Advanced Finite Element Analysis
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Lecture - 13

Last class, we studied what is called as the implicit stress update algorithm.

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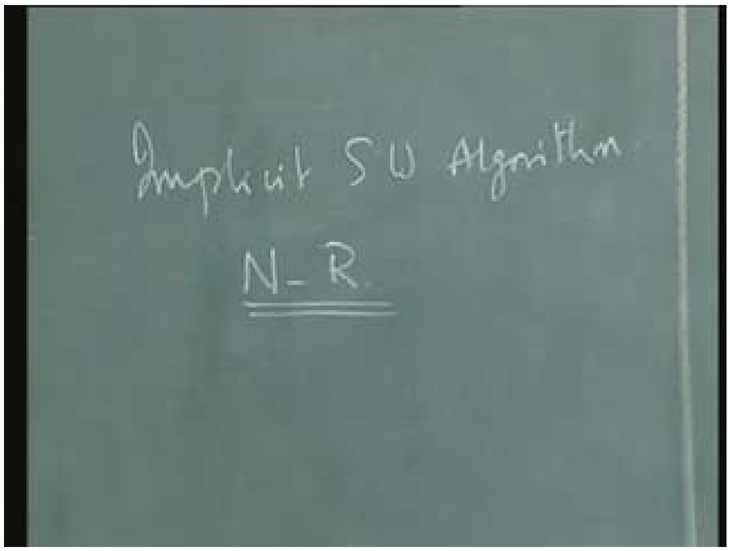

We quickly finish this and go over to different topic altogether that is on what is called continuum mechanics and its fundamentals. As the term indicates, implicit algorithm, what we essentially do is to solve for both the stress state that is in other words, where the stress should ultimately fall as well as where we are going to update the yield surface and again as the name indicates, we may have to then follow what is called as the Newton-Raphson scale.

The procedure that we are just going to indicate is due to Cemo and Artis, paper which came out in 1986 in International Journal for Numerical Methods and Engineering. There are very similar algorithms. There has been lot of algorithms that have been proposed to solve this kind of Newton-Raphson scheme and one of them also appears in the book by Zinkevich's volume 2. I am not going to go into the details of it. I am just going to indicate it and as an exercise, I want you to write down the Newton-Raphson scheme and if you have any difficulty, we will come back and see how to do this.

The procedure for this scheme is exactly the same as what we had said earlier. In other words, we have an elastic predictor part followed by a plastic corrector part. See, people call this as operator splitting as well. You split it into an elastic part and a plastic part.

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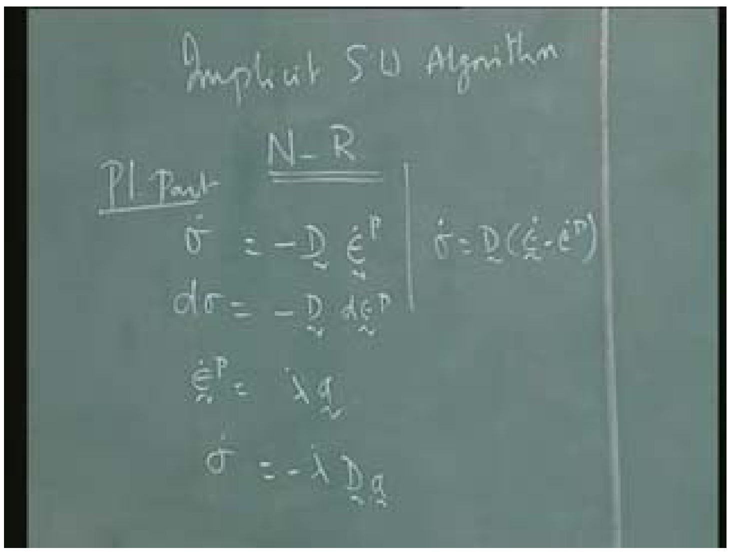

In the plastic part, what essentially you do is to correct the stresses. In other words, the stresses in the plastic part; elastic part, you know. So, we will look at the plastic part of the algorithm; then the correction is carried out by writing this sigma dot to be D epsilon dot P. Remember that this came out simply from the fact that D into epsilon minus epsilon dot P. In other words, for small strain I can use the next line, but I cannot do it for a finite strain. But, this can be, in other words, this can be written as minus D epsilon P and we know that D epsilon P or epsilon dot P is equal to lambda dot a. Substitute it back into that expression, so that sigma dot can be written as minus lambda dot Da. Very straight forward, nothing; there is no difficulty about it. But, the next step is a typical Newton-Raphson scheme.

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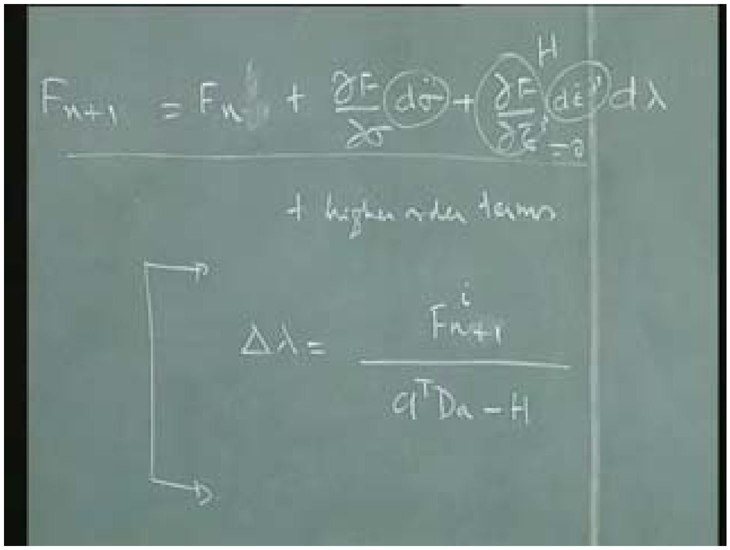

What we do is we expand F_{n+1}. What is F plus F_{n+1}? It is the yield function or the yield surface. We expand that about the current elastic predictor point; the current elastic predictor point and write that down as say F_{n+1} at say j or if you want you can write that as F_n, we will give meanings to this in a minute, plus dow F by dow sigma d sigma plus dow F by dow epsilon bar P, d epsilon bar P. This is, plus of course higher order terms. Essentially what you do is to solve this equation in the same fashion as you had done for the tangent stiffness matrix. There is no difference between the two; you solve this in same fashion. In order to satisfy consistency condition, this has to be zero.

Now, you enter the Newton-Raphson scheme. This is a new, different Newton-Raphson scheme than the iterative procedure and then substitute for this d sigma from here; substitute it here. In other words, in terms of lambda dot or d lambda, you can substitute this there and then d epsilon bar P, we know this to be d lambda; that also we know. You can write this, write the whole, you know, equations now in terms of what you know. In fact, from here you can write down delta lambda or d lambda to be equal to F_{n+1} or F_n rather, because F_{n+1} is going to be zero. F_n for the time being, when you look at it as an iterative loop or if you want you can write that F_{n+1} i, in this iteration, divided by the terms a; so, this d sigma, this term there goes to this place. So, you will write that a transpose Da is what you will get here. Then for this term, this term happens to be H; so, minus H. That is the first expression what you have for delta lambda and with this you can generate a Newton-Raphson loop. I am going to leave it at this place. I would like you to fill it up. Just one comment; in other words, we have now three loops.

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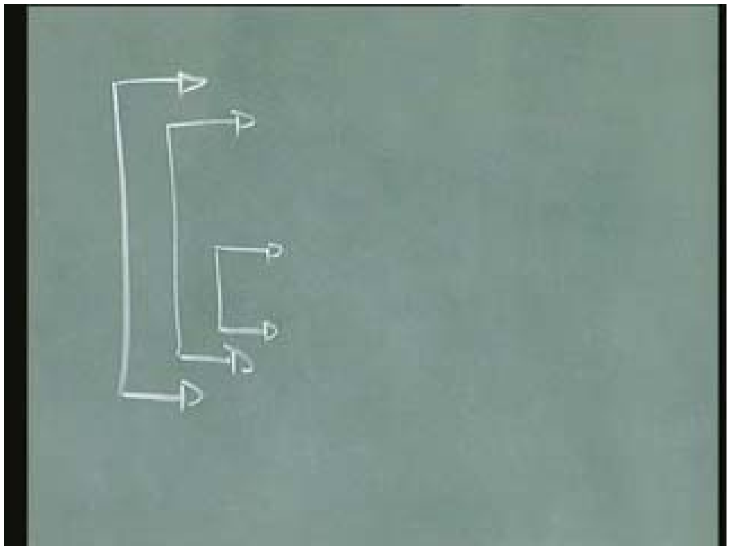

One is a large incremental loop for time step. Within a time step, one large iterative loop for it to converge and within the iterative loop, you have one more iterative loop for this stress update in order that ultimately my consistency condition is satisfied; my consistency condition is satisfied. So, you have three different now loops. This is the typical implicit algorithm. Is that clear?

D is, so, what you essentially do is to calculate delta lambda, calculate D sigma, calculate sigma, calculate epsilon bar P and then see whether F_{n+1} at a particular iteration inside this loop is equal to zero. If not go back and do this. A very straight forward thing; I would like you to complete this as an exercise. You write it down, if there is any questions we will discuss that. Is it clear? So, that sort of completes our first foray into small deformation. I want to quickly shift to finite deformations.

Many of these things, one of the reasons why I am just indicating it and leaving it is because, we will go back again to many of them and modify it. At that time, I will go into more depth into certain of these issues. But, before we do all that, I have to move to certain very fundamental issues which would not have been covered in your earlier courses and this forms the basis of non-linear analysis itself and this is under the realm of what we call as continuum mechanics.

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So, I am going to shift gears and leave finite element for the next 5, 6 classes or more and then do some continuum mechanics and then, we will come back to the finite element and how we are going to solve many of these issues. Basically, if you look at continuum mechanics, there are three things that we are interested in. One is the kinematics - the study of motion or deformation of the body. So, the first thing that we are interested is in kinematics. This is the study of the motion or deformation of the body, independent of what are the forces that are responsible.

In this case we look at, how we characterize deformation, how we define strains, is there more than one definition for strain? Of course, we are going to also see whether they are related to what we know and so on. So, we are going to look at the kinematics of deformation first. It is independent of how it is caused or the reason. The second one that we are going to look at are the balance laws; the balance laws and in that process, we are also going to look at the concept of stress. You will see how this definition of stress itself is going to be very different in this case. We are going to study different definitions of stress and its rates along with the balance laws. Lastly, we need to define the relationship between the deformation and the stress as a concept. So, we are going to look at the constitutive relation.

One of the things which I want to state very clearly right now is that, when we discuss constitutive relationship, though you know that it is stress strain relationship, they are not ones which result from a curve fitting of an experiment where you determine sigma and epsilon; it is not just curve fitting.

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In other words, what I mean to say is you just do not take a sample, you just break it and then just plot a graph, stress versus strain and say that this is the equation for this graph and say that is constitutive equation. No; constitutive equations are deep theories which falls or which is covered by the science of thermodynamics. So, it is very important that you realize right now that constitutive relations are not stress, are not just curve fitting between stress and strain. Why I am stating this is because, many of the packages deal with it, deal with constitutive relationships in two ways.

One is to ask you to give down two columns where you say this is the strain and that is the corresponding stress - one thing and the other packages which are good packages what they also ask you to not only define this stress strain relationship, but also ask you to take a material model and they would say if you are going to define an Ogden model, define it and then give me the stress strain relationship. In other words, this shows that it is lot more theory into the development of constitutive equations. That will take us sometime to go there. But before that, let us now look at the kinematics of the body. Now, first of all there are two things. One is that I have a body. This body has no coordinate system, nothing.

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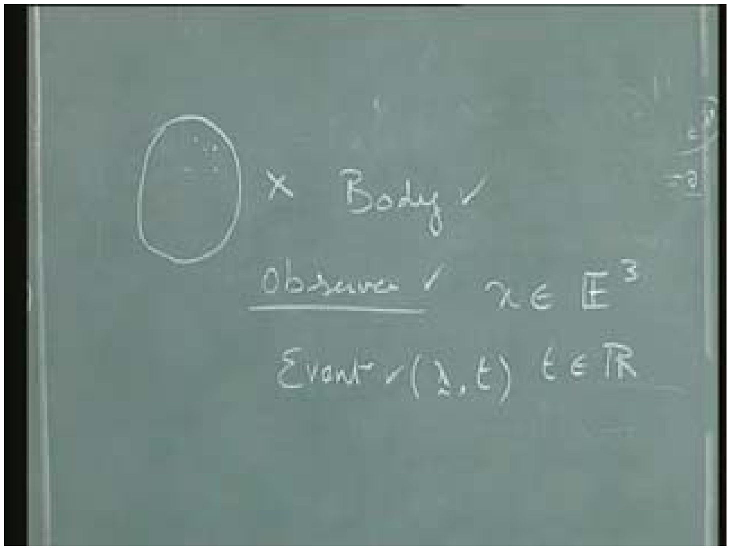

I have this body and if you want to define this body, you can say that this body consists of a set of material points, infinite number of them and those points can be say, designated by X. This is point number one, that you have a body with material points, infinite number of material points; material points distributed in such a fashion that if I go and put my hand anywhere, there is a material point there. That is I would say a loose definition of what we mean by continuum? Wherever I put a point, whatever be the smallest point that I can take and poke this body, I will not end up in a void. That is what we mean by a continuum body. It is continuously distributed over a region; that is point number one.

Point number 2, in order to study this body, that is, the body of interest, in order to study this body we have a person called observer. The observer is as important as the body because, ultimately the mathematical relationships that he is going to develop or he is going to derive depends upon his position, his behavior, his motion - all those things with respect to the body. You have to say that we have two things - one is body and then observer. The observer records, when he studies this body, he records what is called as an event; the observer records what is called as an event. Event can be looked at as consisting of two things. One is a vector and another is time. Now this vector x is actually a member of the Euclidian space, which I would call as E 3 and t is a member of real. In other words, this observer studies the motion of the body using the concepts of Euclidian space and the concept of time. These are the three important definitions that go into the study of the body; that is point number 1 - this body, 2 – observer, 3 – event.

The next issue is we are going to define what is called as the configuration of the body, configuration of the body.

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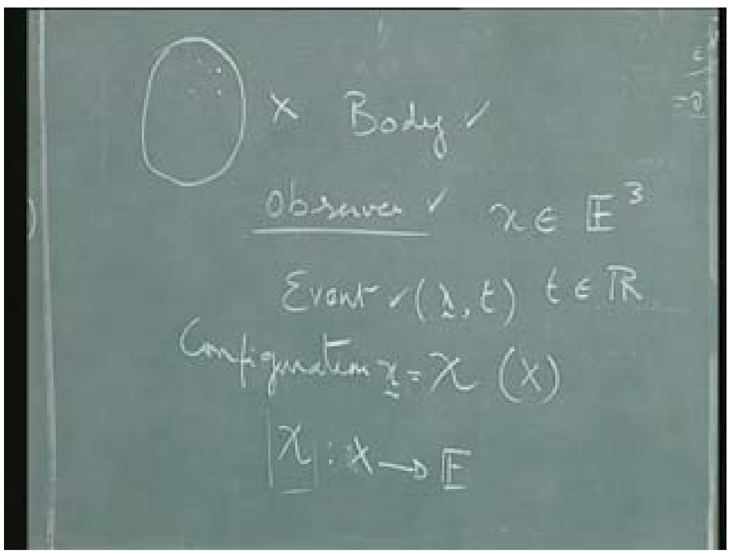

What is a configuration of the body? The configuration of the body is defined as a mapping, is defined as a mapping which maps a position X, which is a material point, to a vector say, small x. Shortly, we will look at this more closely and define the configuration and deformation more closely. But, what we essentially mean to say is that a configuration is the one which maps the points to a point in E 3 and just I will state that as E. This is called Euclidian space. Euclidian space is a vector space with an inner product, a norm defined on it. So, configuration is a mapping; understand that very clearly, configuration is a mapping. It maps; in other words, what does it mean? It means that it maps a given position of a point to vector which is a member of the Euclidian space. The mapping is not unique. There is not one value, but this mapping may vary with time; that mapping varies with, let me write that out here; that mapping varies with time.

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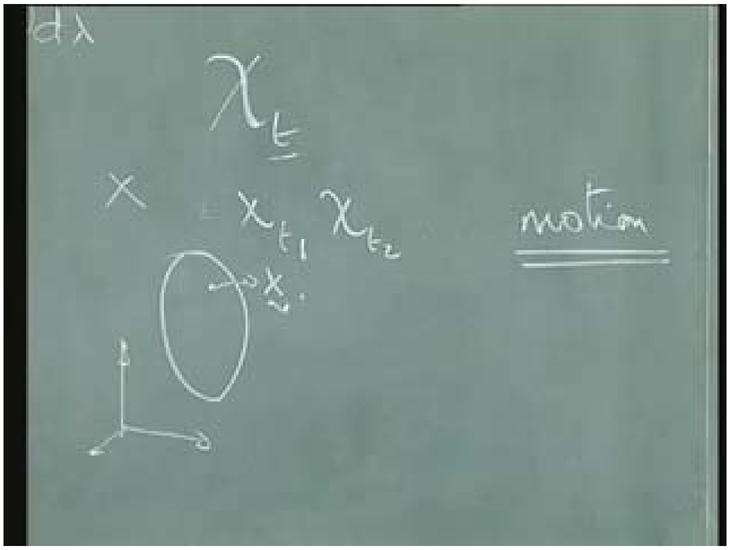

So, in other words, the observer looks at configurations with respect to time and so, say, for example he may look at it as t_1 t_2 and ….. and so on. These change in configuration constitutes the motion of the body, constitutes the motion of the body. So, chi t_1, t_2, etc, they constitute the motion of the body. Note again that the configurations involve also time t and this may change with time. This body, which consists of a number of material points, are analyzed by, the material points rather, are analyzed by the observer by placing it in a particular position, in space; in space.

In other words, if I want to analyze the motion of this, if I want to analyze the motion of this body, then what you do, what do you do with this? I place this body in a particular position and then start analyzing it. So, anybody like this, this body is different from placing the body in a particular position. So, when I place the body in a particular position, what do I mean? I mean to say that for every point in the body, I give a label called coordinates of the body. A point, material point, here of the body which was small x is now given a name called coordinates and so, becomes the x position vector. So, in other words, I will study this with respect to coordinates of the body. In fact, you will have a mapping. In other words, what it means is that every point x, material point, has a mapping to a corresponding position in the body or in this space.

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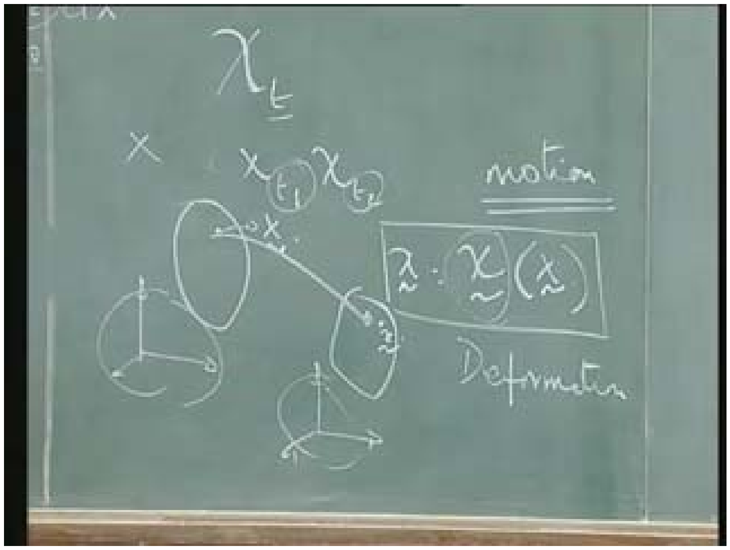

In other words, this equation, this equation what we have here, can now be written, of course this is also a vector, can now be written something like this: x is equal to, small x is equal to the deformation of the body with respect to capital X or in other words, this is a function which takes, which takes a point small x to another point, sorry, capital X to another point small x. Since I had already defined that the configuration of a body is one which takes a point from a point to an Euclidian space, obviously x is an Euclidian space which means that I can look at it as a point which is defined in a coordinate system. Is that clear? Now, this point or this particular equation brings out the concept of what is called as deformation. As soon as I replace the points of the body by means of its corresponding positions, then the mapping function, this mapping is called as deformation of the body.

Note that configuration is one which is associated with points of the body. That means that it is not very definite. Now, I have a body, I have a point here and see how this body moves. Not necessary it should deform. How the body moves, when I study this; so, I am studying the configuration of the body. As soon as I make this definite by fixing it at a position and defining a coordinate to this point and studying how the body moves, then I move from configuration of the body to the concept of deformation of the body. So, there is subtle difference between configuration and deformation. Deformation puts the whole study in a much more formal footing; that is all. Is that clear, any question?

Student: When we are studying that, I mean the point has moved to some other points in the space. The coordinate system remains the same, right?

Yes, very good question. So, does the coordinate system between the two things remain the same? I have not yet talked about it. Just pre… is what I am going to teach. Yes, they can be different. I am going to do that in a minute; I am going to do that in a minute and before that I just want to add one more thing - the point that motion consists of a number of, you know, deformations and hence this equation can be more classically written either as $x_t(x)$ or x the function of capital X comma t.

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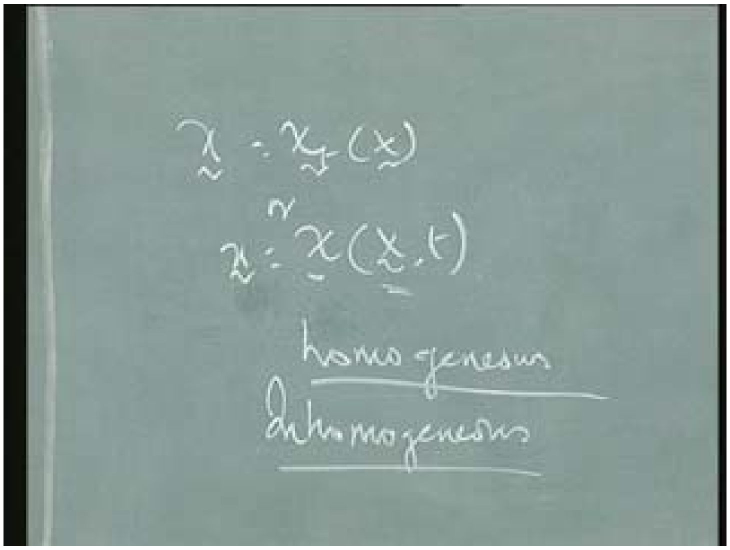

Is that clear? So, I can write this term as capital X comma t. Now, I have two choices before we go into the question which was raised. There are two things that can happen. One is that, if you look at a body like this, then this mapping, mapping function, there is some function, mapping means function; all of them are the same, it is synonymous. So, when I say that this function is one which operates on points, one of the questions that may be asked is does this function remain the same for every point in the body or does it vary from one point to another point. If this function remains the same for every point in the body, we call the deformation, deformation to be homogeneous deformation. If the deformation is such that it varies from one point to another point, then we call that deformation to be inhomogeneous. Please note the distinction between homogeneous, inhomogeneous deformation.

Let us come back to this point again; more details into this. The first thing that we do is to put coordinate system.

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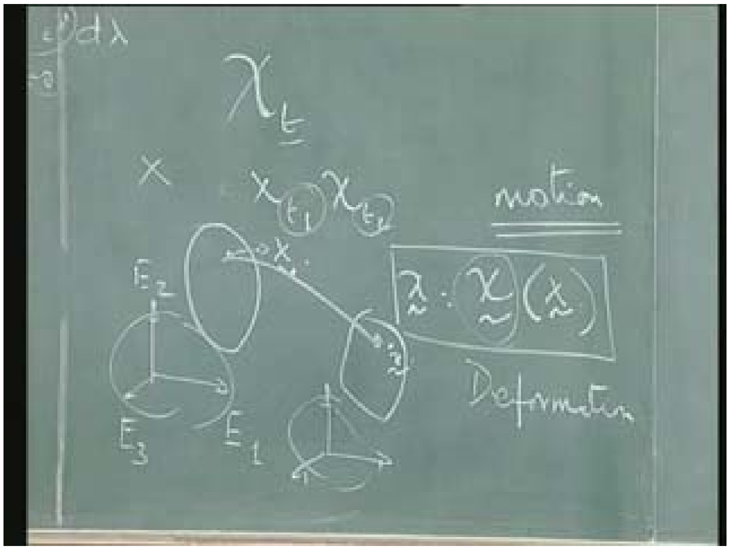

So, let me put a coordinate system for the body in order to specify the capital X and let me call this coordinate system with capital E, with capital E. In other words, this is E_1 say, E_2 and E_3. It is customary to write E_1, E_2, E_3 using capital I, capital J, capital K and so on.

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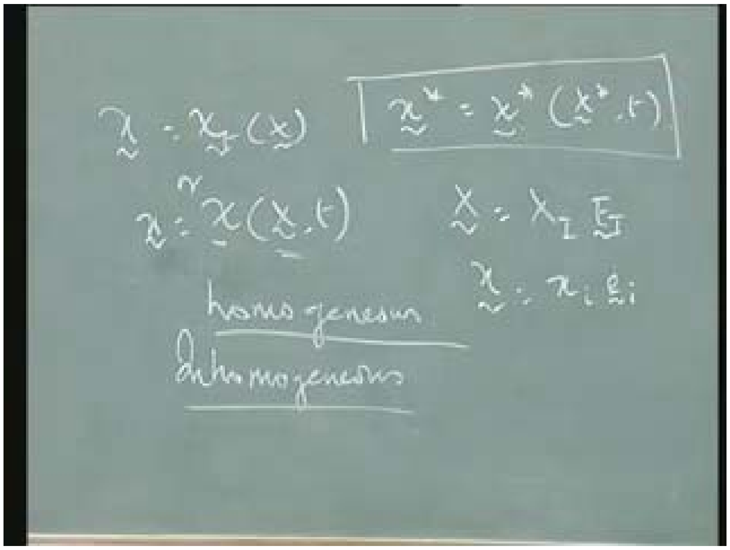

In other words, X, a position vector is usually written as X_I E_I; look at that capital letters there I, where the normal summation is involved for this I. Is that clear? Now, do not forget that we have an observer, who is going to observe all these things - the deformations. This coordinate system, this coordinate system for the body in order to define that Euclidian space in which small x sits can also be made definite with the basis vectors to be say small e_1, small e_2 and small e_3, so that small x here can be written as x_i e_i. Of course, that is also a vector. Note that this chi, what we had defined, is for a particular observer. If the observer changes, then this equation remains the same, but the way we write down the equations are different.

For example you and I are two observers, who are going to observe a particular event that takes place. Let me call you as a stared observer and I am unstared observer. Then, I can write this down as say, x star is equal to chi star x star comma t; x star comma t. In other words, the type of equations are the same. The type of equations that you use is the same, but the ultimate mapping function itself will be different. What is this difference and how it works is a very, very important concept in continuum mechanics. Right now, it is very clear or it may run in your mind that if two of us, two of us observe an event, there has to be a relationship between your observation and my observation, such that the final result on the body is the same.

In other words, what we mean to say is ultimately the body breaks. In very, very mundane terms what it means is ultimately the body breaks, so, there is some damage in the body. It is not that you observe from a different position and I observe from a different position, with a different coordinate system. Of course, we are not talking about the accuracy of the machines which we use, they being the same. It is not that what you observe and what I observe, it should not be very different. Ultimately for both of us the damage that is caused in the body due to the external forces or ultimately the deformations that the body takes should be the same; deformation, within quotes, not this deformation, but deformations what you usually understand. They are going to be the or in other words, it is not that when a load is applied and you observe the body does not break. I observe it from another chair, may be I move the chair a bit and the body breaks and it is not, all these things are not possible.

So, I have to have a relationship between your observation and my observation. We will develop all that, but we need some theory before we do all these things. Nevertheless, I just want to state that that is the type of equation which we will be dealing with in our further definitions. Now you may ask a question whether this capital E and small e, they have to be different or can they be the same? It is not necessary that they have to be different, but as a more general case you can take them to be very different. Now, before we proceed, let us understand this mapping. Let us understand this mapping from the point of view of a rigid body motion or in other words, how this mapping will appear if the body is undergoing rigid body motion? Then, you will understand this whole thing; very simple …...

I mean this concept is very simple. You know that when we say rigid body motion, this body may translate and rotate or rotate and translate.

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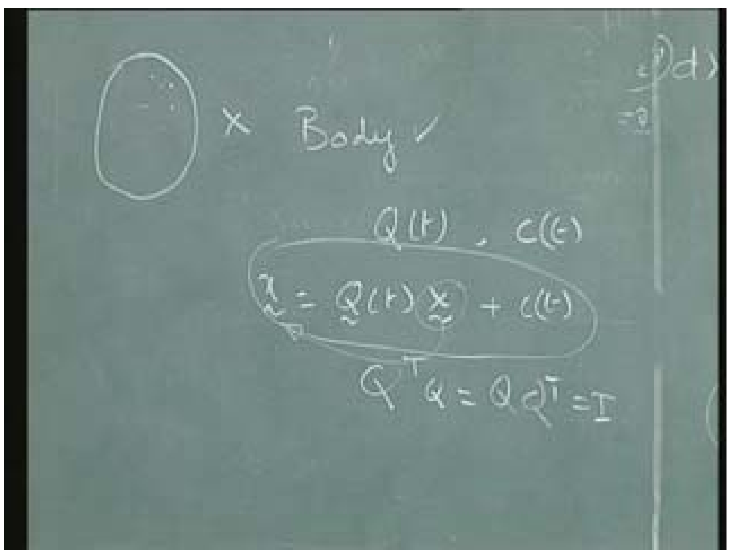

So, there are two things that are involved, so, two vectors - one actually a tensor and a transformation, which is called as C. In other words, I can say that the final result that is x which is the function of capital X, can be written as Q(t) x plus C(t). Note that this is one of the mapping functions; mapping function, in a sense that given this I will get that. If the mapping function is of this equation where, of course, we have a very important property that this Q has to satisfy. Q transpose Q is equal to QQ transpose is equal to I, which means that Q is an orthogonal tensor. Then, this mapping gives me a rigid body motion or the motion that body undergoes.

Note that there is a time there. So, I will get a series of positions of x and hence a series of deformation with respect to time and that this is what we call as rigid body motion. Yeah, this Q what I am saying here is that, this Q has a property that Q transpose Q is equal to QQ transpose is equal to I. We will see that later why this is, but this is what is called as an orthogonal tensor. In other words, you know, may be many of you know that I have to use an orthogonal tensor here, so that, what is the physical property of a rigid body motion? What is the physical property of a rigid body motion? Length of two points does not change, so that length of two points in the original as well as the final, you know, body or in other words, ……. deformation they do not change. Hence using that property it is very straight forward that Q transpose Q is equal to QQ transpose is equal to I. This is one of the examples.

Now, let us look at a body which is undergoing not a rigid body motion. As I just now said, then immediately I look for one important thing. What is it?

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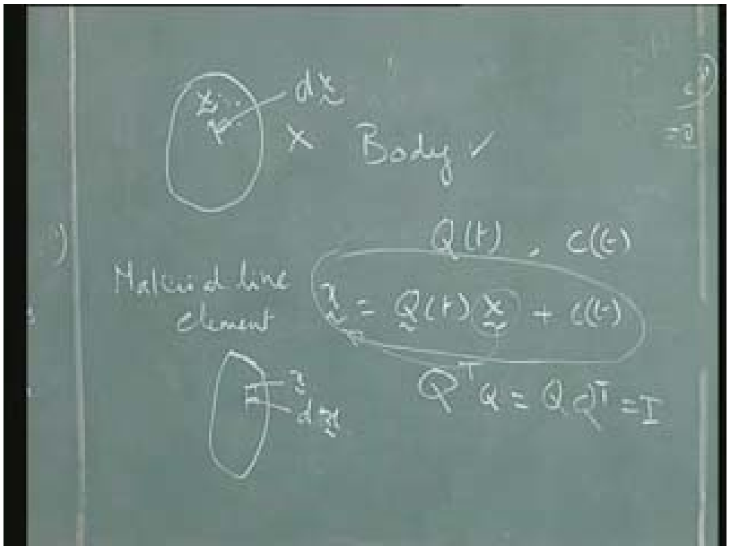

Then, say for example that is the point x. That is what I call as a material line element. This is again a very important thing, material line element. Many researchers and workers are enamored by this term material line element; people call them also as fibers. Material line elements are actually small lines like this, which are drawn, which are vectors basically; dx, infinitesimal lines. Of course, there are material points all along those lines; that is why they are called as material line elements. People also call this as a fiber. This has nothing to do with composite material. This fiber, this concept of fiber has nothing to do with it. It just states that it is a material fiber; nothing but a small line with material points sitting along this line.

As soon as I say that I am going to look at deformations, apart from the rigid body motion, one thing that comes to your mind is how does this deformation or this line element going to change under the motion of the body? Let me call that line element at a particular time; I will just remove the t for a minute at a particular time at that point, so that point let us say that that point becomes small x and the material line element becomes d capital X sorry d small x. Now, you are curious immediately to know what is the relationship between this material line element, its length and so on and this material line element. The concept is very simple. I am going to use here, what I have.

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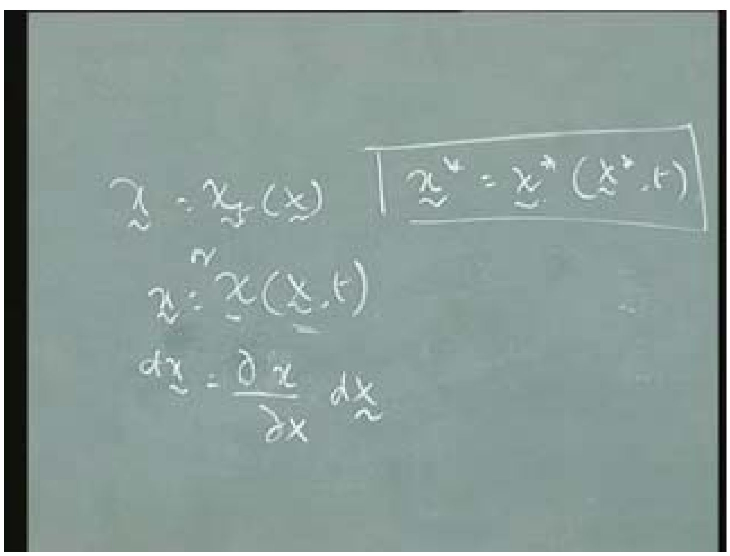

In other words, what I want is dx and that can be defined as dow chi by dow capital X into dX. This being x, you can write that as dow small x by dow capital X into dX.

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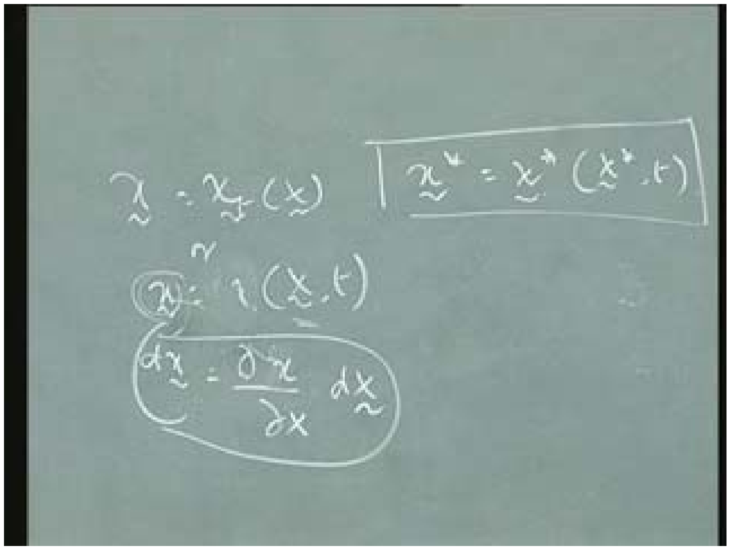

This is nothing but, for this, this being small x, I will just substitute it here. Many text books in continuum mechanics usually use the same x to define, note this carefully, to define the deformation. Many text books you would find that this chi does not exist and you will see small x there as well. Please do not get confused between the small x which is a coordinate, it is a coordinate and small x which is a deformation. Please do not get confused between them; it is usually a practice to express this, this equation in this fashion. It is just a practice.

You are absolutely right; in fact right on dot if you use the function which we had used, but it does not matter. Look at this equation here. That is the most important equation in the whole of continuum mechanics. Yeah, so that is why I said, we will remove it at a time, you know the time, let us remove it; this is at a time.

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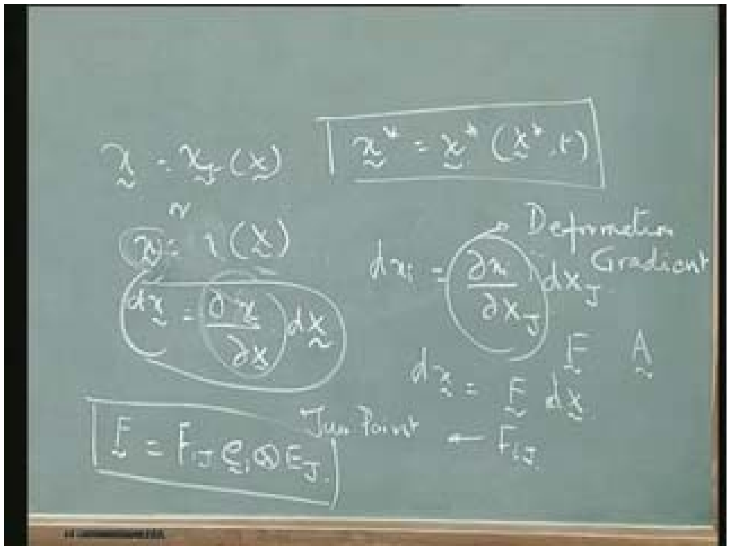

I said that let us, let us consider, let us remove it for the time being and look at this alone. We can apply the same thing at different times. In other words, this is if you want to be very strict, this is at a particular time. So, this quantity, look at that quantity dow small x by dow capital X and let me write down, in order that you understand this quantity, let me write this down in indicial notation, so that in indicial notation dx_i is equal to dow x_i by dow x_J dX_J. Note that my summation is with respect to a capital J, which means that capital J is one which, what is it, which means that I am going to define this quantity with respect to my capital E's.

So, what is this quantity? This quantity is the most famous quantity in the whole of continuum mechanics called deformation gradient tensor; that is called as the deformation gradient tensor. Except the book by Ogden, all other books and all other literature indicate this quantity with a capital F, with a capital F. This book by Ogden on non-linear elasticity is a very famous book and that book alone uses a letter A for it. The rest of the books use F. In other words, again coming back to my tensorial notation, this can be written as F dX.

Now, if we look at F, I have, if you look at F actually from this, I have two indices, small i and capital J. This J indicates X_J, this i indicates this x_i. Hence this quantity is called as two point tensor; two point tensor, because if you look at it in more lighter term, it has one leg in one configuration of the body, the other leg in the other configuration or deformation of the body. So, it has two legs sitting on two different points of the body. Hence it is called as two point tensor. More correctly, if I have to express it in a tensorial form, then I have to express that as F_{iJ} e_i dyadic E_J; e_i dyadic E_J. This is our old course in which we indicate the dyadic notation as the basis for the second order tensor.

Obviously, this is, this can or this qualifies to be called as a second order tensor. I am not going to the details of it, but nevertheless you can see that it is a linear transformation function. It is a linear transformation function which transforms a material line element dX, which is a vector into another element d small x which is also a vector. So, it is a transformation function; it is a linear transformation function, because F does not depend upon capital X. If you go back to the history of how we define, then F is actually a function of, it is not function of, sorry, dX but it is only a function of capital X. It is not a function of dX, which means that it is linear. Only if it becomes a function of dX, I may have to worry about that not being linear transformation. But, F is a function of from here we get, from here; so, F is a function of capital X and so, it is a linear transformation function and also, F has a very important property.

F is also one to one and on two. It is one to one and on two which means that every line element, every line element is taken to another line element and that every other line element has a predecessor. So, if I have these two are the positions of the body, then every line element here has a predecessor here and that is unique. So, this is a one to one and on two which means that F has an inverse which takes me back from d small x to d capital X.

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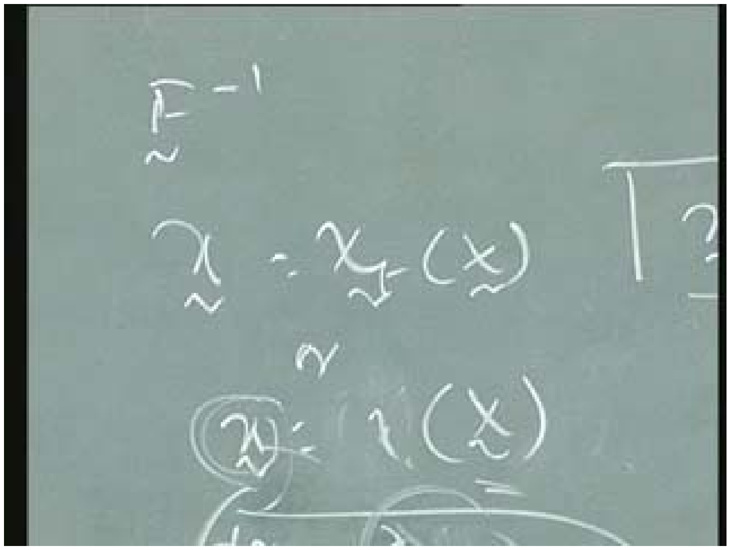

So, the definition of F inverse is that. Because of the requirements that I have on F, F is twice differentiable functions.

We will close this lecture with the last comment. Look at this equation and see whether there are other restrictions which I have to put on F. Say for example, F, can it be a null vector? Can F be equal to zero? Actually I had, I had preempted it by defining F inverse, but F can it be zero? Why? It cannot be zero, why? No; it is not the position, but note this carefully. What is the role of F? If this happens to be zero, then it maps a material line element dX to zero, which means that it disappears, annihilates. It is a like a bomb which annihilates the deformation, sorry, the line element dX. But, no line element is annihilated. If at all it is going to remain there, it may remain with no change, no change in the dimension which means that F takes at the maximum value of i. So, it cannot take a value of zero, because if it takes a value of zero, what you are essentially doing is to annihilate a line element. In other words, determinant of this F is not equal to zero. The determinant of this F is not equal to zero which means that it is going to or it has an inverse attached with it and which we have already preempted, that we called as F inverse.

One more small terminology I want to introduce, which we will use again and again in this course.

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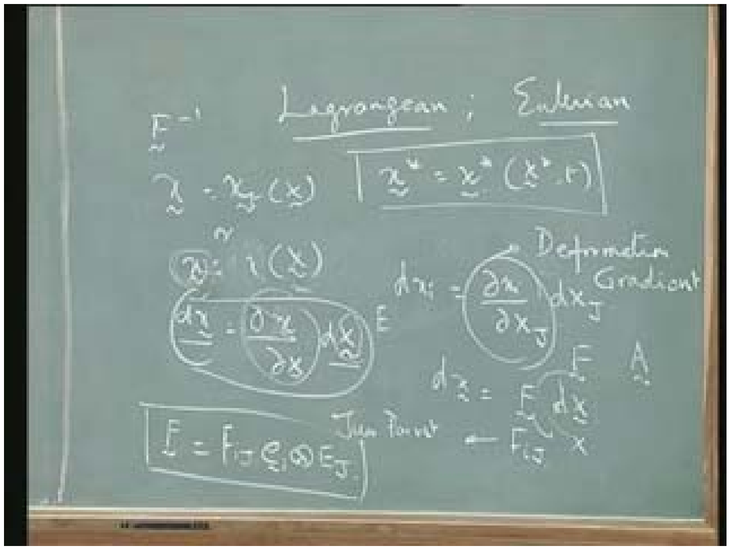

That this coordinates, E coordinates which are used to define this capital X are called as Lagrangian coordinates and these coordinates which are used to define the current position of the body is called as the Eulerian coordinates. Before we go further, we have to define one more thing what is called as reference configuration, which we will do it in the next class. Remember that here we are talking about an Eulerian coordinates and here we are talking about a Lagrangian coordinates. We will stop here; reserve your questions for the next class and we will continue in the next class.