Advanced Finite Element Analysis

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Lecture - 14

In the last class, we started our study on continuum mechanics and we had defined what we called as deformation to be a mapping function and then, we defined one of the most important quantities in continuum mechanics which we called as F and which we named as deformation gradient tensor.

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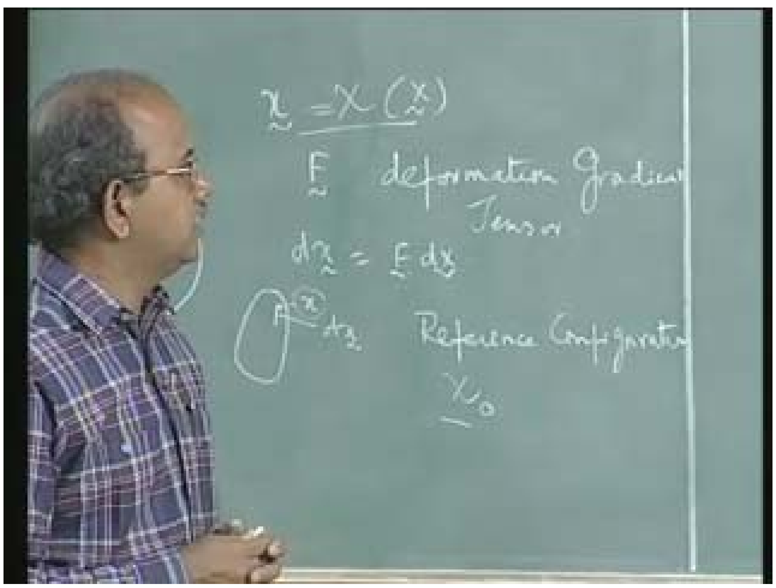

Note that we defined this in such a fashion that any line element dx sitting at x or in other words, if this is the body that we are considering and if this is a point say X and if this is the dX, then this dX, when the body deforms, then this dX becomes small dx, at that point small x and the transformation of capital X and small x, that is the transformation which gets you small x, that is this deformation mapping and the one which gets us this vector d small x from d capital X, that is the d small x is equal to or given by this deformation gradient, F. This is what we saw.

We have not defined the reference configuration. Reference configuration is extremely important in the study of mechanics. There are various definitions for this. One of the major or very important aspects of reference configuration is whether it is a stress free configuration. Most of the cases we treat a stress free configuration to be a reference configuration. It is not necessary that a reference configuration should be reached in a body, should be reached in a body, or in other words it is not necessary that when we start the deformation that is a reference configuration. But, most of the times in normal instances we define the reference configuration by this chi naught, where chi naught indicates the configuration of the body when time t is equal to zero. But, it is not necessary that this has to be the reference configuration.

Most of the times or successful applications are ones where reference configurations are stress free configurations. For example, if you are doing say, analysis of a human heart, now, what is the reference configuration for this? In other words, what is the stress free configuration? That is never achieved. In fact, it is very difficult many times, especially in biomechanics to say what a stress free configuration is, because that kind of thing will not exist, as long as, I mean in the realm or region of time that is when the time of say, for example, where we are interested in beating heart, the time at which it is functioning, at that time it may not even reach this reference configuration. So, it is not necessary that reference configuration should be reached at any time of interest to us. But, you should understand that we need a reference configuration from which we would proceed for further analysis. But, fortunately for us most of the time, most of the time, reference configuration is defined very neatly for engineering structures and it happens to be, the configuration at t is equal to zero.

With that in mind, let us proceed, let us see and of course we had defined F to be a two point tensor and so on. Let us now study more about a deformation and the ensuing quantities.

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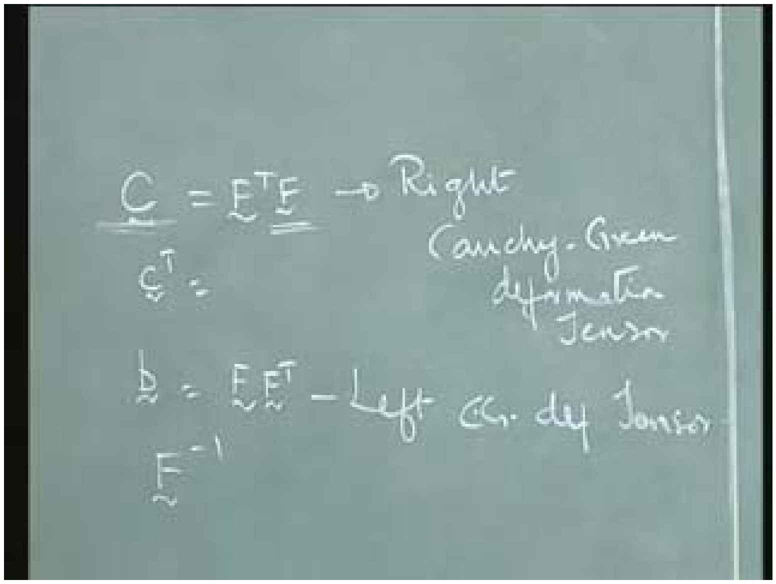

Now, one of the most important quantities that we will come across is what is called as C defined to be F transpose F and is called as the Right, note this carefully, Right Cauchy-Green deformation tensor and we define b to be, of course, these are tensors; we define b to be F F transpose and is defined as Left Cauchy-Green deformation tensor. Now, look at this. Why is it called as right and why is it called as left? The reason is very simple, though people do not understand that it is just that F is written to the right of F transpose or it is in the right and F is to the left and hence they are called as right Cauchy-Green deformation tensor and left Cauchy-Green deformation tensor.

These two tensors are again going to play an extremely important role and it is very interesting also to see that these two tensors are symmetric; it is very easy to see that. So, when I take C transpose, look at that C transpose, then this would become F transpose F and so it is symmetric. Is that clear? They are going to also be positive definite; we will not work on it right now, we will indicate it later. On the other hand, you cannot say about F as, of course as, symmetric because, we do not know yet whether it is symmetric or not; from the appearance it does not look like that. But, of course, we had defined the inverse of F, one which takes us from d small x to d capital X.

Having studied this, now let us go back and look at what we understand by the term strain.

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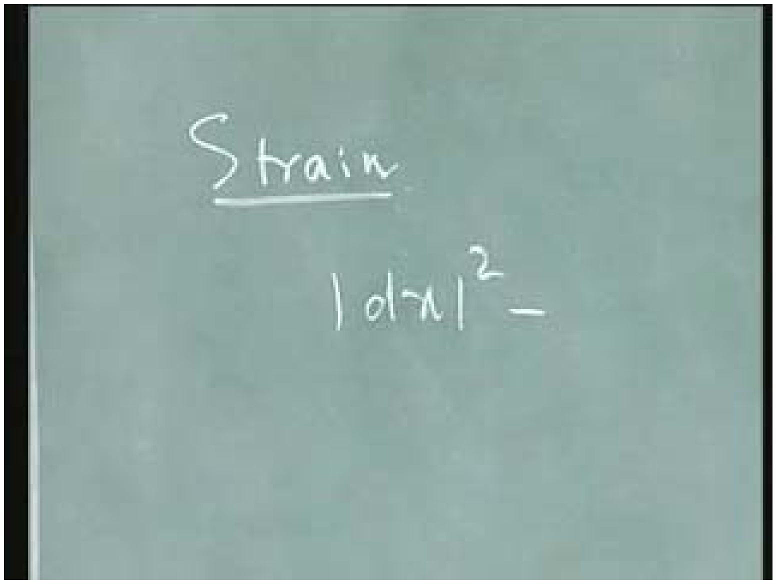

Of course, earlier in this course we had looked at small strains. Now, let us look at the strains in a slightly different fashion. You will see that the definition what we are going to put down will reduce to the small strain case under certain conditions; we will just wait for it. Now, let us define what strain is. Before we do that, we will stick still to the concept that we are looking at strains as something to do with lengths of material line element. Remember that when we define strain in our say under graduate programs, we define strain to be just change in length by original length. So, the concept that we are going to introduce is very similar, but only thing is that that length what we are talking about is not the usual bar which we had used in our earlier classes.

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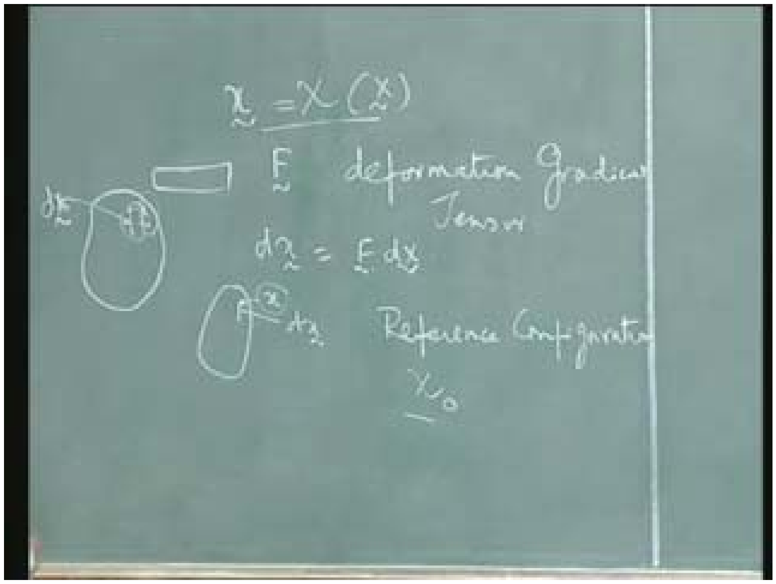

We used to have a bar like that and that is the bar which we used to define the strains. But, here let us look at this material line element closely at every point x and see whether we can do something with respect to the lengths of this material line element.

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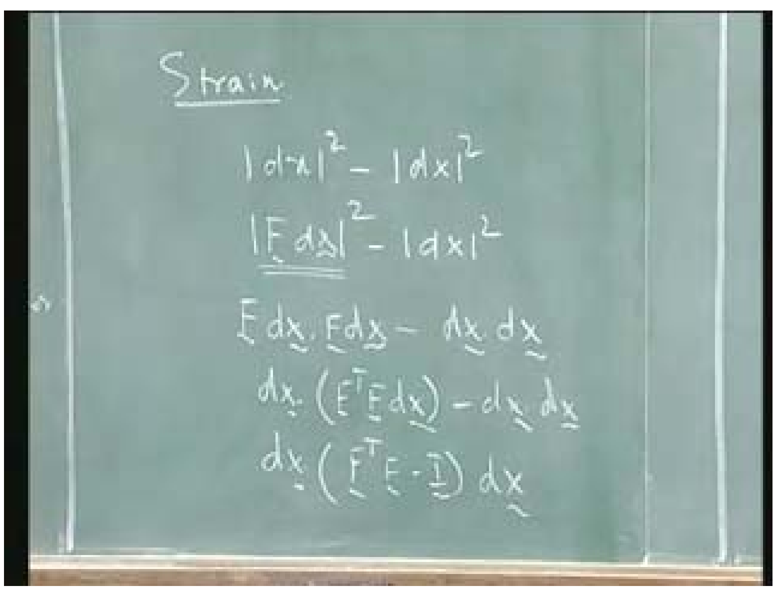

From that point of view, let us see what d small x squared minus d capital X squared is. Of course, this is the norm or indicates the length squared of dx or the magnitude as I said of d capital X. Now, how do you write this down? Of course, I can write that down as F dX minus dX square, sorry, yeah, or in other words, this is nothing but what is this? This is nothing but the inner product and hence I can write that as dot F dX minus dX dot dX and using the property of, of course, of all them are, hope I do not leave this and using the property of transpose of a tensor, this can be written as dx dot F transpose F dX minus minus dX dot dX, which again can be written as dX dot F transpose F minus I dX, where I is, of course the unit tensor.

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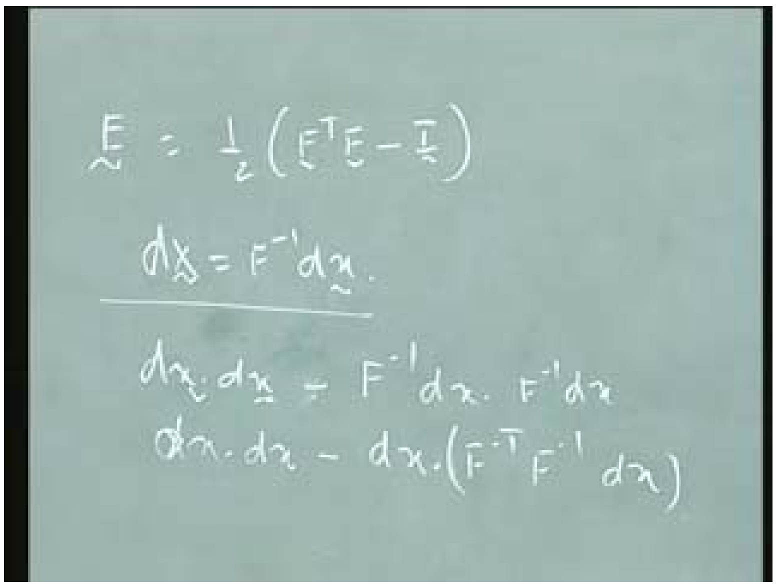

Let me define now E Cauchy-Green strain tensor to be half into F transpose F minus I. I am defining that to be half into F transpose F minus I. Now, look at this term, look at this term and look at this term here and see what we are trying to do. We are arriving at strain tensor purely based on the concept of lengths. Now of course, we saw F transpose F as what? As C; so, F transpose F can be written, this can be written as half into C minus I. Now, this whole, this whole equations can also be written in another fashion; we will see the equivalent of this with our small strain in a minute, but this whole set of equations can be written in another fashion in terms of d small x; d small x.

In other words, I can write that dX is equal to F inverse d small x. Just try it out; let us see what you get. Just look at these equations, substitute that and see what now the definition is that you get out of it. E is defined as half into F transpose F minus I. I am defining E. I will show you how this E is equivalent to that of small strain case when the deformation is small; just wait for a minute. Before that let us look at now the same expression in terms of d small x, d small x. What you do? Simple; so, keep this d small x, so that the expression can now be written as dx dot dx, sorry minus d capital X square which can be written as F inverse dx dot F inverse dx and now you can write that dx dot dx minus dx dot F inverse transpose F inverse dx. Yeah; this is the definition of transpose. So, you just get substituting for dx dx dot F inverse transpose F inverse dx.

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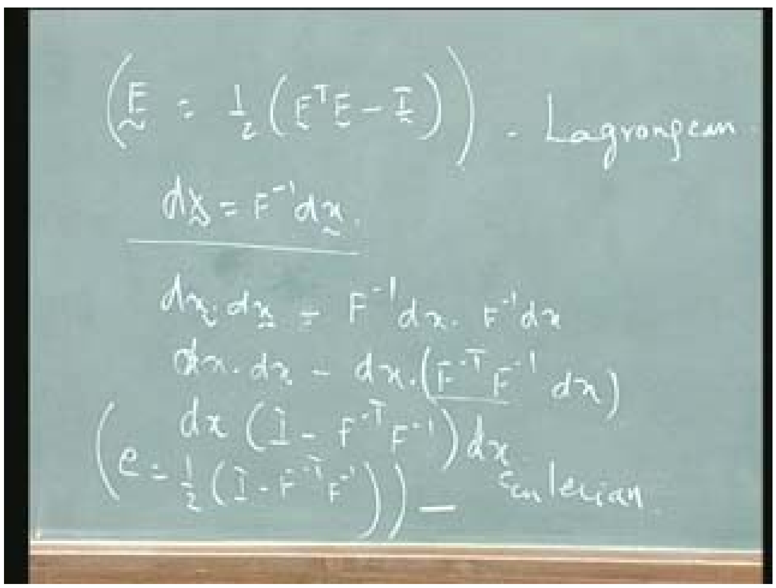

Now, just take that out. dx I minus dx is my original thing and I define small e called Almansi strain to be half of I minus F inverse as the, what is called as the Almansi strain. Look at the difference between now capital E and small e. What I have essentially done is to use the definition for transpose. Look back what we did before; look at these two equations here. You see that the first equation, the Green strain tensor or Cauchy-Green strain tensor, E is defined with respect to the Lagrangian frame of reference and this Almansi strain tensor is defined in terms of the current co-ordinates or Eulerian coordinates. So, this is the Lagrangean and this is Eulerian. So, the first thing we have done is to define strains.

In order to see what the connection is between the large, this we would call as finite deformation strain or large strain, compare this with infinitesimal strain, in order to do that, let us introduce what we call as a displacement vector and talk in terms of displacements and its gradients, so that we can compare, because we know epsilon$_{ij}$ to be half of dow u_i by dow x_j plus dow u_j by dow x_i. In order to compare both, let us now introduce the term called displacement. Is that clear?

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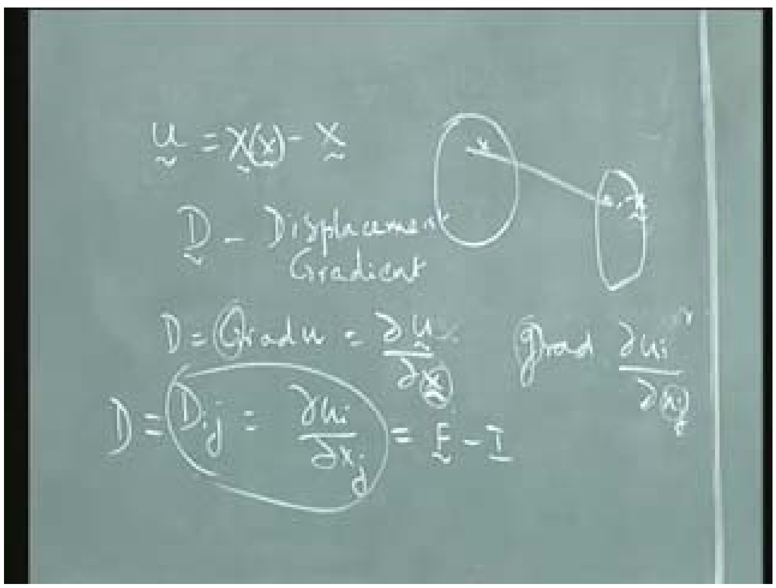

Let the displacement u be, obviously, the displacement u at a point is x minus capital X; that is the displacement. In other words, if that is the body where it is capital X, Lagrangian frame of reference and we start here, say for example, most of the times I have said reference coordinates and we go to this point x. Let u is equal to x minus capital X. This is fine as far as both the coordinates or both the systems of coordinates, that you use for both, are the same or else you can also write that as chi x minus x. Let me define, many places I am defining; let me define the displacement gradient D to be the gradient of u or Grad u. D is equal to the Grad u. Please note that whenever I put Grad with the capital G it means that the derivative is with respect to the capital X. This is nothing but dow u by dow X. Note that Grad u raises the order of this vector from u to a second order tensor or in other words, D_{ij} is equal to dow u_i by dow X_j. If I now put, instead of capital G, if I now substitute this in terms of small g, if I say it is grad, small g, then it means that the derivative with respect to small x which means that dow u_i by say dow X_j. They go together and this and this they go together. Is that clear?

Now, define E in terms of this D. Let us see how you define D in terms of this. It is very simple. I can extend that by differentiating this with respect to capital X. So, the first term becomes what? The first term here becomes the deformation gradient tensor F minus the second term becomes I. So, D which is given by this, D is equal to F minus I. Is that clear or D_{ij} is equal to F_{ij} minus Kronecker delta_{ij}. Now substitute this expression here into my definition for E.

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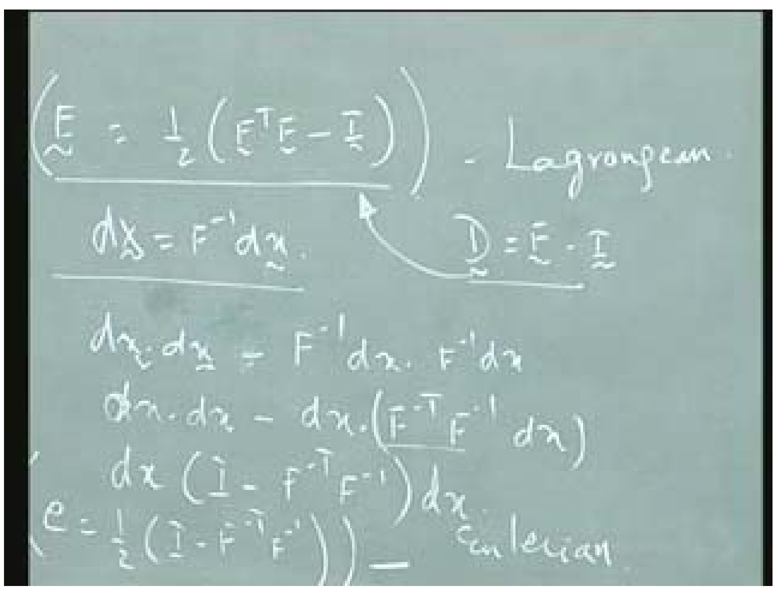

Please substitute that and see what you get as a result or in other words, substitute D is equal to F minus I. Just work it out in a minute, so that it will be a good exercise for you. From here, substitute for F into this expression and see what the result is. What you do is just F is equal to d plus I; so, F transpose D transpose plus I transpose, so, substitute it here and let us see what is the result that you get?

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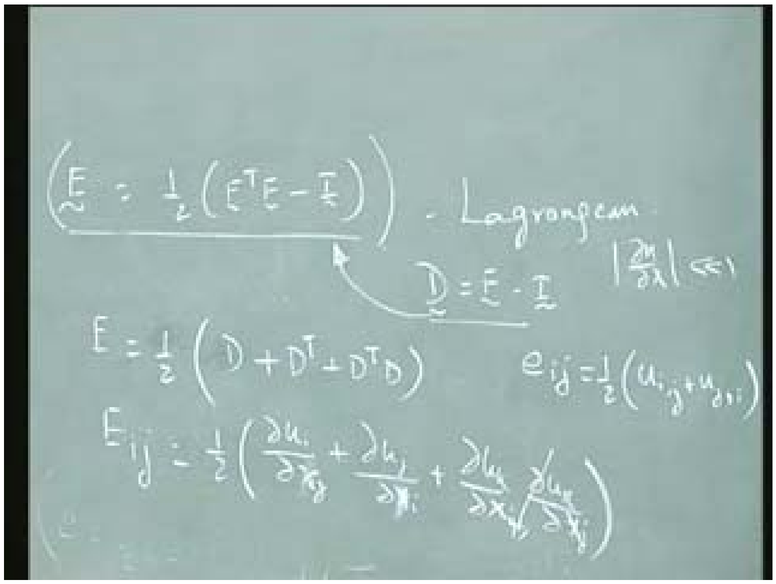

Half will remain the same, so, you will have actually three terms - D plus D transpose and then what is the third term plus D transpose D. So, that is the thing you will have; these are three terms you will have. Now, just contemplate on the difference between this and our definition for small strain. Now, actually what is this? If you want to understand this, let me write down in an indicial notation. Actually I should write capital IJ, but because you are familiar with small ij with respect to small e, let me write that down; half into, what is this? dow u_i by dow X_j plus what is the next one? dow u_j by dow X_i plus, how do you write D transpose D? No; it is dow u_k dow X_i dow u_k by dow X_j. That is the D transpose D.

Yes, you are correct. I should have written it as capital I capital J and because I want it to compare with small e_{ij} with which you are familiar with, I am writing it for the time being like that. You are absolutely right. All these i's and j should be in capital letters. X also is capital, because if you look at how we defined the displacement gradient, look at this; look at that here.

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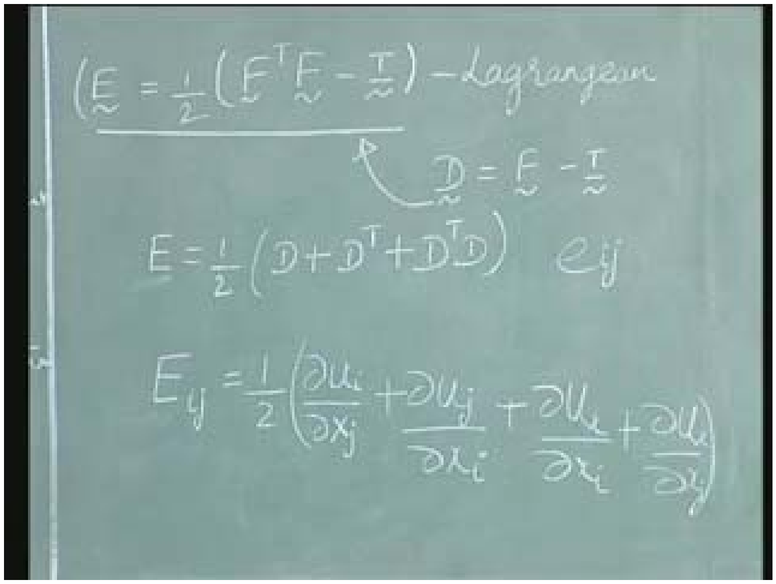

That you had defined the displacement gradient to be capital G; capital G, so, that is the reason why you have this to be like that. Now, look at these terms carefully. All of them are capital. This is capital; capital. I hope it is clear. Now, your definition, if you remember, for e_{ij} was half into u i comma j plus u j comma i. That was the definition for small strain. So, see how it reduces, this reduces to that. Once, if small x and capital X are not very different, in other words, deformations are very small that are infinitesimally small, so that capital X and small x are not very different, then I can replace, number 1, all the capital X by small x and number 2, when the gradients of displacement are also very small such that the gradient is far less than say, dow u by dow x is far less than, 1, then this term goes off.

What we say is we have linearised these quantities. So, linearization of my displacement would result in E_{ij}, capital E_{ij} to be equal to small e_{ij}. So, in other words, the concept though it looked initially as if I had defined half F transpose F minus I arbitrarily, it is not arbitrary. Actually if you look at that closely, I had dx, d capital X on either side. So, there is a length scale there as well and now you see that it actually reduces to my old definitions here. Is that clear? So, both the definitions go together.

Having now defined what strain is, let us now look at what is called as stretch, what is called as stretch. In other words, what it really means is my next concept on stretch means that there are different ways in which you can study how a material line element would deform. One way to study it is to define them using E, another way is to study using stretch. There are so many other definitions for strains. So, strain is not just one quantity. Now itself, we have seen capital E and as well as the other, what we called as the Almansi strain; Almansi strain.

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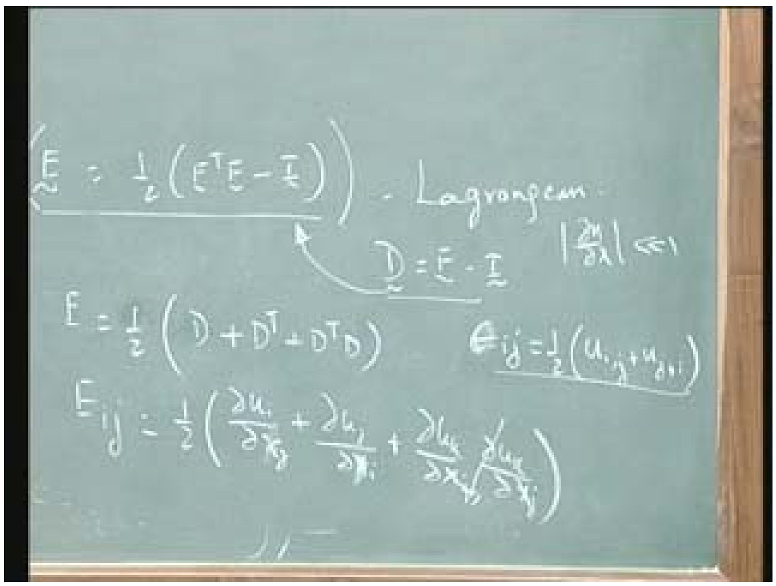

Please note, this small e is from our epsilon. I should put epsilon here, so that there is no confusion here. We should put epsilon here. That small e what we use right now, in this course from now onwards, will be for Almansi strain. Already we have defined two strains. In fact, there are a number strain measures as we call it and we may define them later for use in certain constitutive models. But before we go further, we have to define what is called as stretch. As the name indicates, stretch means that the stretching of this line element d capital X or in other words, when I say stretch, obviously what we are interested in is the amount to which or ratio of dx by d capital X.

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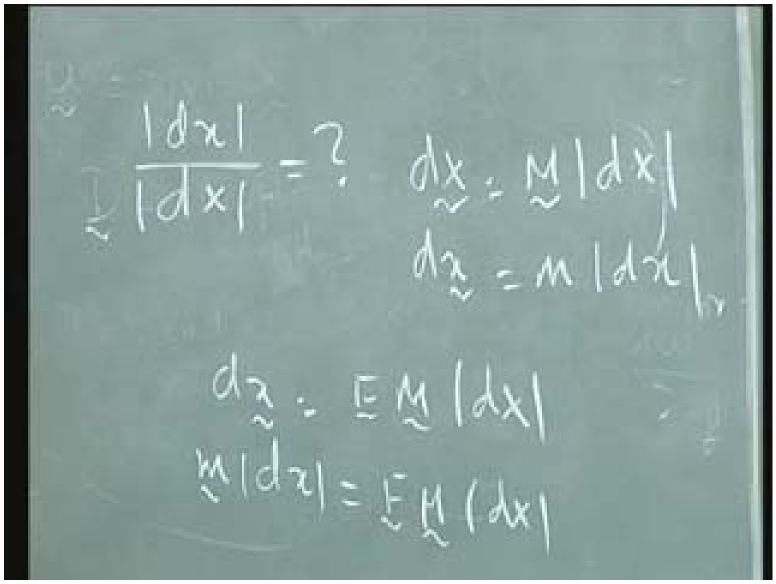

What is that? Now, this d small x and d capital X can also be defined by using unit vectors along their directions or in other words, dX vector is unit vector say, M into the magnitude d capital X, d small x can be defined as m d small x, so that dx using a deformation gradient tensor can be written as F M dX. So, this can be written as m dx is equal to F M dX. M is the unit vector along the direction which we have chosen, d capital X. Is that clear?

Student: is it along the line element?

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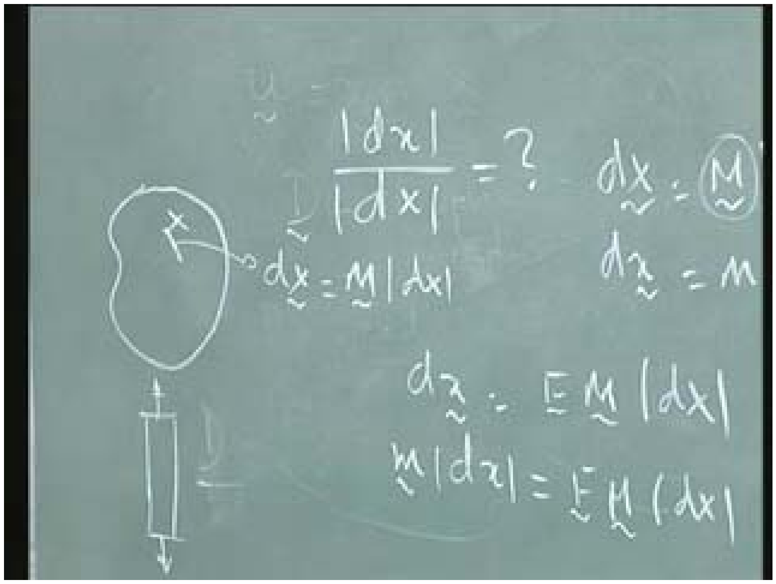

Yes; say for example, if this is the body which you are considering and if this is the point X, capital X, and I am interested in this line element d capital X; that is the line element which I am going to follow. Let capital M be a unit vector along that direction. So, d capital X can be written as M, which indicates the direction multiplied by the magnitude; nothing very difficult about it.

Student: one more question.

Yes.

Student: we can define the strain by taking two parts in the body, sir arbitrarily. So, is there any difference between this and that, sir. Now, we are taking one part and we are taking the line element, like that we are taking. Instead of that we define, if we take two arbitrary points and …...

Yes; please note that, yes I understand. What you are saying is that can I take two different points? Please note that we are looking at the situation it can either be homogenous or non-homogenous. We are looking at a situation where strain is defined throughout the body at every point. So, I have to take an infinitesimal line element. It is not the strain of the total body. It was okay for a uni-directional case. Suppose I take a bar and then apply here, it can be a homogenous deformation. That is a different thing all together. But, when we take a body, general body of interest, then we have to take, you cannot take the whole body; we have to take a very small region around the body in order to define what strain is.

Please note that in our earlier class we did a very similar thing. But, instead of taking this line element here, we took an infinitesimal cube. In fact, what we did was to look at the way this cube deformed at a point. Remember, all our analysis before was also based on the deformation of the cube, whether we define the stress or the strain and so on. In this case we are taking a material line element and the orientation of material line element may change and accordingly each stretch would also change and that is the reason why we have M here. In other words, as I take, I go to this point x and I take this small material line element and as I start sweeping it, I may get different stretches that prompts us to define certain very important quantities like principle stretch and so on.

But before we go further, let us now look at what we are going to get out of this. So, this can be defined here in this case as dx squared, small dx squared will take the dot product, the inner product on either side.

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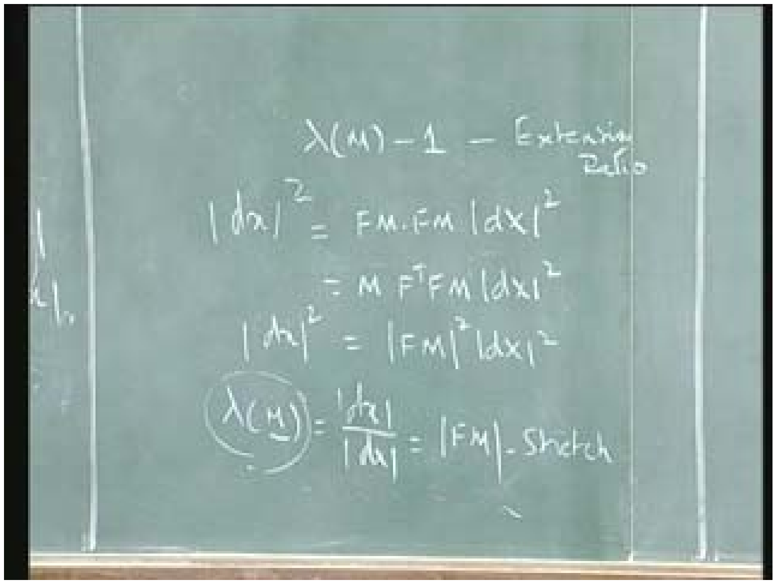

So, this would become, is equal to F M dot F M d capital X squared. This can be written by using the transpose formula M F transpose F M dx squared. No; this is the capital; small m, obviously will go off, m dot m. What is m? Because unit vector; we have been talking about that, unit vector, so, that will go off. Obviously what you will have is only the capital M. So, that is in other words, this is F M dX squared is equal to d small x squared. In other words, we define stretch along M, note that carefully; at a point X along M to be d small x by d capital X that is equal to the magnitude of F M, the magnitude of F M. So, that is what is called as the stretch. Sometimes, lambda M minus 1 is called as the extension ratio; is called as the extension ratio. Is this clear?

So, stretch is defined, note that; again I am repeating it. Note that stretch is defined with respect to M. So, when I change M, that is what I just now said, then the stretch would change. This is exactly how you would look at also the small strains. We had taken a cube such that, the faces of the cube and hence it normals are along 1 2 3. We defined epsilon_{11} epsilon_{22} epsilon_{33} and so on with respect to those faces and its expansion. Of course, we defined what we call as shear to be that change in the angle between these faces. Here, it is much more general than that. So, it is defined with respect to a given M.

There also, given any other plane or any other direction, it is possible for me from this epsilon$_{ij}$ to determine what would be the strain in any other direction.

Now, how do I define what we call as the shear part of it?

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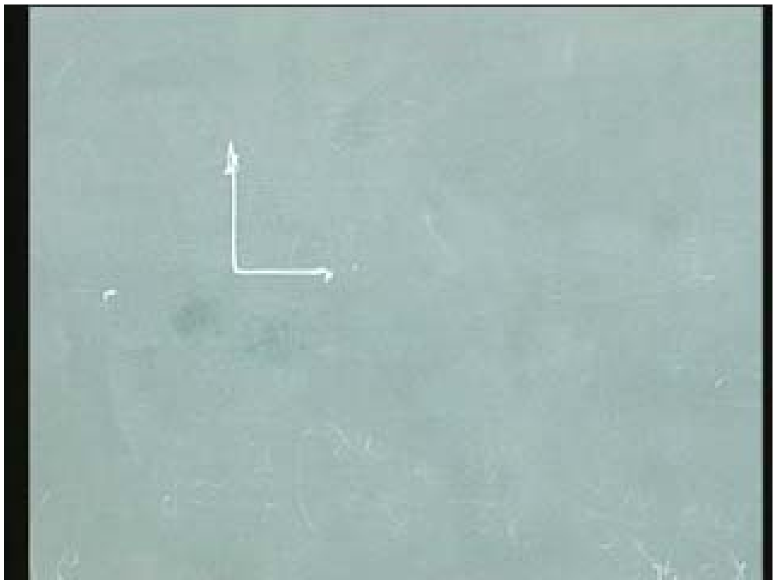

In order to define shear, if you remember, we have, we had say 2D; we had two lines and we were looking at what happens to the angle between the two lines. This is what we were looking at. So, in this case, what we are going to do is to take two line elements. In other words, we are trying to explicitly find out a formula here in this case, what would be the change of the angle of two arbitrary line element.

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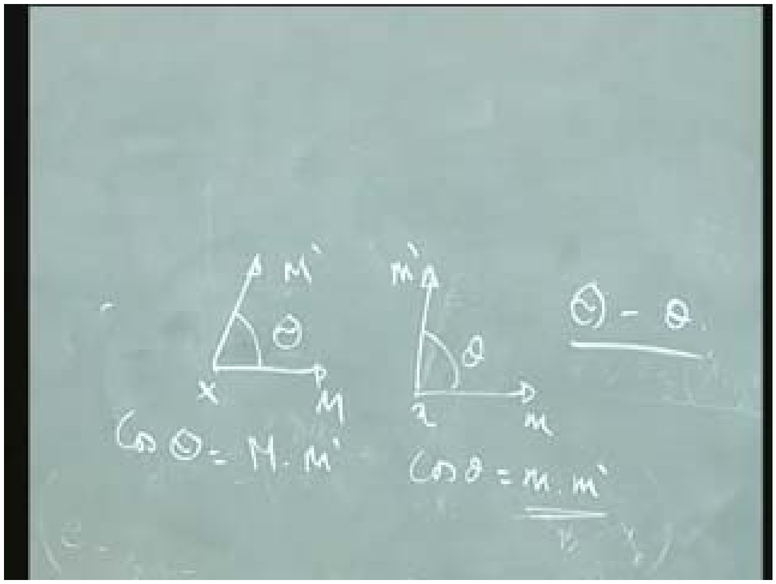

In other words, suppose I have say, let me define that as M and M prime at a point X; actually, I am just zooming into a point X in a body. Let me call that as capital theta. My whole job here is to find out, what happens to this capital theta under deformation or in other words, what is the small theta that is formed at small x or let me exaggerate that; small theta at small x, small x being the point to which capital X is carried to, due to deformation. So, these line elements now become small m and small m prime respectively. Please look at that. Again, I will give you a minute; let us see whether you can think as to how we can find out what is this theta minus theta. I am interested in this theta minus theta, how we can find out.

Yes; dot product. Please work it out; let us see where you come to in this. Look at that and look at this expression. Look at this expression here and then just see how you can do that. You correctly said that you have to use the cos theta definition. Cos theta in this case happens to be, if you want I can write that down, cos of theta is equal M dot, M is what? M is a unit vector; very good; M prime is also unit vector. So, M dot m prime is the cos theta in this case and of course cos theta in this case happens to be what? Small m dot small m prime. But I will give you a small exercise, right now. Let me see or let me check how you calculate that.

Just look at that and see how we can calculate that small m; small m, this quantity. Fantastic; so, that is exactly what I am looking at. So, this can be written in terms of stretch. What is that? That happens to be F capital M divided by lambda M. Please write that down; write it down in terms of the transpose. Have a look at this and write it down in terms of the transpose. This becomes F M divided by lambda M F M capital M prime by lambda M prime and then you can write that down in terms of M dot F transpose F M divided by lambda M lambda M prime. That becomes cos of small theta. That is all; so substitute this, F M lambda M dot F M prime.

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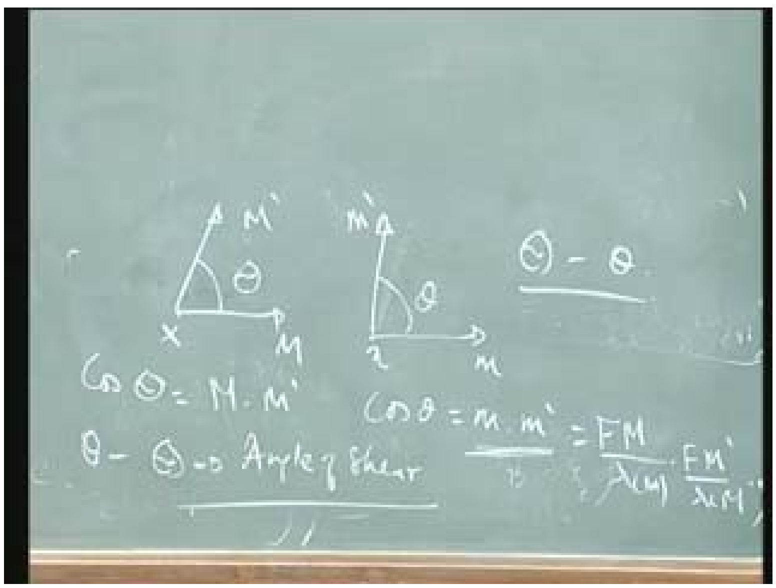

Is that clear? Now, small theta minus or the difference between them, small theta minus capital theta is called as the angle of shear, angle of shear.

Having studied a line element and also having taken into account what would be the change in the angle between two line elements, now let us look at what happens to a volume element and a surface element. They are very important for us later, for certain constraint definitions or equations to write down constraints as well as stresses. Let us now look at what happens to a volume element and how that would change under deformation at a point X. In order to define a volume element, let me take now at this point same capital X, three vectors say, called as dX 1 dX 2 dX 3 or triads of vectors dX 1 dX 2 and dX 3.

Student: In the …….., how we define small m, sir? Because, theta I mean…

Yes, what is small m?

No actually ….

What we mean by small m is the vector this capital M becomes, after deformation. This is before; this is a part of the body before and that is after.

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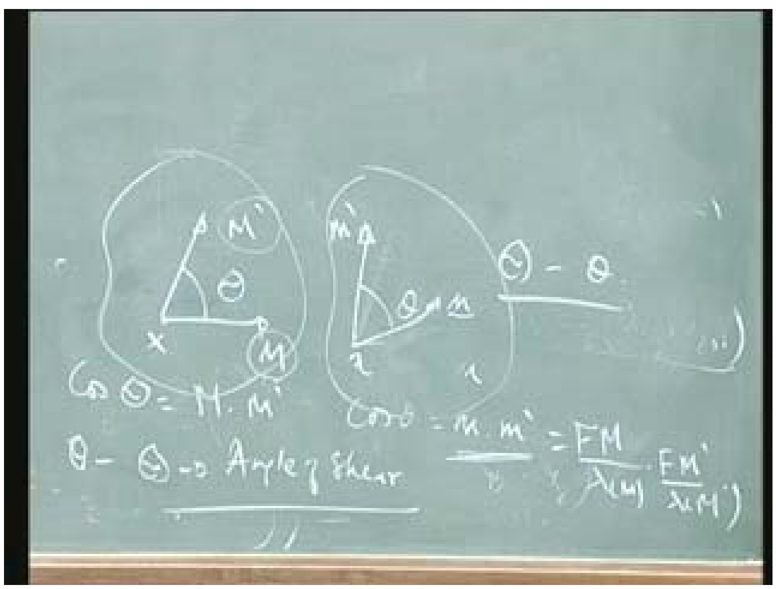

So, small x is the one to which capital X goes to and this capital M goes to small m and this goes to

Student: No; but actually element is, I mean that point is rotating.

Yeah, of course. So, if you want, you need not draw the graph like that; you can draw the graph like this.

Student: So, that way, that way theta, small theta and big theta should be same, because….

It cannot be, because it is not necessary that these two lines …., that is what is shear.

Student: No, because M we can define in any direction.

No, no, no; this capital M, please note that this capital M and this capital M prime are the ones which we are defining. These are the line elements say, line elements it is difficult for you to imagine. Just say that you are drawing two lines at a point, infinitesimal lines at the point X. That is what you are drawing. These two lines you are drawing there and what you are doing is to follow what happens to these two lines which you have drawn, after deformation. So, those two lines now become like this. Let me call that as m and m prime. It is not that small m small m prime, you can draw arbitrarily; it is not arbitrary. But, we are following the deformation of these lines and that is why we are interested in the change of the angle between these two.

Let us now look at volume change. So, volume change, as I told you, we have to see not just two elements, but three elements, a triad as it is called.

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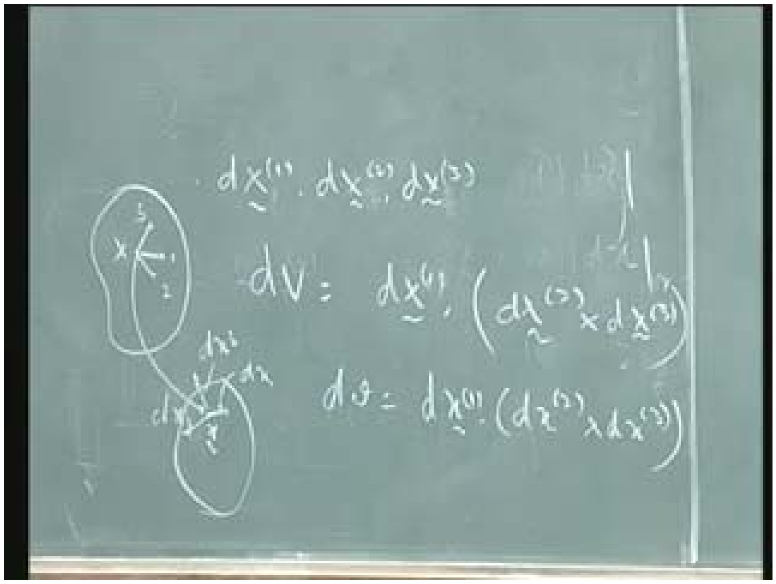

So, let these three line elements at a point X, they are non-coplanar, of course. They have to be non-coplanar; then only we will be able to get volume out of these three elements. The three elements define volume are say, dx 1 dx 2 and dx 3 say, at the point X. They define say, that is dx 1, that is dx 2 and that is dx 3; 1 2 and 3, so that dV from our good old calculus or vector analysis, we know to be, so, let me put this 1 here, so that you do not get confused with the indices. So, that 1 means that it is a vector dot dx 2 cross, this is a simple definition, 3; scalar, the result is a scalar, it is a triple product.

Now, what is that we are going to do? We are going to see or follow like what we did for the capital M, follow what happens to this d capital X 1 2 and 3 at this point, so that we will find out now what would be the volume that is defined after deformation by these three line elements d small x 1 small x 2 small x 3, after say, let us say that the body after deformation becomes like this and that point is now here and that is small x and the corresponding lines now, that line becomes dx1 and that line becomes dx 3 and this line becomes say, dx 2. What I am interested in is to find out what is the relationship between d capital V and d small v? d small v we define now, dot dx 2 cross dx 3.

What I am going to do is again very simple; very simple. What I am going to do is to substitute for F. But before we do that, we remember how we calculate this triple product. That is nothing but the determinant, if you remember.

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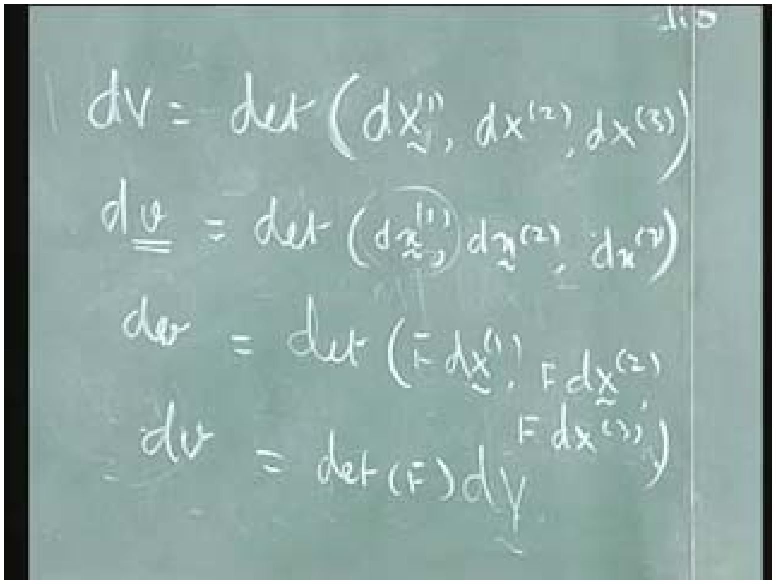

dV is equal to nothing but the determinant of dx 1 comma dx 2 comma dx 3. This is the definition for that, this operation. This is from our vector algebra, may be in high school level. So, d small v is now defined to be determinant of d small x 1 d small x 2 and d small x 3. Now, substituting in terms of my deformation gradient, so, this can be written as the determinant of F dX 1 comma F dX 2 comma F dX 3.

Yes, because we are interested in finding out what happens to capital dV after deformation that is small dv that is what we are interested in. Hence we are writing this down in terms of small dx 1 2 and 3 and what happens to this. We get that directly from our earlier expression on deformation gradient, so, we write it down like that. So, using the concept of determinant, determinant of ab, determinant of a and determinant of b, I can write that down to be determinant of F into determinant of dx 1 dx 2 dx 3, which is equal to dV.

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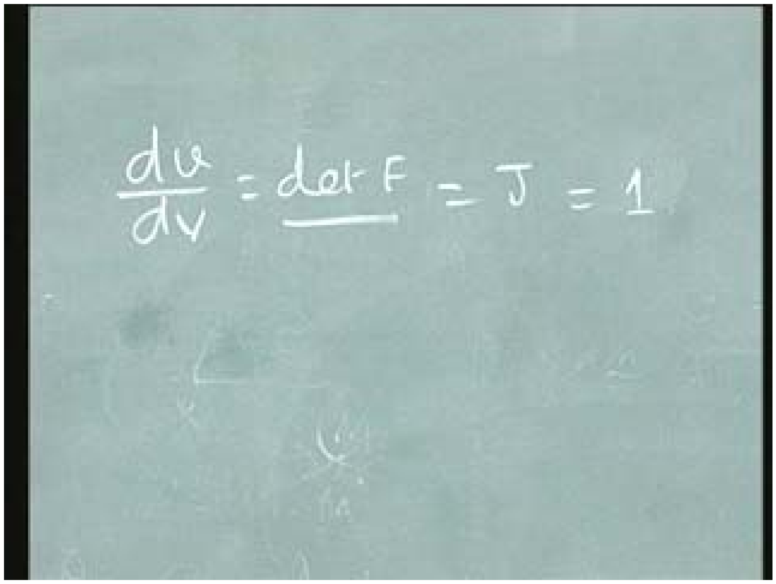

So, we come to a very, very important relationship now, which states that d small v by d capital V, the ratio of the volumes at a point is given by the determinant of the deformation gradient at that point. Please note that all these material line elements are very small line elements. So, you can say that this would give rise to that small volume change. Imagine that if you want to be a cube and what happens to that? Now, if the material happens to be incompressible, then what would happen? d small v is equal to d capital V or determinant F, many times written as Jacobian or J, that it happens to be equal to 1. Having seen how deformation of a small line element takes place, let us look at how a surface element deforms under loading in the next class. We will stop here; we will continue in the next class.