is there. So, what is the length?

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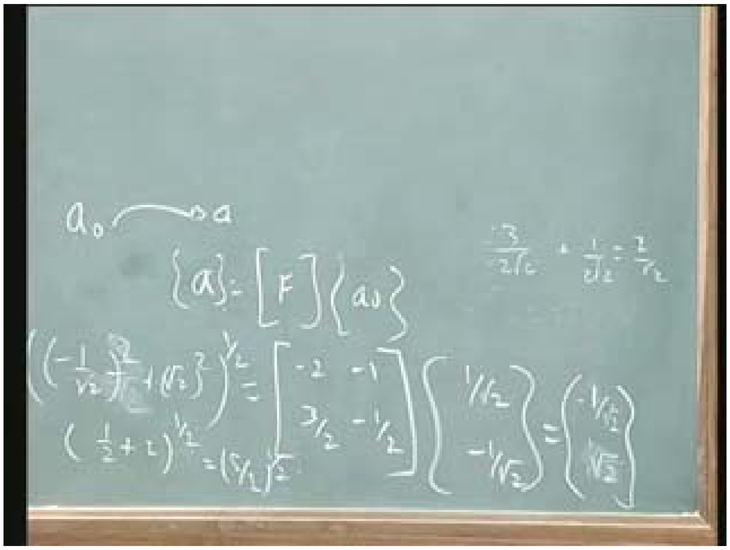

Length is a_0 is 3 by root 2. Is that clear? Is it that, just check that up. 1 by minus 1 by root 2 will become 1 by 2, half plus, no, this is not; very simple calculation. Minus 1 by root 2 whole squared plus root 2 whole squared whole thing power half. So, this will become 1 by 2 plus 2 whole power half. So, that is equal to 5 by 2 power half.

Now, apply for b. Apply the same thing for b, using F inverse. What happens to b? What happens then? By the way, before we proceed, let us look at this F. What is this type of transformation? Look at this F; F does not depend upon X. So, this transformation is called as the homogeneous transformation, very good; so, this is a homogeneous transformation. That means the F that is to be applied is the same throughout the body, does not depend upon capital X_1 capital X_2. So, this transformation is called homogeneous transformation.

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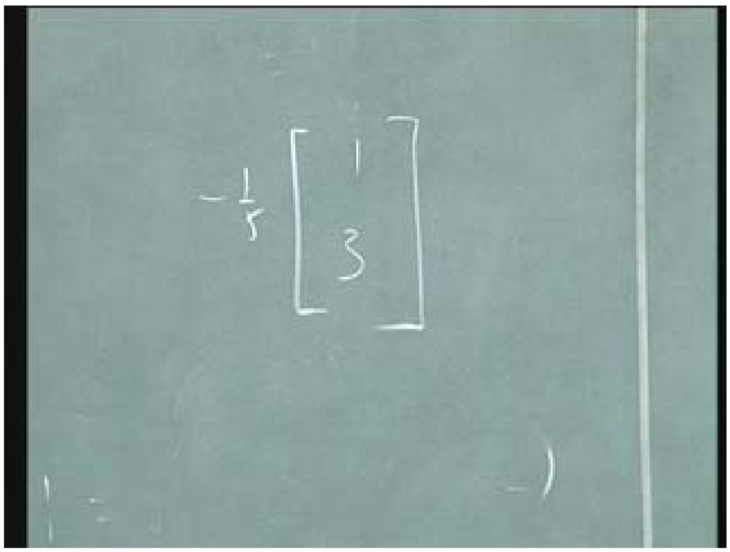

Calculate what happens to b, so, what is the answer. Minus 1 by 5, minus 1 by 5 into 1 3; so, your answers are minus 1 by 5 2 by 5 minus 3 by 5 and I think this is again, it is not correct, 4 by 5.

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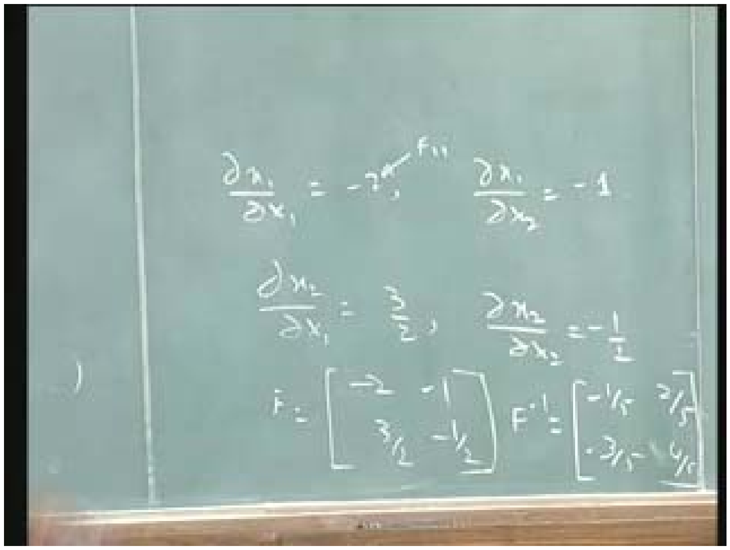

It is not minus 4 by 5, because that minus will go; that is 4 by 5. Be careful with your calculations, so, it should be 4 by 5, not minus 4 by 5. Just check that up, your F inverse calculation. So, ultimately you have to be in touch with simple things.

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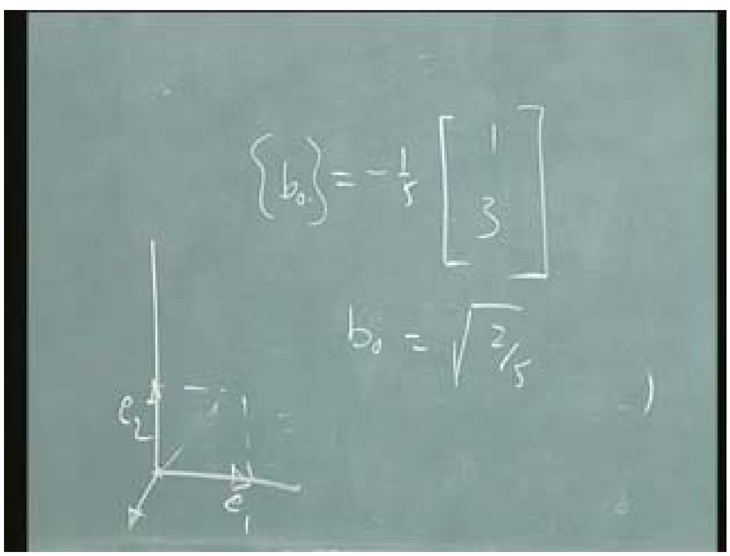

So, that is what I would call as say, b_0 is equal to minus 1 by 5 1 and 3 and what is the length of that? The length of b_0 is equal to 2 by 5, root of 2 by 5. In other words, what it means is that a line element which is now currently of unit length was in fact root of

2 by 5 and vice versa for the other guy, where we had a unit length vector that takes 5 by 2 or root of 5 by 2. That is the first step.

How do you calculate now, what happens to a square like this? Yes, exactly; so, you calculate what happens to these vectors and then do this transformation. Please do that, it is very straight forward and see what happens to this square, plot that square as well. I will leave that to you, it is very straight; now that we have done this, it is not going to be very difficult. Let us move to the next problem. Please take down the next problem.

In the current configuration of a continuum body, in the current configuration of a continuum body, a certain physical quantity, whatever be the quantity, a certain physical quantity is given; a certain physical quantity is given in space and time.

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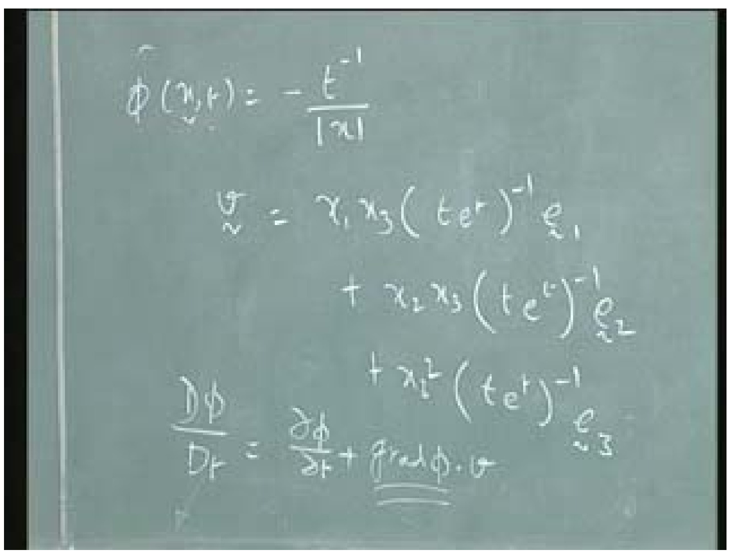

That physical quantity say, let me call that as phi, space and time, is written as minus t inverse by mod of x. I will remove all these things; so, what is, whatever be the quantity, we are not worried about that. Note that the space is characterised by x, time is characterised by t. This quantity is defined at all places except the origin, which means that we do not define it at t is equal to zero and at x is equal to zero, because at that point they become infinity. So, this point, this quantity is defined all over the place.

Now, let an observer a travel with the velocity v, which is given as x_1 x_3 into t e power sorry t e power t inverse e_1 plus x_2 x_3 into t e power t inverse of that e_2 plus x_3 squared into t e power t inverse of it e_3. An observer is moving with this velocity. Determine the spatial, of course, time derivative and the derivative or the rate of change with or which? this observer will see as he moves with this velocity of v. Determine the spatial time derivative and determine the derivative of this quantity as seen by the observer, who is moving with the velocity, small v. In other words, find the time rate of change of phi, spatial as well as seen …… by the observer.

Note that, in this place the only difference between what you have seen before and what I have given now is that in the previous case, when we did the theory, the particle was moving or the line element with the material points that was moving. Here the small difference is that the observer is moving. It does not make any difference. In other words, what I had simply asked is what is a material time derivative?

Look at your notes in the last class. You will see that D phi by Dt is what we are talking about is equal to dow phi by dow t plus grad of phi dot v. Note that that phi is with respect to grad small g or this is dow phi by dow x you have to determine. Note that this guy here is also with respect to the current co-ordinate. Note that v is also with respect to current co-ordinate. We will be very careful in that and see that this v, you look at these things here; x_1 x_2 x_3, all these things are with respect to the current co-ordinates. Your job is now to find out this, this, take the dot product that is all. Now, why I gave this problem is because of the fact that many times in not only in continuum mechanics, in the whole of analysis itself we have to look at relative things.

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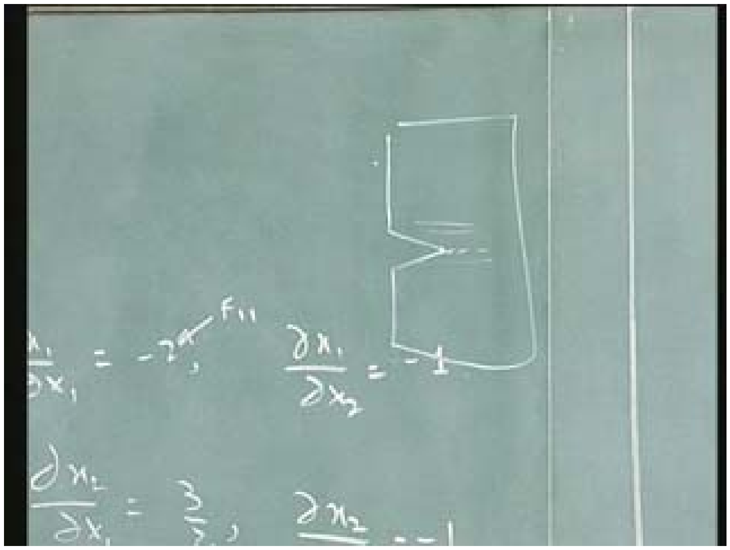

For example, if you are looking at crack growth in a body, crack is growing like that. You can also imagine that, sitting at the crack tip, you can imagine that the material particles are moving right across you. So, these kind of relative things are very useful to study many complex problems. We will talk more about that in the next class.