Advanced Finite Element Analysis
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Lecture - 15

In the last class, we saw how a volume element transforms.

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We said that small dv is the volume of the element after it transforms and we said that dv is equal to J d capital V and J was the determinant of f. If J is equal to 1 or if and when J is equal to 1, then d small v is equal to d capital V, we say that there is no volume change or the material is incompressible or isochoric. We stopped there and we want to look at now, the transformation of a surface.

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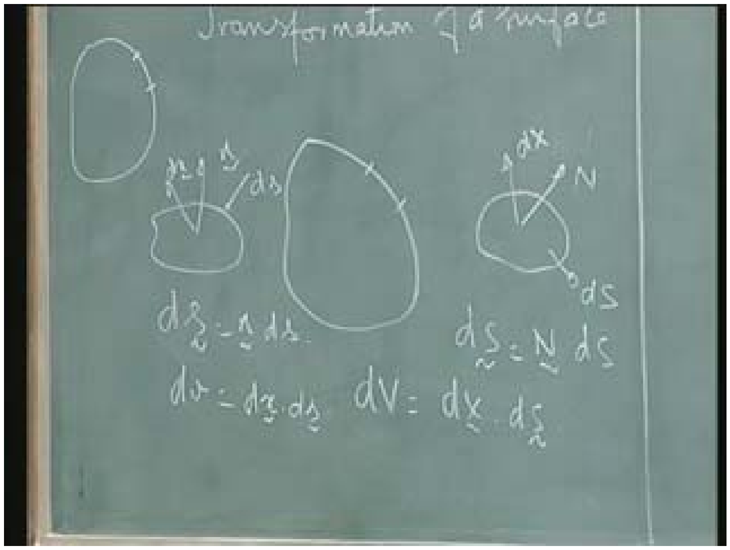

Transformation of a surface is very important, because we know that the stresses are defined through the areas and hence the surface definitions and how it transforms are going to play a very great role in our further definition of stresses. What we are interested in now is that, suppose I take a, that is the body and I take a surface; let us say that that is the surface. Let me call this area of this infinitesimal surface as dS and let N be the normal to it, so that the vectorial form dS can be written as N dS. This dS is the area and this dS becomes the representation of the area along with the normal N to it. Now, let us define dX, such that we can calculate the volume dV formed by dS and this dX or in other words dX dot dS gives the volume of the body, small infinitesimal body formed with dS and dX. Is that clear?

Now, let us say that this surface due to the transformation gets transformed to small ds, such that the normal becomes now small n and now the surface can be written as ds is equal to n d small s and we are writing in the similar fashion, only thing is this is after transformation. The body has transformed, has become like that and now we are looking at this surface where we are concentrating. Please note all of them are infinitesimal surfaces. The corresponding dv that is formed due to the transformation of the vector dX which now becomes say d small x is equal to d small x dot ds. Is that clear? The corresponding transformations are written here.

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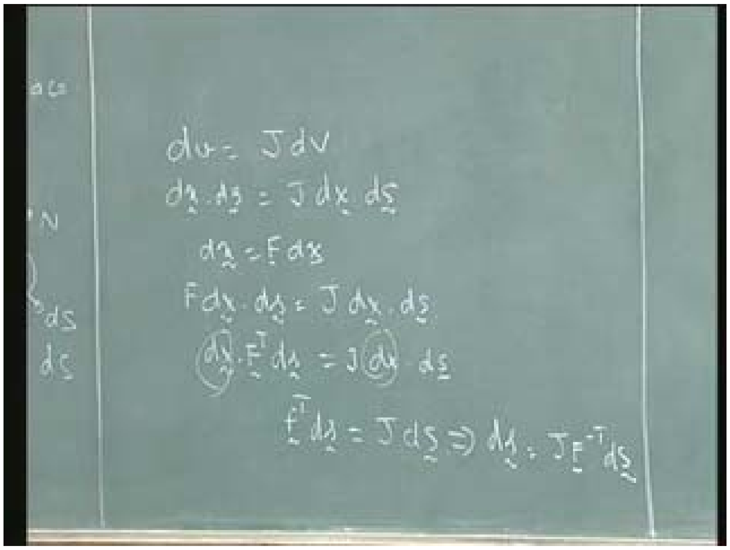

Now, we know that small dv is equal to J d capital V or in other words, dx dot ds is equal to J times d capital X dot dS. Is that clear? Now, this is small. Now, let us replace dx by, we know dx is equal to F dx and so let us replace that, so that this becomes F dX dot ds is equal to J times dX dot dS. Now, using the transpose or definition of transpose that can be written as F transpose ds is equal to J times dX dot dS. Comparing, now look at that. There is a dX here and there is a dX here. Obviously then, what is here to the right of the dot product should be equal to J times dS. Of course, this is a vector. In other words, from this it is clear that f transpose ds should be equal to J dS, which implies that ds is equal to J F inverse transpose dS.

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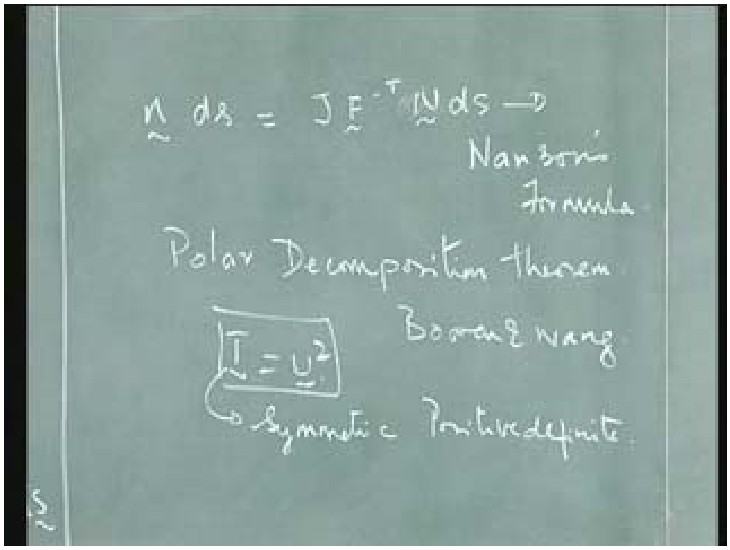

In other words, this means that n ds is equal to J F inverse transpose N d capital S. We are not cancelling ds, we are just writing down d; yeah, dx of course, because this is a dot product, right of the dot product. There are two dot products; this comes from uniqueness of dot products, so, obviously you can see that, you can write the right hand side of the dot products we are comparing. Now, this is the celebrated Nanson's formula, one of the most important formulas in continuum mechanics and will be used later by us for definition of stresses. Is that clear?

Having talked about now the definition as well as the transformation of surfaces and the volumes, let us now look at a very important theorem called polar decomposition theorem. Polar decomposition theorem has its origin in linear algebra. In fact, people who are interested in looking at more about polar decomposition theorem, linear transformations and so on can refer to the book by Bowen and Wang on Linear and Tensor Algebra. So, what I want to state, number 1, very important thing, is that polar decomposition theorem is not peculiar to continuum mechanics. It comes out of the result of linear mappings or transformations, which is the subject or which is the topic in linear algebra.

Before we go, we state this polar decomposition theorem, we have to state a very important lemma, which I am not going to prove, but very simple to understand. What this lemma states is that, for any symmetric second order tensor T, for any symmetric say second order tensor T, there exists a unique symmetric second order tensor, positive definite second order tensor, such that T is equal to U square. T is actually symmetric; we should add or had that also positive definite. We had already defined what positive definite is; second order tensor. There exists a unique U, which is again symmetric positive definite, such that U squared gives you T. With that in mind, let us define the polar decomposition theorem.

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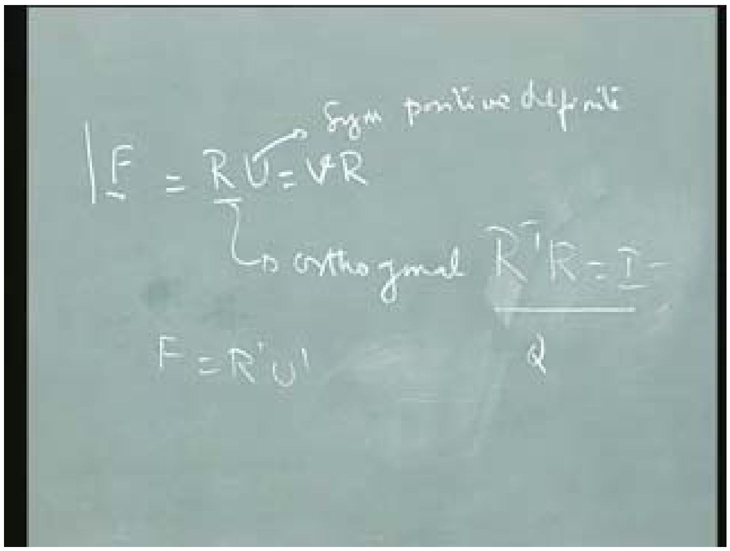

Polar decomposition theorem states that for any non singular that means F, if I say F is one of them, second order tensor that means that its inverse exists, if I take a tensor whose inverse exists, then this tensor, second order tensor can be multiplicatively decomposed into R U or is equal to V R; we will put that small v R, where R, this R is an orthogonal tensor and U is similar to what we have defined here - a positive definite symmetric second order tensor. In other words, what it means is that given any second order tensor which has an inverse, I can multiplicatively decompose it into an orthogonal tensor and a symmetric positive definite tensor.

Yeah, what orthogonal tensor is, we had already defined. So, R transpose R is equal to I. That means that R transpose is equal to R inverse. This we had seen, for example in the case of Q when we defined, remember when we defined Q, we defined this to be an orthogonal tensor. One of the important properties of this kind of orthogonal tensors is that when you transform, when you use them as the transformation tensors, then the lengths do not change. Remember that this is one of the tensors which define the rigid body motion, rotation. Of course, there is one more say, translation which can be defined by means of our C(t) say, or C a vector which does translation. This is a very, very important theorem. Though I had put here the symbol F, R and U, I am going to continue with these symbols, but this is a very important theorem and can be used, as I said, generally for any other purpose also. But, in continuum mechanics, the decomposition is applied specifically for F, the deformation gradient tensor.

Not only is that we can decompose, but we can decompose that uniquely. In other words, if F can be written as R prime U prime, then, R prime is necessarily equal to R U prime is necessarily equal to, U. So, they are unique. Let us prove the theorem and look at the implications and see why this theorem is important. So, first let us prove the theorem in the sense that we are going to show, first part is very simple; we are going to show that you can write it in this fashion and the second part, we are going to show how this decomposition becomes unique. Is there any question?

Student: Sir, is that R and U are the rotation gradients and displacement gradients?

No, no; I will talk about that in a minute. Let me just wait for a few minutes; you are absolutely right, but let me first prove it. We will look at the physical picture, a beautiful physical picture will evolve out of this decomposition, you have to wait; you have to wait for some time. Let us go ahead and now first prove the theorem.

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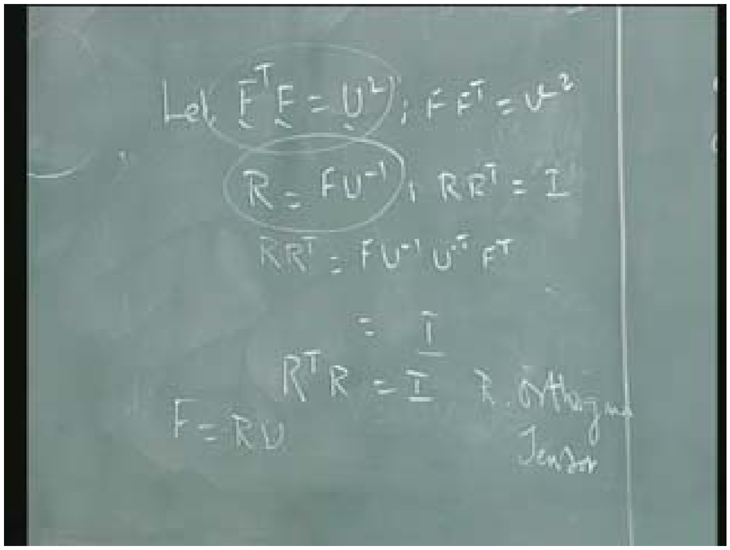

Let F transpose F is equal to U squared; let F transpose F is equal to U squared. Obviously, I can assume like that, because my F transpose F qualifies my theorem, the first lemma which I stated. So, obviously, I can say that let F transpose F is equal to some U squared and F F transpose is equal to v; both of them qualify, sorry v squared, both of them qualify for the first lemma. Is that clear? Now, let R be equal to F U inverse, let again. Then, my first job is to show that R R transpose, first job is to show that R R transpose or R transpose R is equal to I. If I define like that is to, is this R orthogonal? Please check up.

R R transpose is equal to F U inverse. So, again F U inverse transpose, so, that becomes F U inverse transpose F transpose; U being symmetric, so, U F U inverse U inverse F transpose, which is U squared. So, F U squared F transpose, which according to our definition now becomes what - is equal to I. So, in the same way, you can show that R transpose R is equal to I. In other words, if I define F like that, then my, in fact, R transpose R also becomes I. In other words, when I define like this, then R becomes an orthogonal tensor. So, obviously now if I define U to be like this and R to be like this, then it is quite straight forward that my decomposition of F R U gives me an orthogonal tensor and a symmetric tensor; both of them second order tensors.

Now, let us see how, we will see the decomposition of the second one later; let us see how we can prove the uniqueness.

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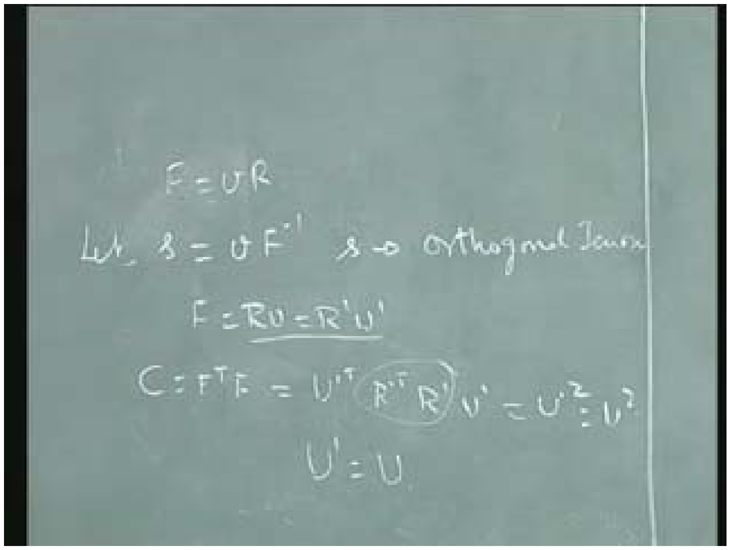

If you want you can write down right away s also to be say, let us say, let us say, s to be equal to, yeah, F. We defined F is equal to v R and so, let us say, F we defined, s is defined as v F inverse. We can again show very easily, straight forward, that s is again an orthogonal tensor. Let, I am saying let, now my job is also to prove that s is equal to R; I will do that later. The second part of the story, we will just keep it for a minute; we will first prove the first part F is equal to R U, by showing that R has to be unique. In other words, what it really means is, as I told you, if I write F is equal to R U that is equal to R prime U prime, then R prime is equal to R and U prime is equal to U. It is very simple. C is equal to say, F transpose F; we know that, and that write it down in terms of this, in terms of this, F transpose F. That means that U prime transpose R prime transpose R prime U prime.

What happens to this? So, this goes to I, so, that becomes U prime squared. We already know that F transpose F is equal to U squared. That is how we had started the whole thing. So, this becomes U squared. So, U prime squared is equal to U squared, which means that U prime is equal to U. Once U prime is equal to U, obviously this condition gives me the fact that R prime is equal to R. So, the decomposition is unique. The next point is very simple.

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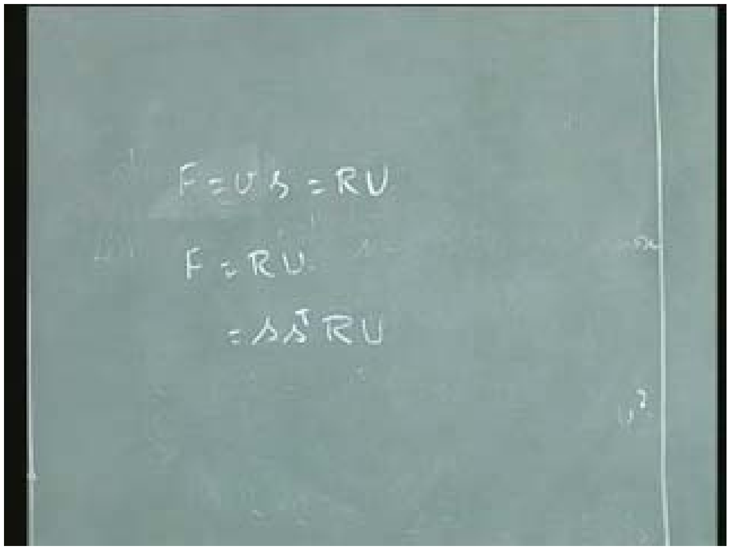

Suppose I write this. What I am saying is, let us write that as v s. Let us look at the second part v s is equal to R U. So, from this, we have to show or this is equal to F, of course. We have to show that s is equal to R. How do we show that? Just think about it for a minute. s is of course, an orthogonal tensor that is there. See how we can show how s is equal to R? R U; it is quite simple. Say, let me say, F is equal to R U. That is equal to s s transpose R U; s s transpose R U and then that is equal to I or you can in fact, write it in a slightly different fashion.

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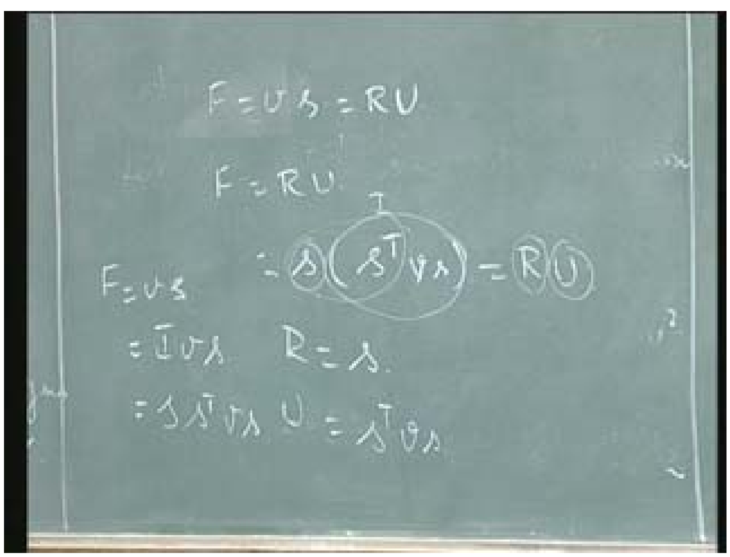

We can write, F is equal to s into s transpose v here s; that, we can compare that with R U. What essentially I have done is s s transpose is equal to I. So, this one, first just I have pre multiplied by s s transpose, so, this becomes I and that is equal to R U. Both the equations are the same which means that since the decomposition has to be unique, this R has to be necessarily equal to s and this U is necessarily equal to s transpose v s, so, U is equal to s transpose v s. What I essentially did was to just pre multiply this by I or in other words, F is equal to v s and that is equal to I v s that is equal to s s transpose v s and from there I got this relationship. That is a very straight forward way of showing that s is equal to R.

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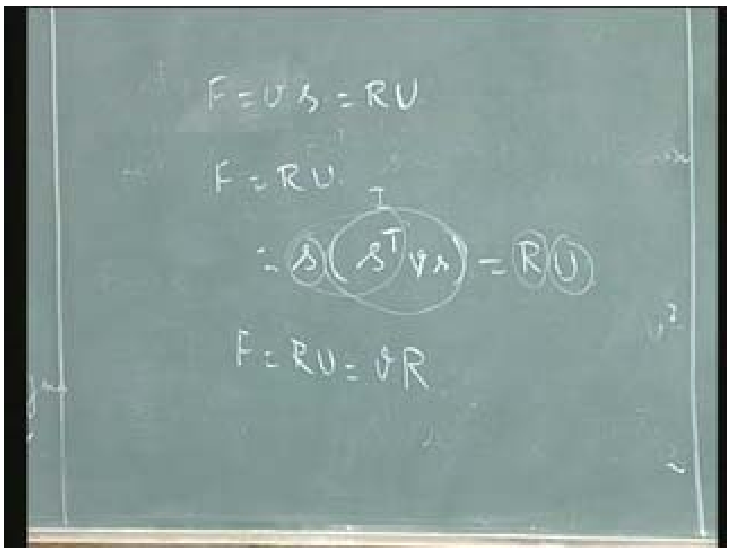

Final result I will write here. F is equal to R U that is equal to v R. Now, just to understand what this U is, what this R is, let us now look at the definition for our stretch and whether we can use this decomposition at that place. Is this clear, any question on this? We will now show what the physical meaning of this is. Please write down the stretch equation from your notes and substitute there, this decomposition.

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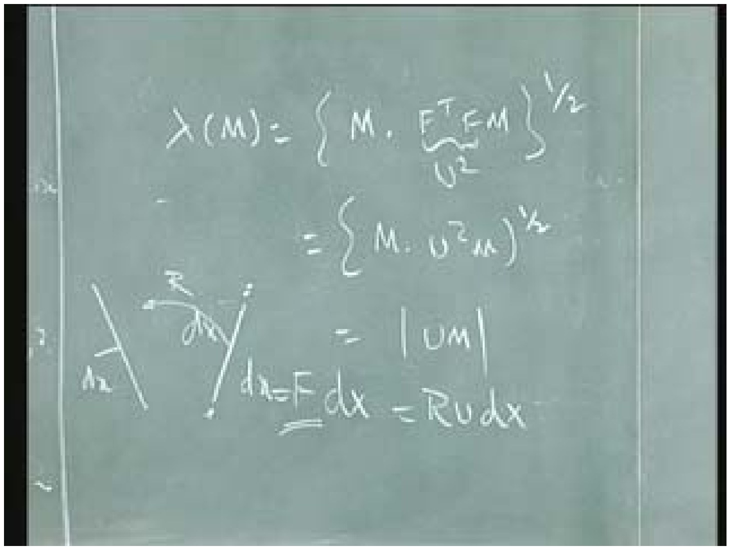

Remember what the stretch was, say, lambda of say, M; we wrote like that if you remember. So, that was M dot F transpose F M. Please go and have a look at what …. Yeah, correct; whole power half, this is how we wrote. This is d small x by d capital X, mod of them. This is dot product. What is this, what is this, from what we had defined now? U squared, very good; so, U squared. So, this can be written as M dot U squared M whole power half and what is that? That is nothing but the magnitude of U M.

Now, what does this indicate, what does this indicate? It indicates that if I have a material line element dx, the stretch part of the material line element has only the U part of F and hence the lengths are affected only by U, only by U. So, if I have now F dX which is equal to R U dX to give me that small dx, what it really means is that this dx is stretched by the U part of F and then in other words, this fellow now undergoes say stretching, stretching. Stretching is, I mean, expansion in length is one of the condition of stretching. Stretch does not mean that looking at it all the time. Then, this guy gets rotated by R say, to become like this. So, this now is d small x. This transformation F involves a stretching followed by a rotation. Is that clear? F cannot be treated, look at this very, very carefully; F cannot be treated as a measure of strain. Now, go back and look at my definition for strain.

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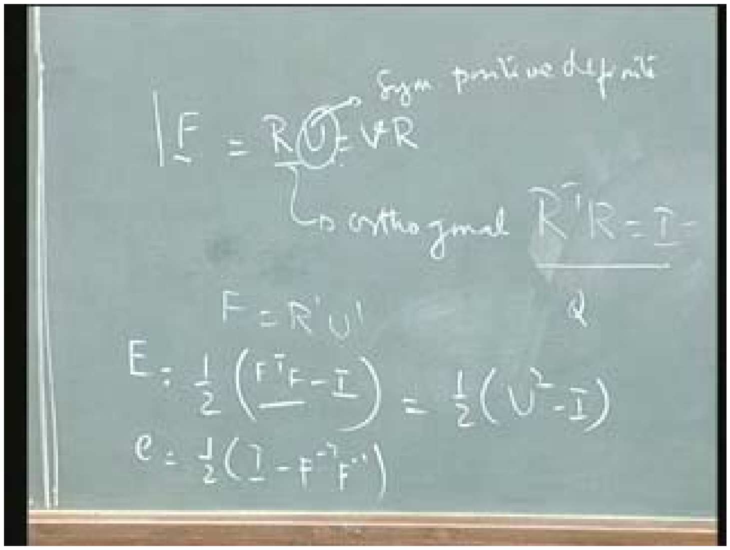

What was my definition for strain say, E? E is equal to half F transpose F minus I, very good. Now, what is the connection between this U and here? This is equal to half into U squared minus I. Now, look at this condition here, what we had written for E. What has happened there? We have actually been very careful in defining the strain to be one which is affected by stretch or by U alone. So, this is called pure stretch, because U does not contain any R in it. Hence the strain is not affected by the rotation part of it, but is affected only by the stretch part of it.

In the same fashion when we defined Almansi strain what did we do? We did that as F inverse, remember that, I minus, what did we do? I think, if I remember right, I used small e half into I minus F inverse transpose F inverse. Now, you will see that this is V inverse squared and hence we see that this again depends upon the stretch. On the other hand, when you use V as or v R, what it simply means is that first you rotate it and then apply the stretch; rotate it and apply the stretch. By this decomposition, we are clear about that component of F which causes stretching and that component of F which causes a rotation.

Yeah, see, what we mean to say is that suppose I have, let me explain that again if you have a doubt.

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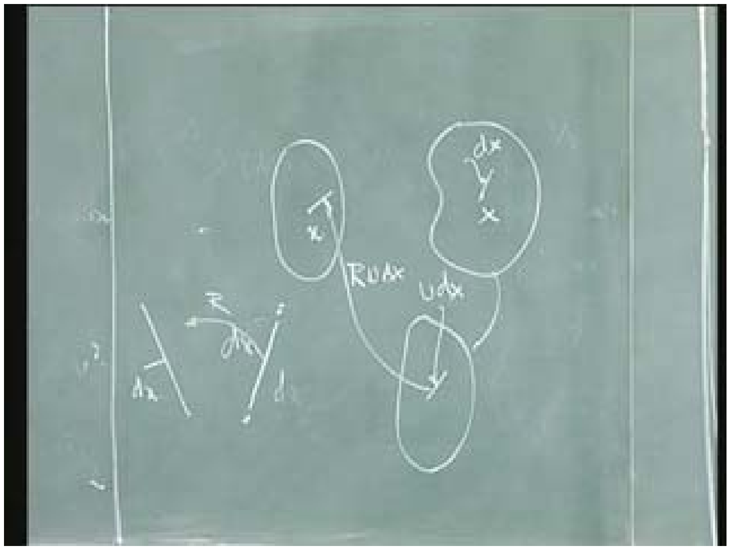

Suppose this is the body, this body undergoes transformation. So, it goes like that. That is the point X, so, that point becomes small x and let us now see what happens to this material line element say, dX. This undergoes a transformation first to a body like that, a body like that, so that that dX now becomes U dX, which is the stretching part and then that gets later rotated from here to here, so that that becomes R U dX. There is a stretching first followed by rotation; stretching followed by rotation. If it is v R, then it is rotation followed by stretching. Note that again this is different from the decomposition you would have done for U I comma J in the small strain case.

We are going to see an equivalent of it, may be in this class or next class that U I comma J, if you remember, U I comma J you have split it up into a symmetric part and a skew symmetric part. This skew symmetric part you would have called that as omega and that was called as the spin tensor, spin tensor or spin; just this omega which is half of U I comma J minus U J comma I, that is the skew symmetric part. So, you would call that as spin. You might have a doubt; what is that omega which was something like a spin or rotation and what is the difference between that rotation and this rotation? Interestingly, you will see in a minute may be later in the class that this spin has something else attached to it. It is not pure spin, this spin here and that this R is a pure rotation. We will see that in a minute, so, do not get confused now; we will sort it out in a minute. Is that clear?

Having defined now the decompositions and R U decompositions as it is called, let us come to one more decomposition called spectral decomposition. Now, what is spectral decomposition? I think we have done that or you would have done it in earlier classes on Eigen value problems.

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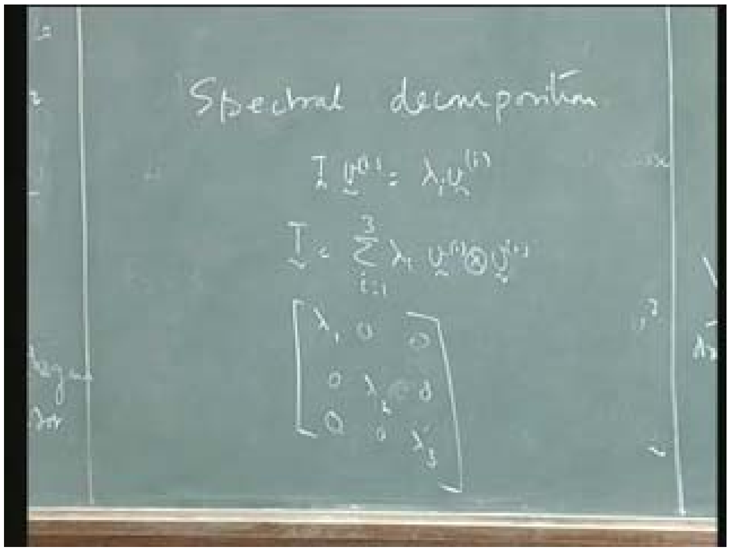

All of you know that if I have say, a tensor T, then there exists for a symmetric second order tensor T, there exists a v i, such that v i is equal to lambda_i and you know all the properties of v i which you would call as the Eigen vectors and lambda_i, you would call this as Eigen values. I am not going to the details of all this, I amsure all of you know these things; we had covered it in the first course and you would have studied that in linear algebra. In other words, what I just want to emphasize is the fact that T can be written as sigma lambda_i v i dyadic v i. In other words, this v i dyadic, sorry v i dyadic v i, i is equal to say 1 to 3 can be used as the basis or Eigen vectors can be used as a basis for definition of any tensor.

In other words, when the basis is shifted to these Eigen vectors, then T can be written as say lambda_1 lambda_2 lambda_3, the others being zero. Is that clear? Of course, the question is what happens when lambda_1 is equal to lambda_2, then how do you write this, how do you write this? You say, for example, lambda_3 is unique and lambda_1 is equal to lambda_2, then what is the condition of the Eigen vector? Then any two, yes absolutely; any two perpendicular vectors in a plane, which lies in a plane, defined by the lambda$_3$ to be normal or v 3 rather to be normal, will qualify for an Eigen vector. We can define this using a projection theorem, as it is called in a very simple fashion and may be we will go into details of this when we come to it, may be in the next class or so. Now, the point is that having so many tensors, is it possible to get their Eigen values and use that for any of this analysis.

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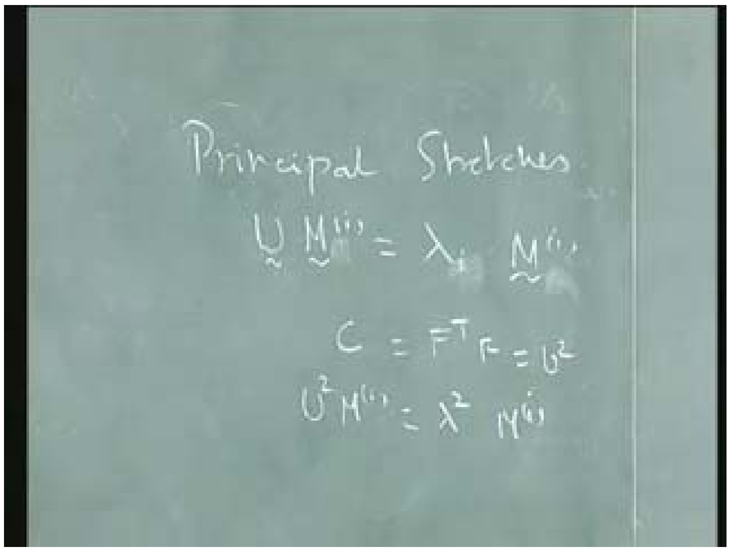

That is what is done in many cases and many of the analysis are based on what are called as principal stretches, principal stretches. In other words, in the same fashion as we define the Eigen value problem, this T can be replaced by my U and U M_a. Remember that U M_a was the guy who was in my stretch. Remember that is what we ultimately landed up in a few minutes back. So, I can define an Eigen value problem using U, because it is a symmetric second order tensor into lambda say ai M_a. So, this gives rise to, this problem gives rise to three Eigen values and three Eigen vectors - lambda say, a1 or lambda$_1$ lambda$_2$ lambda$_3$ if you want and write this as M 1 M 2 and M 3. I had put that as a, for a particular M, because you remember that M we had used it previously to denote any unit vector. This is a normalized version that is all. If you want to remove it, you can remove that; there is no harm in it. Just to distinguish that M is not any vector that is why I had put it. But, if you know the context, then obviously you can remove it and you can write it something like that.

What happens to, what is the relationship between the Eigen value and the Eigen vector of U and C? What is C? C is F transpose, remember what is C? Right Cauchy; so, F transpose F, F is in the right hand side, so, that is equal to U squared. What happens now if I put U there? U of U M, so, in other words, U squared of M, what happens? Lambda squared, so, U squared M of i is equal to lambda squared U of again lambda M i. So, lambda squared M i. How did I get that? It is very simple. If you do not understand it, I will put that step again.

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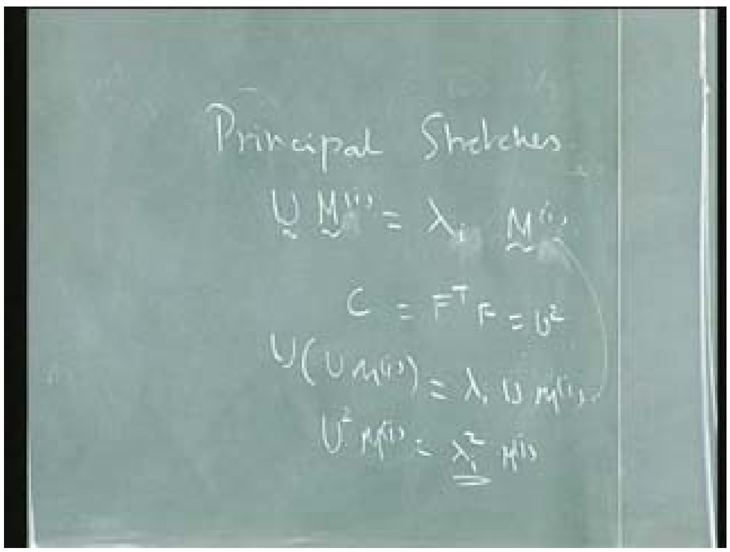

U of U M i is equal to lambda_i U of M i. so, this becomes U squared M i is equal to U of M i. Substitute it from here, so that it becomes lambda i squared M i. In other words, the Eigen values of C, the square of Eigen values of U and the Eigen vectors do not change. So, these lambdas are called as the principal stretches and many, as I told you many, elasticity problem people indicate it or use what are called as principal stretches. So, principal stretch is again one of the ways of defining strain.

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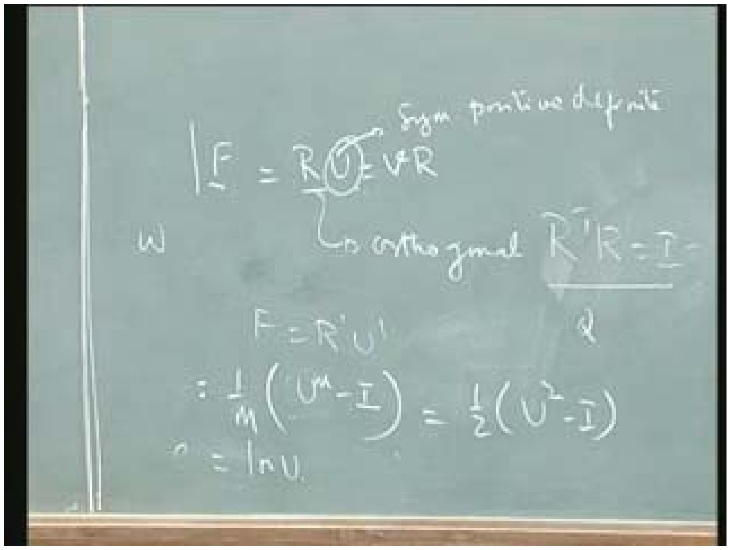

In fact, I should have mentioned it that a general way of defining strains, a general way of defining strains, is by putting 1 by M U power M minus I, where M takes different values. For example, M takes a value of 2, this equation reduces to our well known green, Cauchy green strain. When M is equal to zero or M is equal to 1, then this reduces to U minus I. This is also one of the strain measures that are used. In fact, another strain measure that is used is ln or natural logarithm of U and so on. These are generalised, other generalised strain measures; we may not use all of them in this course. When it is necessary I will define it, but I just want to state that E - there is nothing very holy about this E - definition of capital E, you can define various stress measures by using various powers of U as well. So, this again, one of them is the principal stretch.

The advantage of principal stretch is that immediately my stretch would be just a diagonal lambda$_1$ lambda$_2$ lambda$_3$ and they are the stretches along what are called as the directions of the Eigen vectors. Now, having done that, what is the relationship between this principal stretches of U and v? Is there any relationship between them, between the principal stretches of U and v? Remember that v is a measure, which can be used along with Almansi strains. That means that when you are dealing with the current or deformed coordinate system which means that you are the Eulerian coordinate system, then the measure that you would use is Almansi strains and so it is important that we also understand what this relationship is between the lambdas of U and lambdas of v. Are they the same are they different and how the Eigen vectors also get transformed? No, no, no; do not guess. Please look at it; please look at this, go from here, go from here. No, no, no; wait a minute. Do not be in a hurry.

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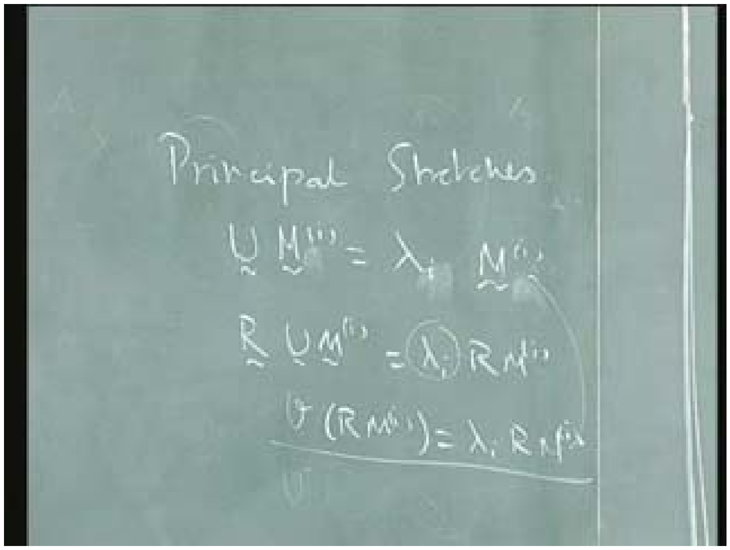

Just multiply this by R is equal to lambda$_i$ R M i. R U is equal to v R, so, v of R M i is equal to lambda$_i$ R M i. Now, look at this expression here; that talks about the Eigen value problem or Eigen value equation of v. The only thing that has changed here is that this M has become R M and hence the Eigen vector has now become R M. So, Eigen value here has remained the same. What does it mean? It means that the Eigen value of v or the principal stretch or the principal values of v is the same as that of my U, but the Eigen vectors get transformed or is rotated by R. Is that clear? That is another important result that we have to see. Please note that, similarly C's, C has lambda squared as a principal stretches.

Having defined now these quantities, which I will use it later in the course, we have to define quantities or we have to calculate what are called as rate quantities. What does the rate quantity which means that how does many of the quantities, what we are looking at now, will change with respect to time?

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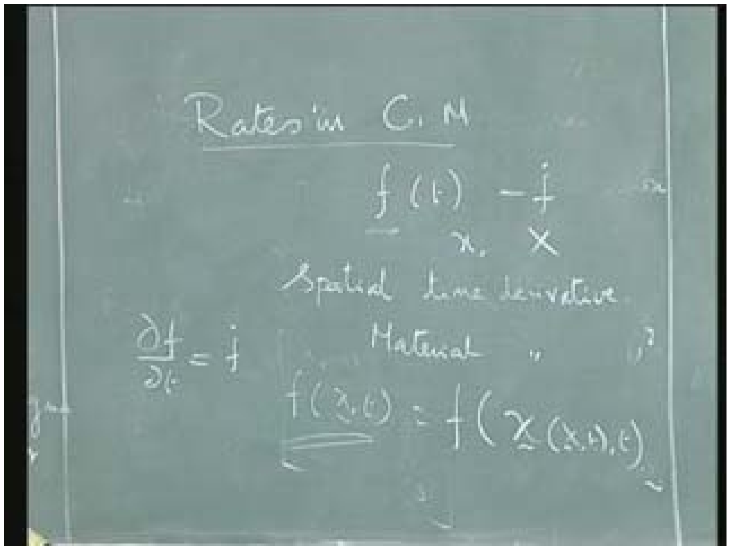

Rates in continuum mechanics - what is a big issue there you may think, but there are quite a few issues which we have to sort out. What do we mean by rate quantity? Suppose I have some function f of t; obviously, what I am interested is in f dot. Now, what is the issue here? The issue is that this quantity f may be a function of small x or may be a function of capital X. When f is a function of small x, then they call this as spatial time derivative and when f is a function of capital X, they call this as material time derivative; spatial and material time derivative. Suppose I write that f as a function of x comma t, note that this x can be replaced by capital X, by my deformation function t. In other words, the issue is that when I define the Eulerian co-ordinate, that Eulerian co-ordinates, that small x now changes with time, but whereas the capital X which strictly speaking is our reference, first time when I marked it, it was there; that does not change with time.

So, you can look at the derivatives in two aspects - one is the material derivative of a spatial function; material derivative. That means that how does, actually this particular quantity is going to transform or how do I take the rate quantity of a spatial time derivative? In other words, is it that dow f by dow t is equal to f dot or is there something more to it, something more to it? Very good; so, what is that is what we are going to see. May be, we will derive that in the next class and then derive certain quantities which are associated with this spatial and the material time derivative and what is its relationship in the next class.

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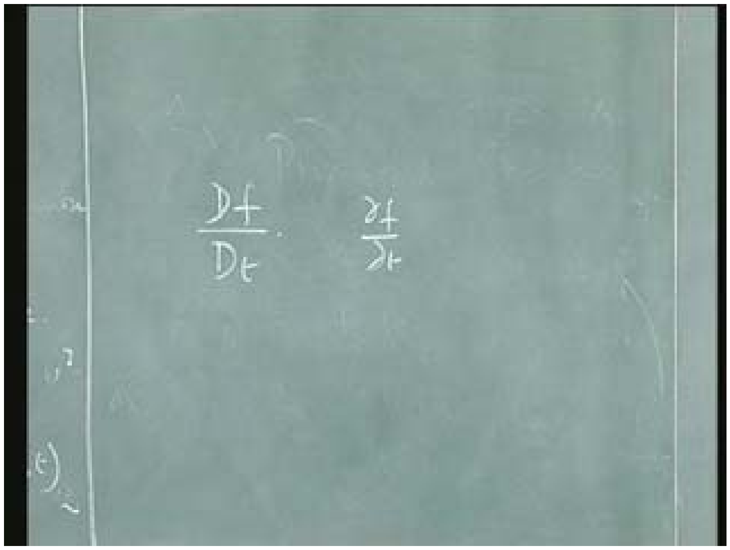

The only thing I just want to point out that, usually the material time derivative is written with the capital D, Df by Dt and spatial time derivatives are usually written as dow f by dow t. Is that clear? Before we go to the derivation, yes, I am going to derive that in the next class; do not have enough time; there is one comment I want to pass is that many of the constitutive relationships, which we are going to use, is rate quantities. Hence, determination of rates becomes very important. Do you understand that? The determination of rates becomes important, so, we have to necessarily know how to calculate this and distinguish clearly between spatial and the material time derivative. We will stop here. If there are no questions, we will stop here and continue with this derivation in the next class.