Advanced Finite Element Analysis

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Lecture - 16

Before we start that derivative, which I said we will do in this class, I just want to make a comment on what are called as projection tensors, which are very useful in many applications.

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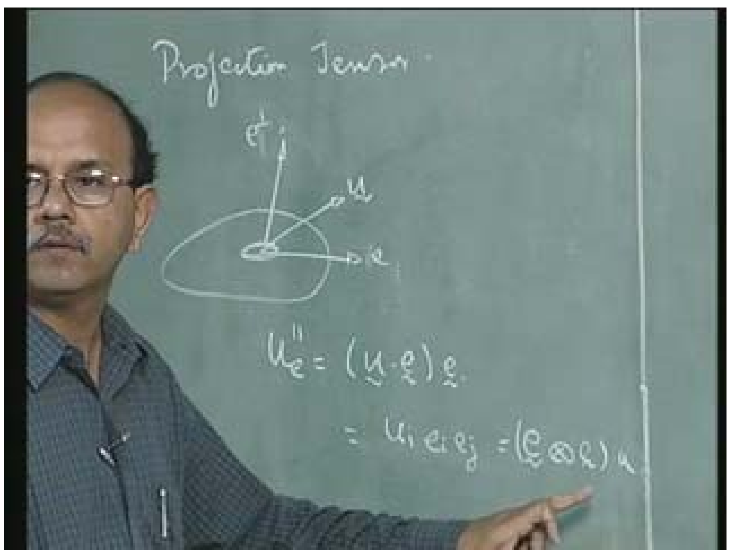

So, let us see what this projection tensor is. Let us say that I have a vector u. Now, I want to resolve this vector say, in the direction of e and in a direction perpendicular to e; u parallel to e and perpendicular to e. So, what I do is very simple. If I want to have u say, parallel to e, then this is nothing but u dot e e. If you look at that in indicial notation, then it becomes, this becomes u_i e_i e_j, which can be written as e dyadic e u. Note that, e dyadic u is a second order tensor where, you take a vector, another vector and do this operation, so that say for example, if you have a dyadic b, suppose I get say, capital C, then the component C_{ij} is equal to a_i b_j.

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This is an operation which is opposite to what we do by contraction. So, we see that u parallel to e can be determined by this expression.

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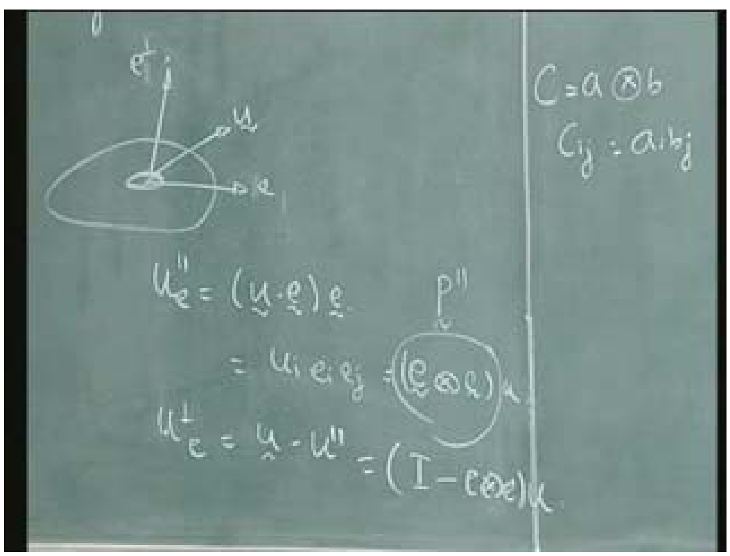

This is called as the projection tensor parallel to u or sorry e. Now, if I want u which is perpendicular to e, then this is nothing but from simple vector algebra we know that this is equal to u minus u parallel which means this is equal to I minus e dyadic e u.

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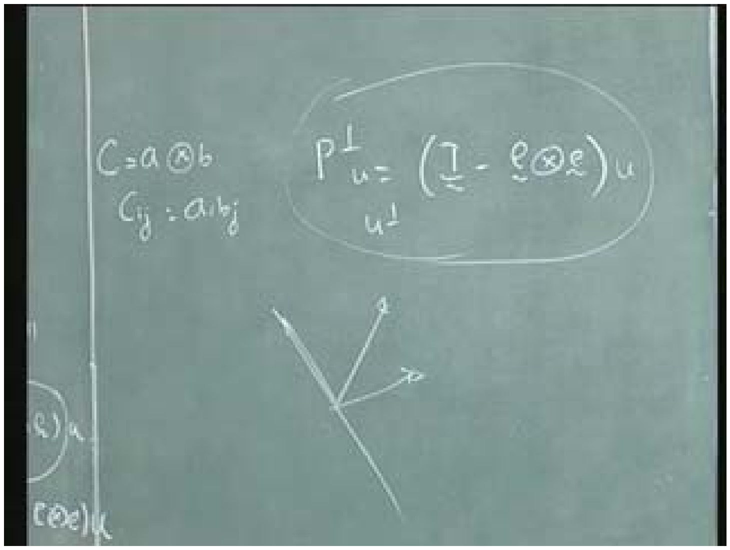

The projection tensor in the perpendicular direction is given by I minus, I of course is a unit tensor and that when operated or u will give me what is perpendicular or u perpendicular. This is important in many respects say for example, if you want to look at the stresses that are acting say, normal, if this is the stress that is acting and I want to resolve this into a normal stress and a shear stress, then you can use the projection tensors in order to do that; you can use that in, also in Eigen value problem, then this projection tensor; so, it has quite a few things lined up for it later; we will do that when we come to that place.

Now, we will proceed with what we called as the rate quantities.

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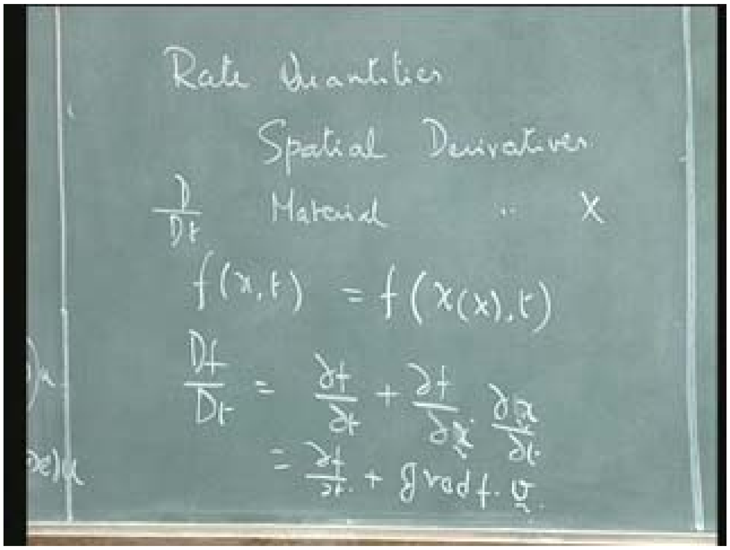

We said that you can distinguish derivatives into two categories - one is what is called as the spatial derivatives and another is called as the material spatial derivatives and what we defined as material derivatives. What is a material or material derivatives in a sense, material time derivatives and spatial time derivatives? Now, the material time derivatives are the derivatives of a quantity, keeping, keeping capital X a constant; keeping capital X a constant. For example, if I have a quantity say, f of x comma t, if f of x comma t, I want to find out the material derivative of this, that means that I can write this down as f of say chi of X comma t, then the material time derivative is obtained by keeping X a constant at a particular X. We will look at this physically in a minute after we derive what this means.

As I told you in the last class we usually write that as D by dt; D, capital D by Dt. So, D of f by Dt, which is the material derivative of f by chain rule is written as dow f by dow t plus dow f by dow x dow x by sorry dow x into dow x by dow t. Yes?

Student: we are not considering the range of materials that will …...

Let me answer that question, what do we mean by this? Just wait a minute; let us understand this and then I will explain to you what this means, which means that the material time derivative has two components, two components and let us try to understand what these two components are. But, before we do that, let us write this down more neatly in another fashion. This can be written as dow f by dow t plus, what is dow f by dow x? Grad F; grad F dot v. Note that now this is a vector, grad F dot v. Now, let us understand what these two means, these two terms mean.

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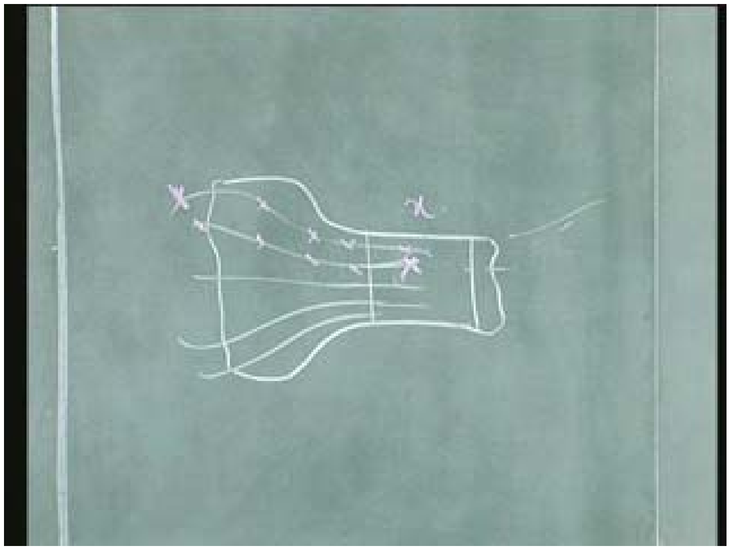

Now, in order to understand that let us look at a physical picture from fluid mechanics; can be extended to solid mechanics for example, an extrusion problem and so on. Now, let us see how we are going to study say, fluid flow through a channel like this. Let us now concentrate in this region generally and more specifically, let us look at what happens to that point here, to this point. Let us now follow the velocity of the material at that point.

Now, there are two things that can happen or that can be envisaged. One is, you focus at that point, see whether that point in space has any change in velocity; that is one perspective. If you really look at it what happens to material point, a material point may start here, may flow through this channel, may come here, may come here, may come here like that; this point may come here like this and come to this point. So, at that point it might have the velocity at a particular instant what you have noted. Further down it may go, there may be a change in shape, change in velocities and all these things. So, the first point or the first way, first perspective where you look at a point and watch what happens to that point is an Eulerian perspective.

The other prospective where you sit on the material and as it goes you capture its velocity is a Lagrangian perspective, Lagrangian perspective. Of course, whether it is an Eulerian prospective or Lagrangian perspective, for a material point, at that place the velocity is the same. I am sitting there say, let us say I am travelling in a boat, I am coming here; you are watching this point and my boat comes to that point, you capture a velocity, I capture a velocity locally with some device; we will have the same velocities. Now, look at it from another point of view. For you as you travel, this is no more a steady state; this is no more a steady state. So, you are the material derivative. As you travel, your velocities may change or does change, so, your velocities change with time. But, at this point of time, here, for you watching only that point your velocity may or may not change with time. So, from one perspective the velocity changes with time; from another perspective, the velocity need not or may not change with time. That is the difference between a spatial and the material derivatives.

You have a point here and the derivative at that point is given by this term. If this points velocity changes with time, dow f by dow t changes. Suppose it remains constant as well, then there is a convective term, the second term, which carries that velocity or change in this thing and so, in the second term, it is a convective term and is due to that velocity there. That is the difference, physical meaning of the difference, between a material time derivative and a spatial time derivative. Spatial time derivative, just to summarise, we are looking at small x; material time derivative you are looking at capital X and it is very important to realise that as far as the velocity is concerned.

Yeah, dow f by dow x dot dow x by dow t; it is just chain rule. So, that becomes grad f dot v.

This is what, so, if the velocity here is what we call as dow f by dow t, suppose there is with respect to time, that point the velocity changes, that will also be reflected in the material derivative. It is a very straight forward definition. Now, what happens if I have, this is okay for a field f. What happens if it is a vector field? Note that grad f makes this tensor to jump one order.

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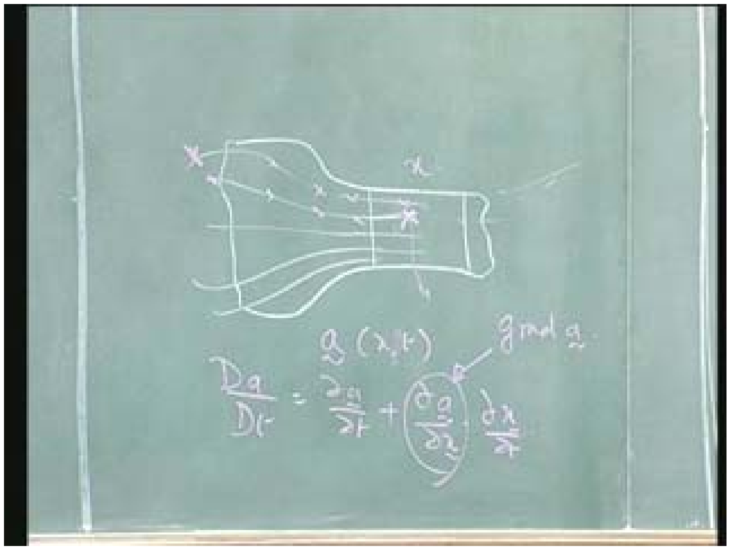

What happens if this happens to be or this f happens to be say, some function of say, a vector function of x comma t, then what happens? You would see that Da by Dt term here is equal to dow a by dow t plus dow a by dow x dot dow x by dow t. What is dow a by dow x? This is a second order tensor.

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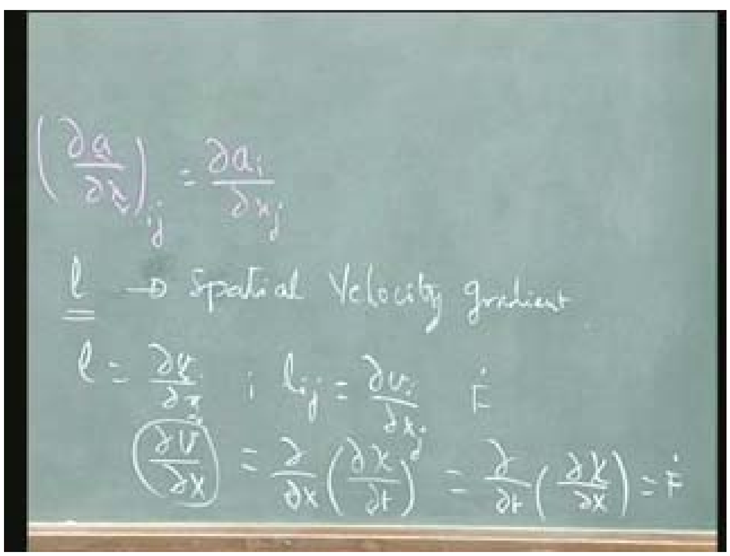

In other words, dow a by dow x, if I want to write it in component terms say, it becomes dow a_i by dow x_j, please note that. There is or in other words, this can be written as again grad a, pushing the order of the tensor. Is it clear? Now, let me define a spatial velocity gradient l; spatial velocity gradient or spatial velocity gradient tensor, however you want to call it, l to be dow v by dow x. Note that it is dow v by dow x, v and x being vectors, so, obviously, l is a second order tensor. l say, let us say that l_{ij} is equal to dow v_i by dow x_j. Note that this is different from material velocity gradient which is written as dow V by dow X.

What is this material velocity gradient, let us see that. Let us see what this material velocity gradient is. Write that down; let us see what that is. Please try this, what this is, which is of course you can write that as grad, this is nothing but say, capital grad V; you can write that down as capital grad V. What is the relationship between say, F dot and this? You can check, just check that that is what I wanted. Check what the relationship is between F dot and that. Very simple, there is nothing there. That is equal to dow by dow X of, what is V? dow small x by dow t, which can be written as dow chi by dow t. Since X is independent, I can switch these differentials, so that I can get that as dow by dow t of dow chi by dow X. So, what is this? F dot.

What is F? This is F; dow by dow t of that. So, that is F dot. Now, what is the relationship between l and F dot? Very straight forward, it is nothing difficult, just try that. What is the relationship between l and F dot? Just do a chain ruling and then, yes please.

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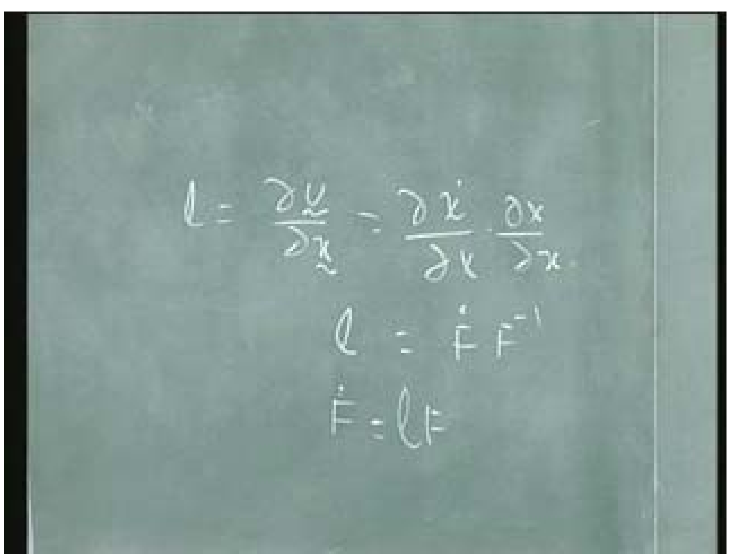

Note that, l is equal to dow v by dow x and from here do that, then see what that is. I am deliberately giving you, because I want you to remember what all we did. Please look at that. If you want, I can help you in one more thing, dow chi dot say, that is what we wrote there, by dow X dow X by dow small x. F, is it, just have a look at that; dow chi dot, this is what? We just now saw it, F dot and what is this? F inverse that is all. It is very simple, you know. That is why I wanted you to do it. So, F dot is equal to l F. It is a very important relationship, F dot is equal to l F. Is that clear?

Now, this l or velocity gradient tensor is a very important quantity and in fact, this is very similar to your u i comma j, which you would have seen in your earlier classes. In fact, you look at that, what it is? v i comma j; so, l_{ij} is v i comma j. So, it is very similar to u i comma j. If you remember, in earlier classes you had split u i comma j into an epsilon and an omega part, symmetric and a skew symmetric part. You can do exactly the same thing with respect to l. So, let me remove this; let us see how we do that.

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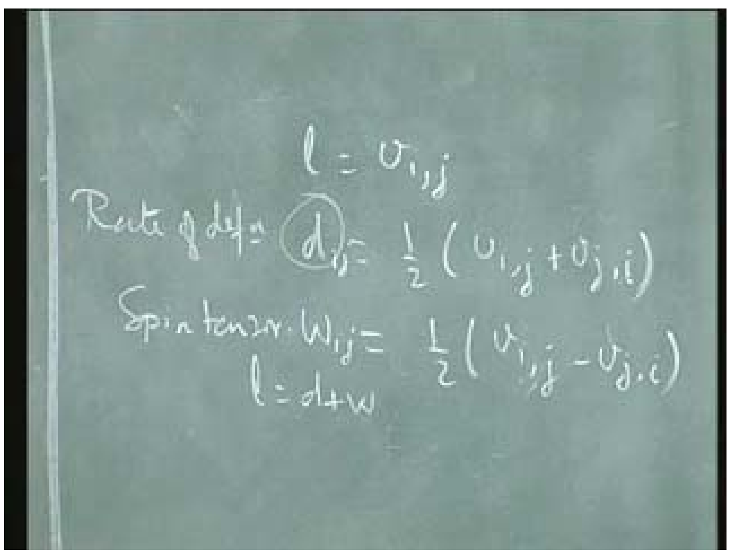

L, as I told you, is v i comma j. Let us say that the symmetric part is given by d and d is equal to half of say v i comma j plus v j comma i and the skew symmetric part is given as half v i comma j minus v j comma i, so that when you add this up, so, l is equal to d plus W. d, this d is an important quantity and is called as rate of deformation tensor; is called as rate of deformation tensor, rate of deformation tensor and this omega is called as spin tensor. Note that both d has d_{ij}. In fact, you can write that as d_{ij}, written in that fashion, $omega_{ij}$ written in that fashion; so, rate of deformation tensor and spin tensor.

Now is rate of deformation tensor the same as that of the strain rate say, E dot or what is the relationship between them? That is a very important thing again for us to remember. As it is, you see that this guy is more closer or why more closer? It is a spatial quantity. Now, what is its relationship with E dot, the rate of our Green strain? Is there any relationship between them? Of course, there is going to be a relationship and what it is? Yes; you do that. It is very simple, let us see. Some of the small derivations I would like you to attempt, because that makes you familiar with what all we have done so far; application becomes very simple, they are all very easy.

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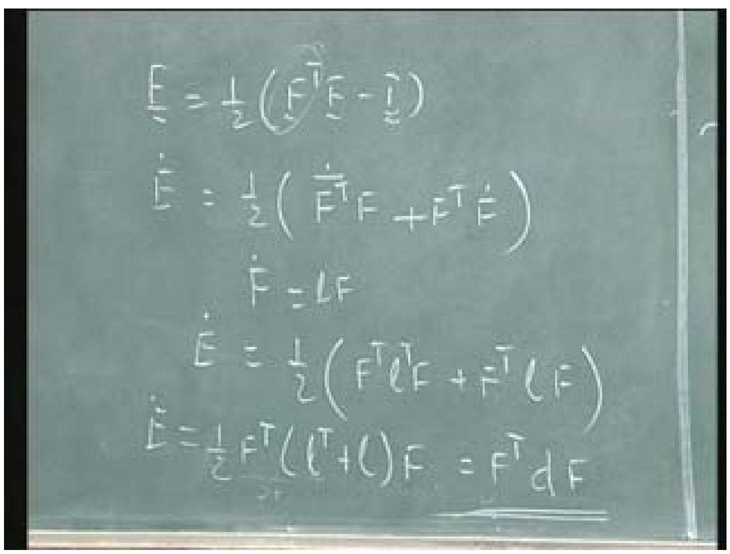

Yeah, I am leaving out these squiggles. I think you are, I am sure by now you know which are squiggled and which are not. Calculate E dot. Now, substitute in terms of the relationship F dot is equal to l F; substitute that and check what is that you get. Very simple, is equal to half of F transpose l F plus F transpose l sorry l transpose F transpose l transpose l F. Yeah, that is dot of F transpose. This is differentiated with respect to time, dot quantity that is all. Now, look at that quantity. This can be written as E dot is equal to half F transpose l transpose plus l F. What is this? F transpose, you can bring the half that side here, l transpose plus l divided by 2 F that is equal to F transpose d F. E dot is equal to F transpose d F; so, E dot is equal to F transpose d F. Is that clear?

Now, I have another important thing to derive. I will give you two minutes, you just try it out; it is very simple again. What is the relationship between this spin tensor and rotation R or is it that W dot is equal to R dot. See, if you remember, we had seen yesterday or may be in the last class or before that, we had looked at F very closely. In the last class, we said that we can split this polar decomposition and we have F, we have that split up into R and U and we said that R is a rotation and U is a stretch and so on. Now, that is one rotation. So, rate of rotation if I ask, you may say that it is R dot. Is R dot the same as W, which is a spin, which I called a spin tensor or in another fashion or in another way we can say that F, I had very neatly decomposed into a rotation tensor and a stretch tensor.

Is it that this l had been additively decomposed in a very nice fashion like that and is it that we have in d, a quantity which represents purely stretching and do we have that quantity W to represent only the rotation? That is a very important thing to understand. Does that, does these two, do these two types of deformations or splitting of the deformations, does it lead to the same result? Let us see that. Let us see what is the relationship between R dot and W? So, I have, I will give you two clues, let us see; you start that.

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R R transpose is equal to I and that F is equal to R U. That is the first step. Then, substitute in l and the next relationship, l is equal to what? No, no; we just now derived in terms of F dot, F dot F inverse. Now, we have three, these three equations. Let us see how you can calculate that, these three expressions. This is a problem which I am giving you. Let us see how you do that. In other words, if you want I will, F dot is equal to R dot U plus R U dot. That also I am writing, substitute it back into l. l is equal to R dot U F inverse. So, l is equal to R dot U F inverse plus R U dot F inverse.

What is U F inverse? What is U F inverse? We had F is equal to R U, so, U F inverse.

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So, this can be replaced by R transpose R U dot F inverse. From the first expression here you have R dot R transpose plus R R dot transpose is equal to zero. Substitute it, let us see. I will just stop here for a minute, continue and let me know. Now, what I want you to do is very simple. Now that I have written l, now that you have an expression here, write down the expression for d and this omega or w, small w. Write down an expression for these two and tell me what the expression is. That is all. If you want, you can replace this by U inverse R transpose, because R inverse is equal to R transpose is what we are using there. If you want, next step also I will give you a clue.

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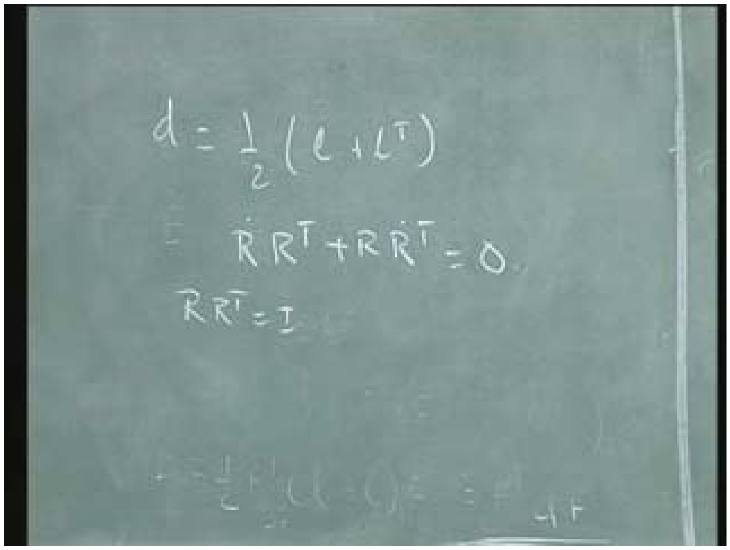

What is d? Yes; half of l plus l transpose is what it is. Calculate l transpose and then use the first equation R dot R transpose plus R R dot transpose is equal to zero. This one obviously, because R R transpose is equal to I. Differentiating with respect to time and being I, obviously it is very straight forward. This is what I defined. l transpose plus l or l plus l transpose, whatever it is, it is all the same.

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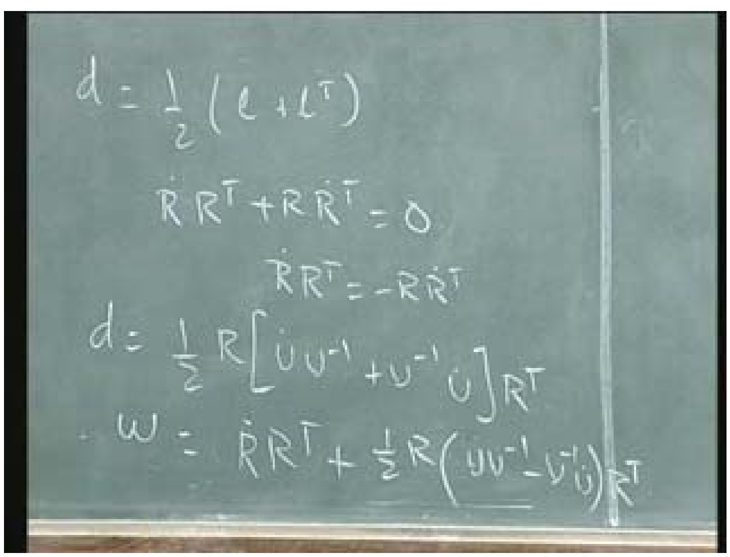

This is R dot R transpose, this will be minus R R dot transpose. Write down say for example, d. What would be the values or what would be the ….. half R U dot U inverse, yes, plus U inverse, because it is symmetric; U inverse U dot R transpose. Right? Very good, fantastic. What is w? Write down w; same way you write down w. That is all, nothing. First term, if you want I will write down; R dot R transpose plus that is the first term which you will get. That term went off because of this when I took this symmetric and anti symmetric case. R dot R transpose plus half of R, yes, U dot U inverse minus U inverse, U inverse U dot, very good, R transpose. That is all very simple substitution.

Now, what I want to do is to focus on say, this term. Look at this term. This term here involves not only R, but also U. If I had stopped it with this term, first term, then w depends only on rotation. So, w is not a pure rotation, but it involves stretch. So, d is not pure stretch, but also involves R and R transpose. So, both of them are not pure stretch. It is not stretch rate, it is not U dot. What I want to bring out by this is clearly that d is not just U dot, w is not R dot. This is a very important thing to remember that the spin tensor is not only dependent upon R.

Student: what is that spin tensor actually means in the physical context?

This spin tensor is not, is not the same as that of just stretching and rotation. It has rotation plus stretching - terms which are involved like this and then a rotation. So, it is a combination. You cannot give a simple physical explanation. It is a mathematical thing, you cannot give simple physical explanation like what we did yesterday, but it has a rotation part and a stretching followed by a rotation part. So, both of them are involved. That is what this means.

Before we go further, I just want to define one more small thing with respect with F, because we have to go and define stress tensor. I want to define one more term called F bar. You have lot of terms, I understand that. One more term, which we will define, what is called as F bar.

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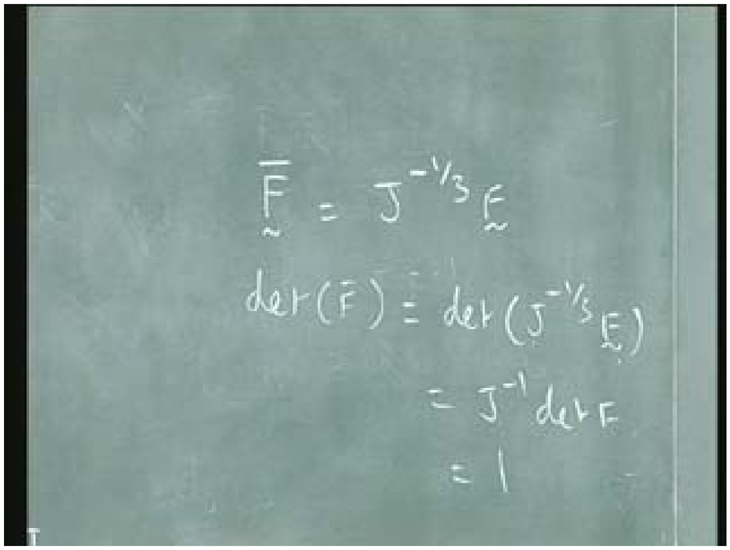

F bar, you will see what it is in a minute, has a very important connotation in incompressibility conditions. If you really look at determinant of F bar, this is determinant of F and determinant of J minus 1 by 3 F now becomes, this become, this being 3 by 3 will become J inverse F; so, J inverse determinant of F. Now, determinant of F being equal to J, determinant of F bar is equal to 1. From simple theory of determinant we get that determinant of J power minus 1 by 3 F, because this is being 3 by 3, you will multiply every term by J, so, J inverse determinant of F, so that equals 1.

What does it mean? It means that F bar is the one which is responsible for dilatation and when there is incompressibility, when determinant of F is equal to 1 which means or J, I mean this in other words, this is the one which talks about or we can, let me put it like this; we can split F into two parts - one is the dilatational part and the deviatoric part and when determinant of F bar is equal to 1, we say that this is the deviatoric part of F and the other part J power minus 2 by 3 is the one which gives you the dilatational part of F. In other words, F can be split into two parts like what we do for stress - dilatational and the deviatoric part. So, this can also be split into two parts - the dilatational and the deviatoric part. So, its pure isochoric deformations are defined by F bar.

With that background, let us now look at what are called as stress tensors.

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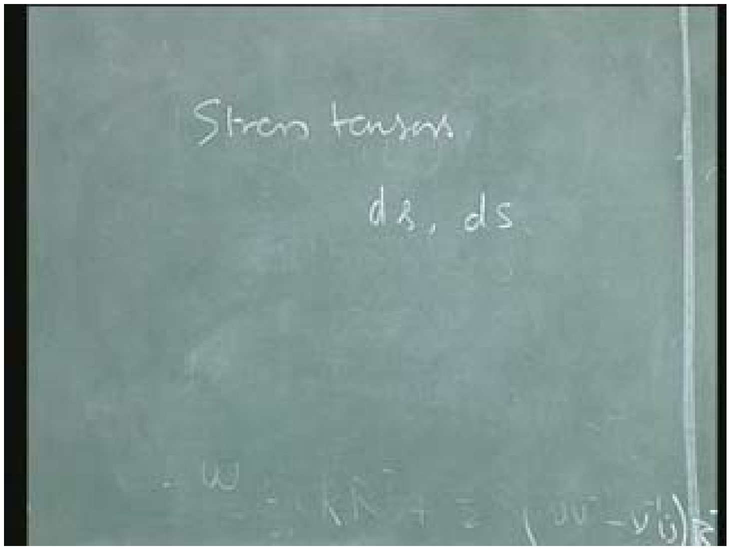

We have seen that there are number of or various strain measures that we have used; number of strain measures. In fact, we generalised the whole thing. We said that half U power n minus 1, where n takes different values can be used as a strain measure. In fact, when n is equal to zero, people define the strain measure to be ln of U and so on. In fact, why all that? Stretch also, we saw can be very useful to us to define deformation. Like that, there are a number of stress measures that are now going to be defined.

All of us are familiar with the Cauchy stress, sigma; we had used that in earlier course. We were not worried about change in the area and so on, because all of the deformations which we had considered till now are infinitesimal deformations. But now, since we are looking at finite deformations, finite deformations in a sense that, there is a change in area and in fact, we had a formula which connects ds and capital DS, capital DS being the original area, small ds being the final deformed area. We now know that this area difference, before and after deformation, has also to be taken into account. In fact, you would have done in your under graduate course say, strength of materials or may be in material science, you would have done defined what is called as the true stress and engineering stress.

What did you do there? What you did was very simple. Whether you took the original area or the current area, you called that as engineering stress or nominal stress and if you have taken the current area, you would have called that as the true stress. So, Cauchy stress is nothing but the true stress; Cauchy stress is nothing but the true stress and we have an equivalent stress measures in the multiaxial case for the nominal stress, for the nominal stress. Number one, we have Cauchy stress well defined. We have an equivalent stresses for the nominal stress that is the second definition of stress and I call that in the multiaxial case as the first Piola-Kirchhoff stress, first Piola-Kirchhoff stress.

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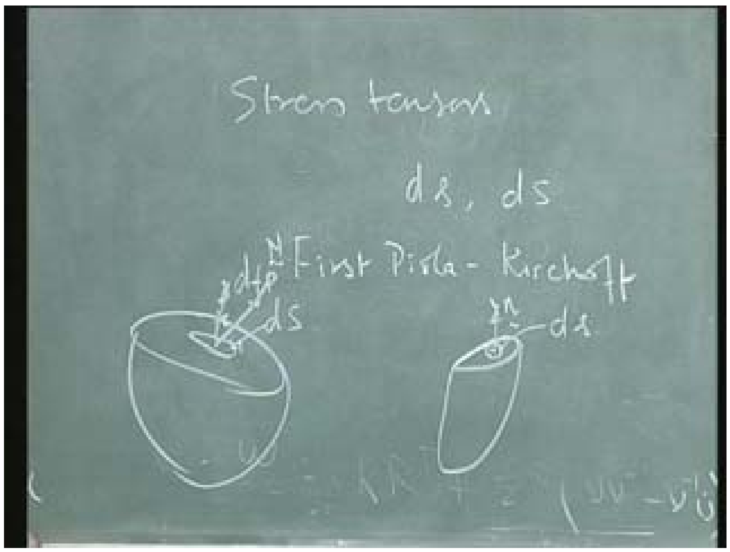

Let us say that I cut a body like what we used to do in last course on the definition of stress and let us take a small element. Let us define the area, same fashion as we did in the last class when we derived Nansen's formula; let me call that as say d capital S. Let there be, of course, this is X, let there be deformation and let us say that deformation takes place and this point goes to small x and let the area be d small s. Let the infinitesimal force that is acting in that area be force vector df and we know how to define the stress; in earlier class, we had defined this as delta F or we had defined it as delta F by delta s limit s delta F, delta s rather, goes to zero.

In this case we say that, let this be the normal. Of course, this also has a normal, let me call that normal as say N.

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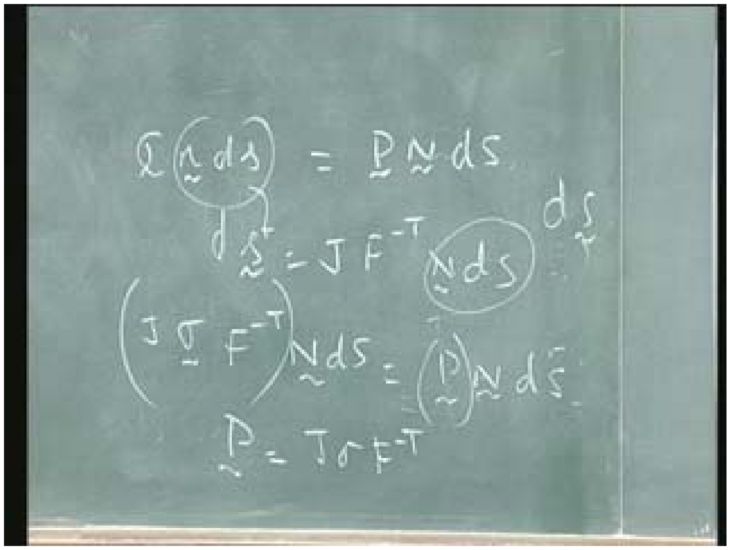

Then, df being the same, that is acting in that small area, because it is an infinitesimal area, you can define sigma n ds which gives me df, the deformed coordinate to be what? We call that as P N dS. So, sigma is a quantity which is associated with the current or the deformed coordinate and P, of course this is also a tensor, is one which is associated with the reference coordinate system. But, what is ds? We derived it in the last class. What is ds? J F inverse transpose, so, N dS or d capital S. I mean d capital S which is a vector. So, d capital S which is a vector can be, this is nothing but dS. This is actually d small s. Substitute that into that expression, see what you get. Sigma, this J can be out here, F inverse transpose N dS is equal to P N dS. Comparing the left hand side and the right hand side obviously, P is equal to J sigma F inverse transpose. So, the Piola-Kirchhoff stress can be calculated from the Cauchy stress and the vehicle for this calculation is nothing but the Nansen's formula. So, P is equal to J sigma F inverse transpose or sigma is equal to J inverse P F transpose.

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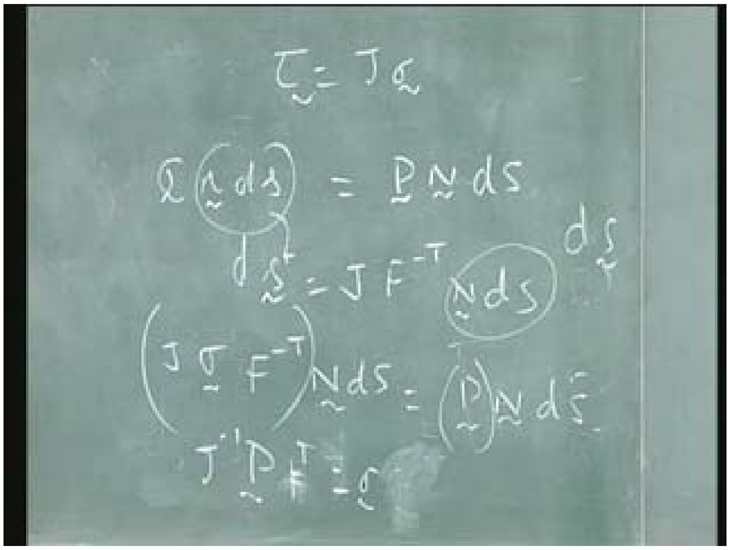

If you want to write that this side, bring this this side, so, that becomes J inverse; bring it the other side, so, this becomes F transpose that is equal to sigma. The second stress measure now we have learnt is first Piola-Kirchhoff stress.

The third stress measure which comes directly from here is called as Kirchhoff stress. Note the difference between first Piola-Kirchhoff stress and Kirchhoff stress. In fact, I am going to define in the next class what is called a second Piola-Kirchhoff stress. Kirchhoff stress, which is usually denoted by tau is defined as J sigma. So, tau is called as Kirchhoff stress. Now, we know sigma is symmetric. By this definition, you know Kirchhoff's stress is also symmetric. The question is whether P, Piola-Kirchhoff stress is symmetric. This being symmetric has very important connotations in our computational techniques, because you can save lot of space and efforts and so on. So, let us now check whether first Piola-Kirchhoff stress, which is nothing but an equivalent of a nominal stress, is it symmetric.

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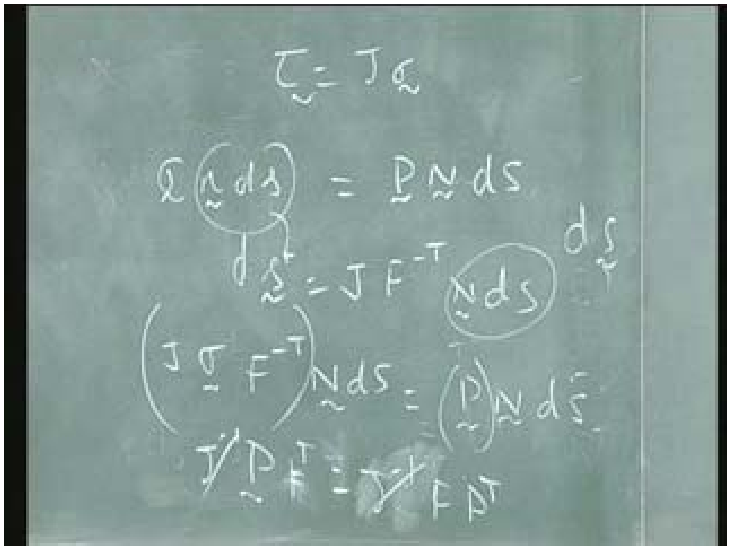

In other words, since sigma transpose is symmetric, this becomes say J inverse F P transpose, which means that P F transpose is equal to F P transpose; not, P is not equal to P transpose, but P F transpose is equal to F P transpose which means that the first Piola-Kirchhoff stress is not symmetric. Though the definition is very straight forward and nice, first Piola-Kirchhoff stress is not that very useful to us in computation situations, but the concept is very important to understand and we will continue with our concepts of stress, other stress measures and so on in the next class.