Advanced Finite Element Analysis

Prof. R. KrishnaKumar
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture - 12

In the last class, we had come up to the stress update algorithm and we said that there are three types of stress update algorithms.

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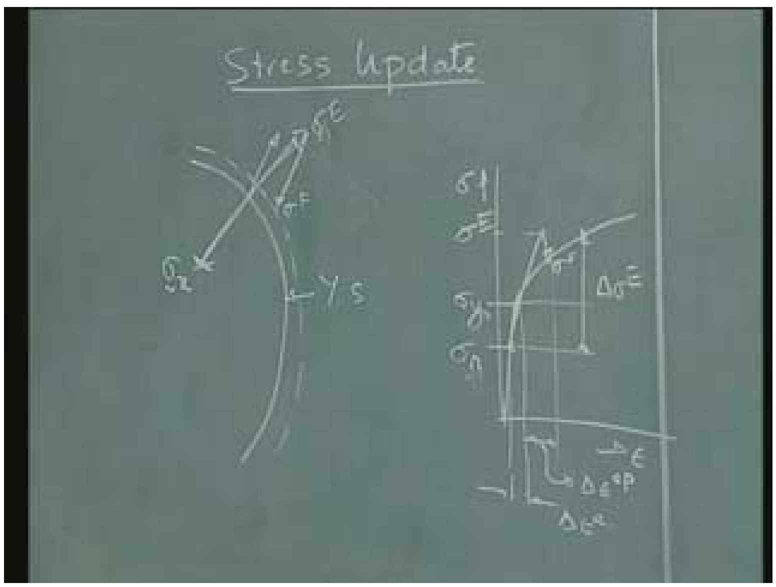

One is the explicit type; another is the sub with sub incrimination and third is the implicit type. Let us now look at the explicit stress update algorithm, which will be executed in every iteration. We had seen about that in the last class and in fact we had seen this type of algorithm when we had, when we were doing the one-dimensional case. Remember that we had predicted using the elastic properties; in other words, there is an elastic predictor followed by an plastic corrector. The game that we are going to follow in this case is exactly going to be the same with say, for example, starting from this point we had called that the sigma$_m$ last time. Since it is going to be inside an iterative loop, let me call that as say, sigma$_x$. Of course, it is a vector; sigma stress vector. Strictly speaking it is a tensor. Let us not worry about that; we know about it. From here, what I am going to do is to do an elastic prediction and take this point outside the yield surface to sigma E. That is the first step.

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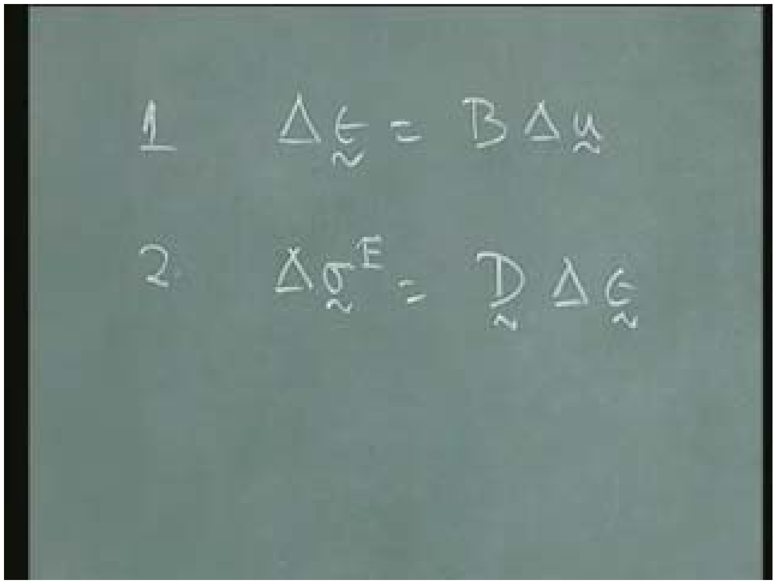

Even before that, of course you would have come to this stage after calculating u or in other words, delta u and you would have known at this stage what is the increment in strain. Let me call that increment in strain as B into delta u. Using this, I calculate what I call as delta sigma elastic; that is the elastic predictor part of it and that is equal to D times delta epsilon. So, that is the second step. Here I check up, of course whether I am inside the yield surface or I have gone outside the yield surface.

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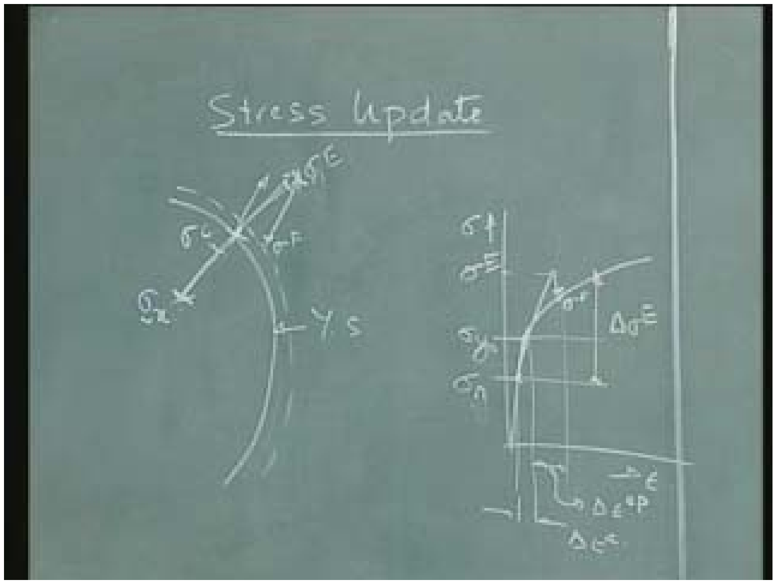

Now, if I am still inside the yield surface that means that my increments have taken me only from here to here. Then, I am not going to follow the rest of the steps, the situation is very straightforward. But, if due to this incrimination I have gone outside the yield surface that is in other words, I am at this point, I have to bring it back into the yield surface. So, that is exactly what we did here as well. How am I going to bring it back to the yield surface?

Of course, the first thing that I have to do is to find out, like I did in the one-dimensional case, where I hit the yield surface. That point, let us call it as a sigma_C where I am contacting the yield surface. Remember that we had 1 R and 1 minus R and so on in the one-dimensional case. Same way, I have to find out what is the part of delta sigma E or in other words delta epsilon that I have to correct or that I have to reserve for the elastoplastic case and what is that part that I can add straight away with the existing stress. How do I do that? It is very simple. I know the equation for yield surface.

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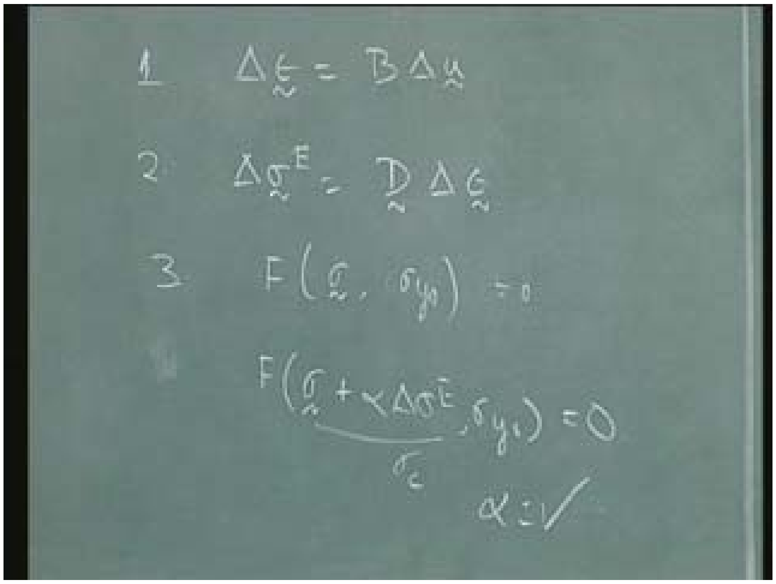

I had already written down that equation as F of say, sigma comma $sigma_{y0}$ is equal to zero. Of course, after first hitting the yield surface, first time hitting the yield surface, the $sigma_y$ becomes a function of $epsilon_p$; $sigma_y$ becomes a function of epsilon bar P that is equal to zero. But, right now for the time being we are still not come to the stage, so, I am just putting that as $sigma_{y0}$ only, because we are going to cross the yield surface for the first time.

Now, this is what I am going to solve. How do I do that? Quite simple; I am going to state this, restate this, as a function of or in other words, let me remove this; sigma plus say, alpha times delta sigma E comma $sigma_{y0}$ should be equal to zero in order that I calculate this contact point. This is what will give me the contact point; that is this point here. Remember that this would result in a quadratic equation and we will solve for alpha from this. This alpha is that 1 minus R term what you had seen in the one dimensional case. So, once I know alpha that means that alpha times of delta sigma E is what I should not correct.

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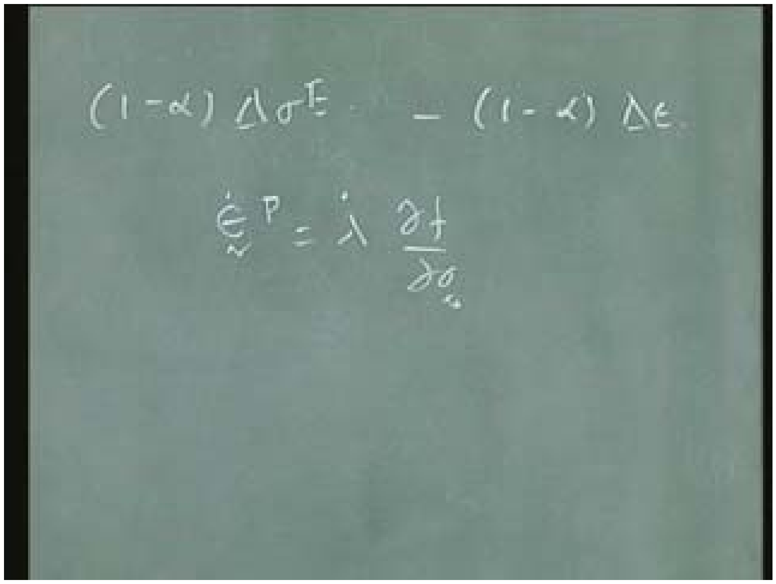

Then, I take the rest of it, 1 minus alpha term and then multiplied by delta sigma E and that is what I have to now correct. Absolutely right; so, I will have a corresponding delta epsilon, delta epsilon which goes inside the elastic part and elastoplastic part and that is also, as we know before, 1 minus, so, the corresponding things will be 1 minus alpha into delta epsilon is what we are going to use as the elastoplastic case. How do I do that and that is where this explicit scheme comes into picture. I have to use of course, the flow rule; the flow rule or normality flow rule. Remember that the normality flow rule was written as epsilon dot P is equal to lambda dot into dow f by dow sigma.

What is dow f by dow sigma? That is lambda dot dow f by dow sigma gives epsilon dot P and what is dow f by dow sigma? That is the normal at the point of contact. Now, there can be confusion at this point. How am I right in taking the normal at the contact point, how am I? Because, the yield surface is also going to progressively change or expand as I now calculate the stresses using the elastoplastic conditions or in other words, I will calculate the elastoplastic strains and this will help me to expand the yield surface. So, how am I correct to take that as the point where I calculate the normal?

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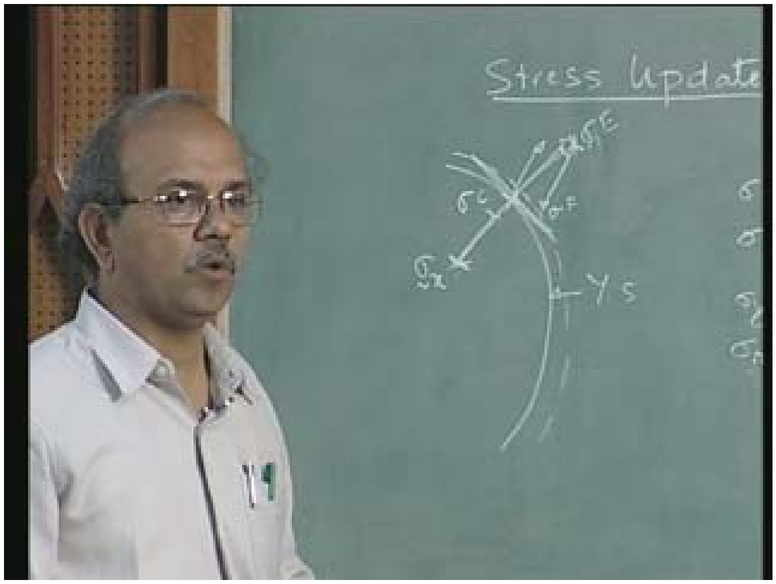

Absolutely, very valid question. But, since it is an explicit scheme, I start of with the calculation of normal at that point. If I have to implicitly solve it, then I have to go for Newton-Raphson iteration in solving together both epsilon bar P or delta epsilon bar P as well as the normal. So, that is the difference between an implicit and that explicit scheme. In explicit scheme, in the beginning you would calculate the normal and use that normal in order to calculate further quantities. Is that clear?

So, that is the normal that I calculate. Let me, let me call this as D_T. There are two ways of doing it. One is calculate delta epsilon P that is here by using this relationship, so that you can write sigma trial.

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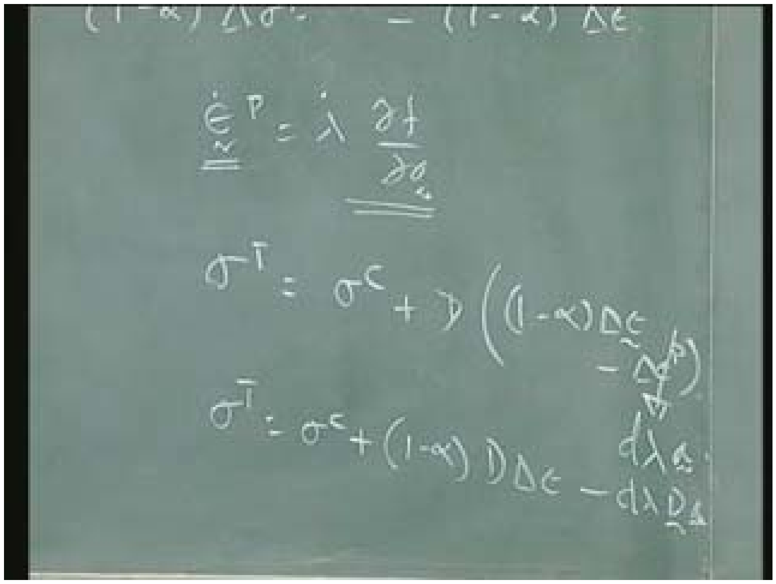

I am going to write that as sigma trial, not sigma final. There is a difference between sigma trial and sigma final; sigma trial to be sigma C contact plus D times 1 minus alpha into delta epsilon minus delta epsilon p. With the contact stress, I am adding whatever correction I am going to make. Now, situation is simple; delta epsilon$_{p}$, I am going to replace this by what? Delta epsilon p, I am going to replace it with here D lambda. What did I call this dow f by dow sigma in the last class? a, the flow stress; so, D lambda a, so that you will have sigma C plus D into 1 minus alpha into epsilon. Of course, both of them are tensor quantities or matrix multiplication is involved; minus D lambda Da. Is that clear? So, what is this? Sigma T is equal to sigma C plus 1 minus alpha D delta epsilon minus d lambda D B.

By the way, how do I calculate d lambda? Take a look at what you have done before, tell me how you calculate d lambda; a good revision for you. In terms of D and a, I calculated d lambda. What is it? Where did we do that? We did this to calculate the tangent modulus and what is that d lambda? Does not matter; even if it takes a minute, go back and see where we did it. Yeah, I am not going to say that, you just figure out what is it. I will give you a clue. We did this during the calculation of tangent stiffness matrix and we derived a very important relationship between lambda and epsilon P as well. d lambda is equal to delta epsilon bar P. I am leaving it; you figure out how to do that. So, I am going to calculate sigma T using this expression.

Now, why is that I have said sigma T and not sigma final, sigma trial and not sigma final? Let us come back to this graph here and what is the sigma trial?

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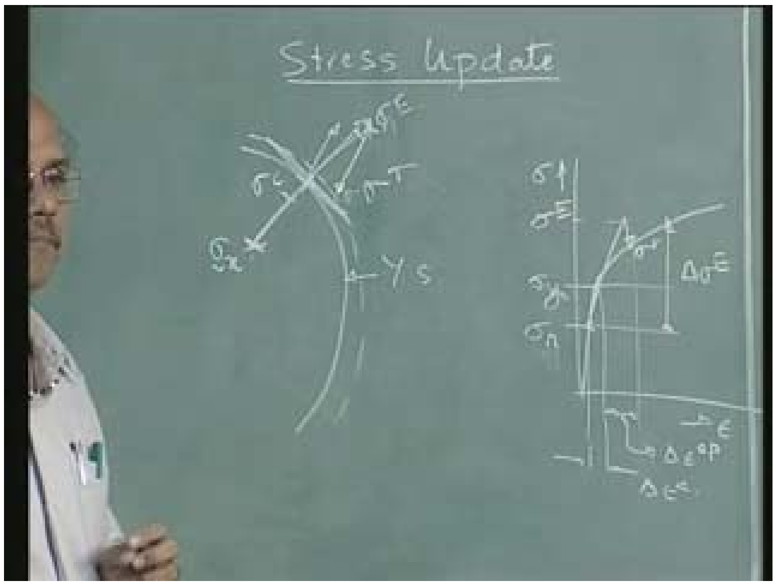

That point may be what we call as sigma trial. What essentially we are doing is to predict here and then do this correction. If you really look at this, you can split this up again into sigma C plus D into delta epsilon, together which will give me sigma E minus alpha D delta epsilon minus d lambda Da. So, this part is actually the correction part, this correction part. Now, why I have put that as sigma T is because, I am not sure when I correct it, it will lie on the yield surface. Is that clear? It may not lie on the yield surface, because I have not updated the yield surface and delta epsilon bar P I have not yet found out. That is what I will do as the next step.

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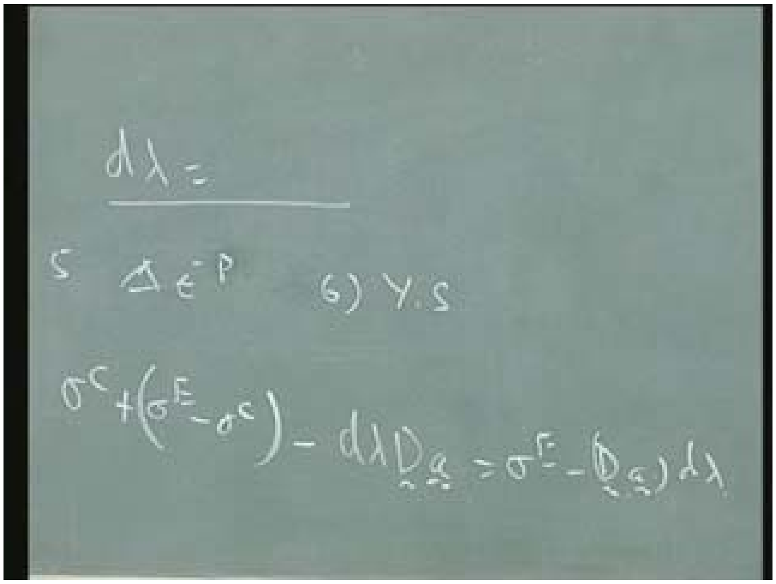

If I call this as the fourth step, then fifth step is the calculation of delta epsilon bar P. Once I get delta epsilon bar P, I update my yield surface and get the new yield surface. Is that clear? Look at this equation; that tells it all. In fact, if you want, explicitly if you want me to write, look at this term. What this term indicates? This is D into delta epsilon; D into delta epsilon, what is that term? Sigma E. So, if you look at this, if you want me to write that I will write it again plus the first term here is delta E. What is the second term? Minus alpha into D delta epsilon that is the sigma contact term minus sigma C. 1 minus alpha is what we are going to correct. So, this term alone that is alpha into D delta epsilon is D into alpha delta epsilon, which we are correcting it. So, this term becomes sigma E minus sigma C; minus of course, this term here d lambda Da, minus d lambda Da. In other words, this becomes sigma E minus Da into d lambda. Coming back to this point here (Refer Slide Time: 16:20), you will see that that is the second term; that vector is the second term here, second term here and so the totally we come back to sigma T. Is it clear, any question, you have a doubt on this?

This is a major step. Then, I calculate delta epsilon bar P and then what do I do? Update the yield surface. Since the point now which I calculate sigma, what is it, sigma trial does not lie on the yield surface, I do a correction. How do I correct it? Why you not think about it? How do I now correct? What is meant by correction? I have to pull that point back on to the yield surface, somehow. If I do not do that, then the stress will not be such that it will not lie on the yield surface. The stress will be outside the yield surface or consistency condition will not be satisfied. So, this is the major problem with the explicit scheme that consistency condition will not be satisfied.

So, I have to somehow pull it back on to the yield surface. Is it clear? Why is it? Because, I am updating the yield surface with respect to delta epsilon$_p$, separately; if I solve it together there will not be this problem. So, how do I do that? I have to correct it.

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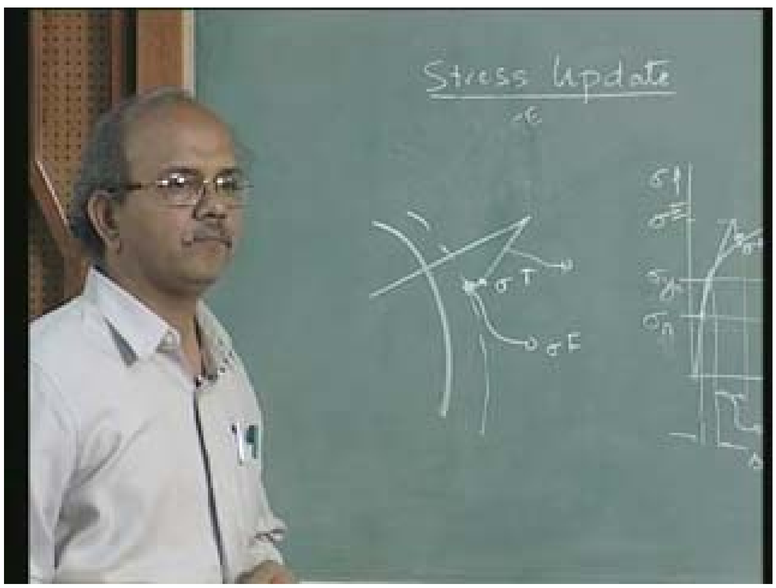

In other words, to give an exploded view of that what we mean to say is this point, this yield surface may be like, this I may go out, come back and be at this point. That point may be the sigma trial, I have to somehow pull it back and put it on to that point, so that that point will be my sigma final. Is that clear and that is my second term vector d lambda Da. Now, how am I going to pull back? When I pull back, it should be such that my yield criteria is completely satisfied.

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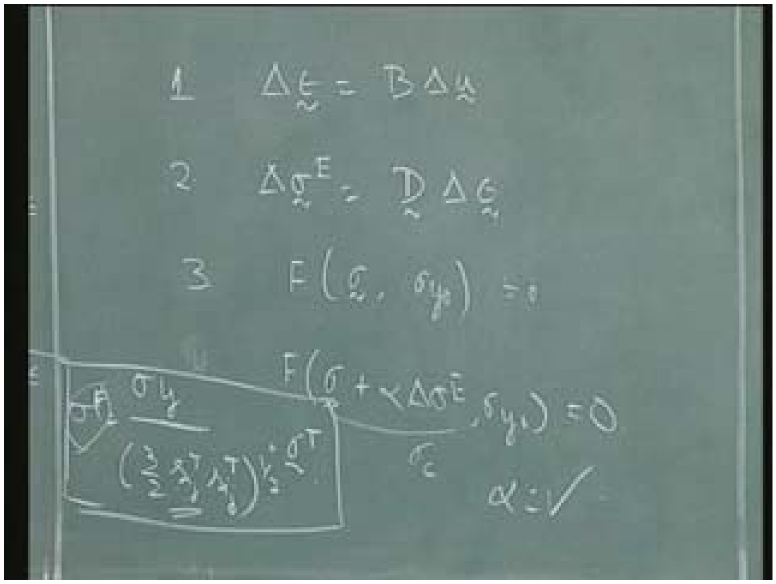

What I do is to have a ratio of $sigma_y$ divided by my s_{ij} terms; 3 by 2 s_{ij} T s_{ij}T whole power half and then, multiply my sigma T that is equal to sigma final. Look at this equation carefully. I use a ratio to pull this back. With this new sigma F if you calculate the yield function, you will see that automatically you will satisfy the yield criteria, because this function would have done the trick. Is that clear? So, this ratio checkup, it is very simple straight forward. This ratio is the one which is used to correct sigma T, so that the sigma final will lie on the yield surface. That means that what I mean to say is, this is now pulled back and that pulling back is, what is that? Pulling back is the correction of the trial sigma T.

So, I have to correct it using a certain ratio, such that the final sigma F will be one which is lying on the yield surface. So, I do that by dividing by $sigma_y$ whatever I have calculated. I have calculated now currently $sigma_{y0}$ plus H delta $epsilon_p$ that divided by this s_{ij}, this ratio I have multiplied by sigma T, so that it lies on the yield surface. In other words, your question is what is this s_{ij}? s_{ij} is the deviatoric part of the trial stress, sigma T; the deviatoric part of sigma T is what I mean by s_{ij}. Is that clear? Please do a small calculation; you will see how this would be the same as that which you will have it, which will satisfy the consistency condition. Is it clear?

Having done that, having now understood, let us do a small problem to completely understand every step here, so that that will give us confidence of how to solve this problem. One of the things which I want to just emphasize is that we will do the stress update at every Gauss point, at every Gauss point. Right now, we will do a problem. We will calculate the sigma T, sigma F and so on. Some of you, who have the calculators, please keep it ready. I will require your help to solve this problem.

Let us say that we start from a point which is say inside the yield surface.

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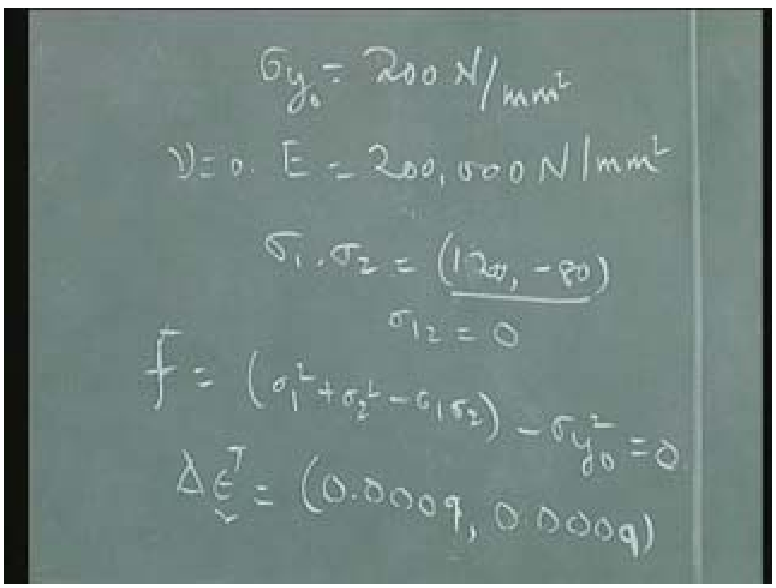

Let me say that sigma_y is equal to say, 200 Newton per mm square, sigma_{y0} is 200 Newton per mm square. For the time being we will assume that there is no hardening factor. Let us assume that there is no hardening here. You can introduce hardening that is not a problem. Let us assume E value to be 200, 000 Newton per mm square and nu value to be zero. We will operate with principle stresses sigma_1 sigma_2 or in other words, sigma_{12} terms are all zero, just for ease of doing the problem or else it will take lot of time to do the problem. So, let us say that sigma_1, sigma_2 are 120 and minus 80 respectively. In other words, since I said it is principal stress, obviously sigma_{12} is equal to zero.

Let me write down, for ease of doing the problem, the f in terms of principle stresses. So, that can be written as $sigma_1$ squared plus $sigma_2$ squared minus $sigma_1$ $sigma_2$ whole power half or in other words, if I want to write the capital F minus $sigma_{y0}$ square is equal to zero. Let us say that I have calculated delta epsilon and let me call and that is where I am going to or in other words, delta $epsilon_1$ and delta $epsilon_2$, let them be 9; this, both of them are 9, that is delta epsilon. I have already done the first step. Now, if I calculate F using this, I have done that as well; from there, we will start.

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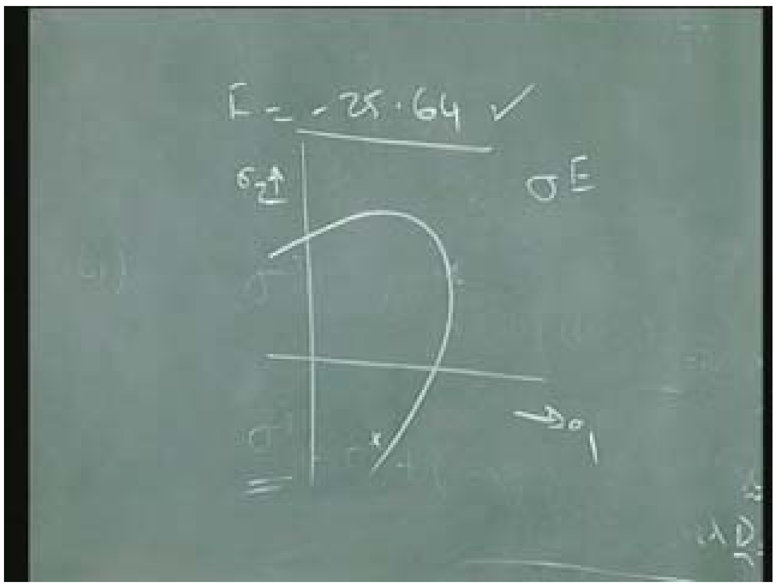

F happens to be minus 25.64, which means that we are inside the yield surface. F is less than zero, which means that we are inside the yield surface; you do not require actually a calculator. Let us start this. I know that I am inside the yield surface, so, what is my first step? My first step is to do, what is my first step? Where am I? Say, this is $sigma_1$ say, $sigma_2$ and that is $sigma_1$; I am somewhere. Yes, because with this stress, how can you say whether we are inside the yield surface? With this stress, I calculated F. F is, that is what I gave this as minus 25.64. That means it is less than zero; so, we are inside the yield surface.

Yeah, if it is zero, if F is equal to zero, then I will be on the yield surface; less than zero, we are inside the yield surface, very obvious that, so, we are here. So, the first step is to do an elastic prediction. Please calculate the value of the elastic prediction sigma E, see whether we are inside the yield surface or outside the yield surface. It is not very difficult. What you have to do is I have E here, that is the D. D becomes just 200, 000 into 1 0 0 1 matrix. Now, calculate sigma E and tell me where we are, whether we are outside or whether we are inside? So, delta sigma, it is very straight forward calculation; this multiplied by this.

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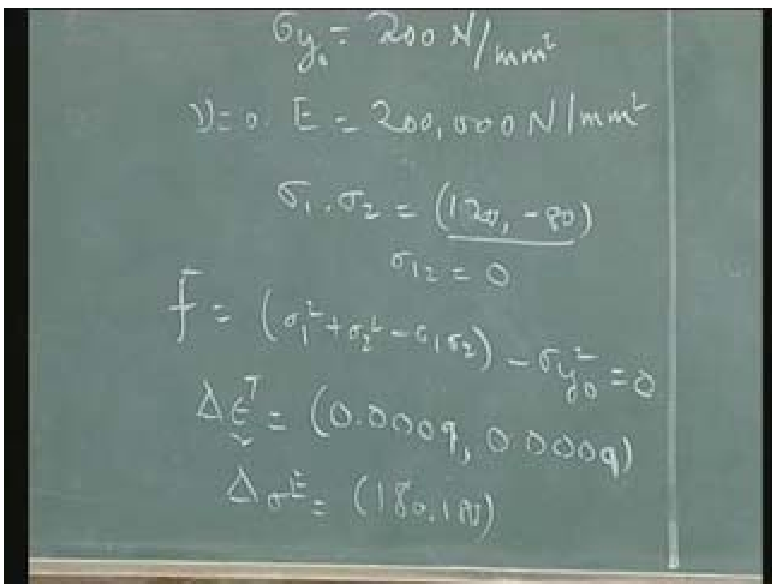

This multiplied by this that will become, so, delta sigma E, I will write it here, will become 180 comma 180. So, 180 comma 180 will be added to this, which will be 300 comma 100.

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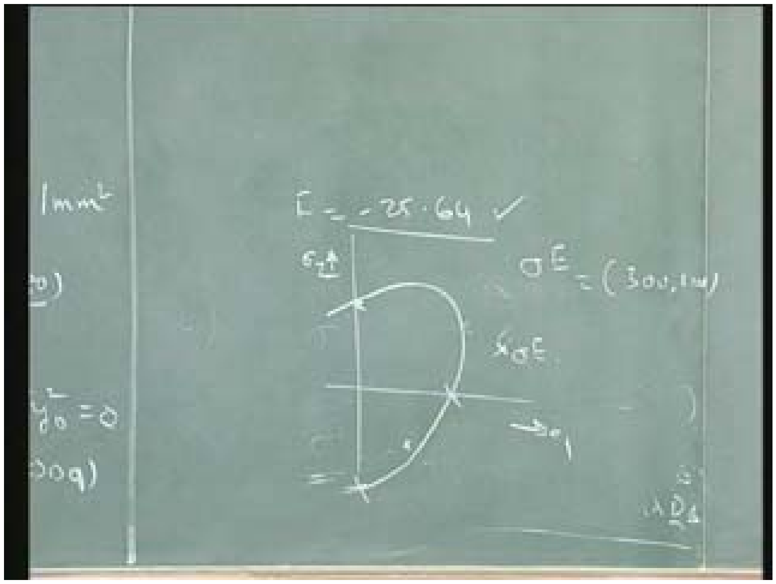

So, sigma E is equal to 300 comma 100. Since the yield is 200, I am hitting the yield surface, so, that is 200, that is 200 and that is 200; so, 300 100 is somewhere here; may be or even slightly less. So, it is outside the yield surface. So, you are going outside the yield surface, so, that is sigma E.

Once I know that I am outside the yield surface, now I calculate alpha. Let us see how you calculate alpha. I will give you a minute. Just look at these equations. See how you can write down especially this equation and see how you can write down a quadratic form for solving of alpha. What do I do? I rewrite this equation now in terms of sigma_1 plus alpha delta sigma E 1; exactly.

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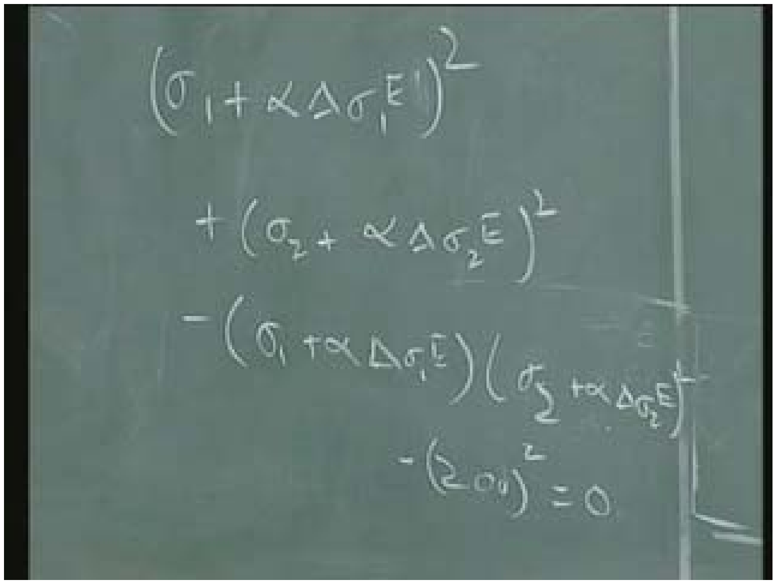

This equation can be written as sigma$_1$ plus alpha delta sigma say, E 1 whole squared, sorry, let me write it here; that is the first term plus sigma$_2$ plus alpha delta sigma$_2$ whole squared that is the second term minus sigma$_1$ plus alpha delta sigma$_1$ E into sigma$_2$ plus alpha delta sigma$_2$ E whole squared minus 200 squared is equal to zero. Please expand this and tell me; all of you can do this and tell me what would be the resulting equation. That is the third step; we have to solve now for alpha.

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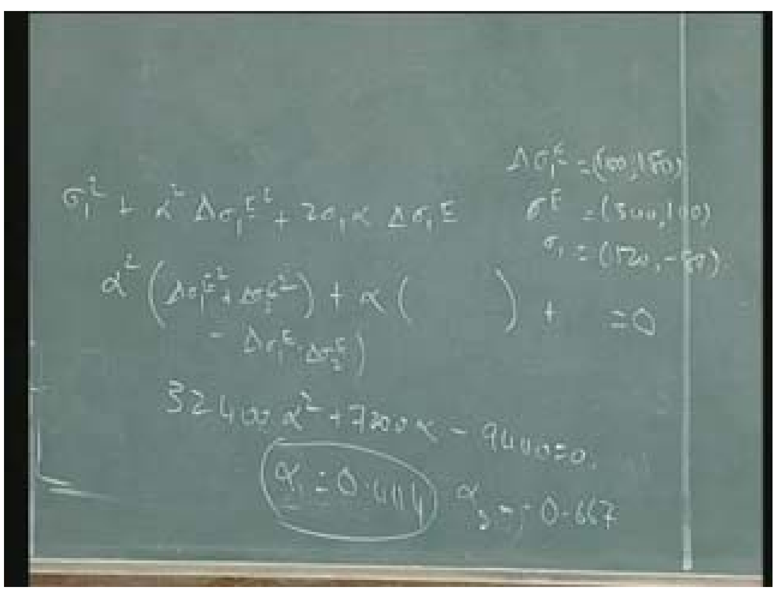

Look at this, look at this equation, expand it and write down; quite simple, $sigma_1$ squared plus, I can write that down; $sigma_1$ squared plus alpha squared delta $sigma_1$ E squared plus 2 $sigma_1$ alpha delta $sigma_1$ E, first term. Write down the second term as well as the third terms and tell me ultimately what you will get in terms of, see you will get something like this - alpha squared into something plus alpha into something plus a constant is equal to zero; that is what you will get. Please calculate that. Rearrange the terms. You know, I hope you have written down what delta $sigma_1$ E is. What is that? That was 120 or 180 comma 180. It was 180 comma 180. So, write down and tell me what the values are. Substitute the values also. Yes, it will take couple of minutes; does not matter, just do that.

By the way, you know also $sigma_1$. What are the terms that will come? If you have any question, I will answer it. So, what are terms which will come for alpha squared? Delta $sigma_1$ E squared will be there, you write plus delta $sigma_2$ E squared will be there, then $sigma_1$ $sigma_2$, so, minus or plus yeah, that is this last term here minus, yes, delta $sigma_1$ into delta $sigma_2$ that will be the first term. Write like that for alpha and the final term as well. Then, substitute for these values of delta $sigma_1$ E and so on and tell me what would be the quadratic equation.

If forgotten, delta $sigma_1$ E 180 comma 180 and the other one sigma E is equal to 300 comma 100 and $sigma_1$, what was $sigma_1$? We started with 120 comma minus 80. Yes, is it clear, any question? Everything is clear? What happened? Yeah; so, let me give that answer, you can substitute it, you can work it out. This happens to be 32,400 alpha squared plus 7,200 alpha minus 9,400 is equal to zero. So, that is the value. Yeah, this expansion, you can expand it, substitute the values and I hope, I am correct. This is the value, so, I will solve for alpha; $alpha_1$ and $alpha_2$ and $alpha_1$ happens to be 0.444, $alpha_2$ happens to be minus 0.667, which means that that is where I cut the yield surface.

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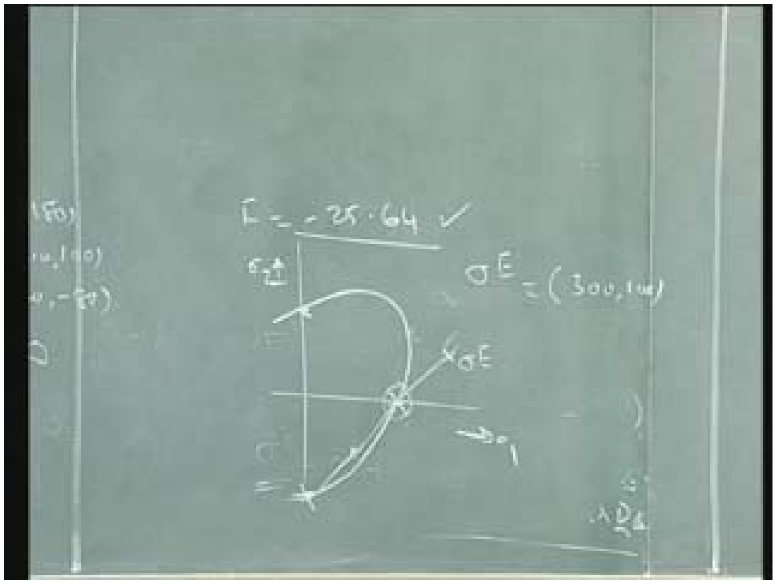

Since I am moving in this direction, obviously I need not take this value. I have to take only the positive value and so, 0.444 is the $alpha_1$ and that is what our current alpha is. So, alpha becomes 0.444. Which is the value that is given on the curve? Yeah, what we are saying is that $alpha_1$ is this value, $alpha_2$ is in the opposite direction that is minus value, which is in the other direction. How do I find out? Simple; substitute that into my sigma equation, find out where you cut the yield surface. In other words, you can contact at two points. This is not the correct one. The positive alpha is the right one, because that is we are finding out the alpha between this and this. So, that is the correct point and so I take that point. So, I take alpha is equal to 0.444.

My next step is to find out what? Delta lambda, yes, elastic; so, delta lambda and so, what is delta lambda? I asked you that question.

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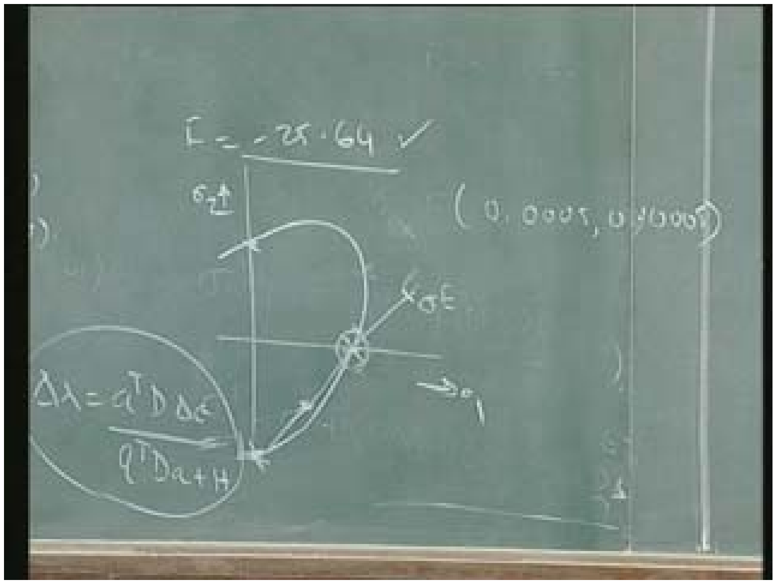

So, that happens to be delta lambda happens to be a transpose D delta epsilon divided by a transpose Da plus H. Very good; so, a transpose Da plus H. This is what we derived a few classes back. Even before I go there, I have to find out using alpha, I have to separate out the delta epsilon into two parts, what I have to correct and what I need not correct. So, tell me what is it that I have to correct. Yes, correct; 1 minus alpha into and what is that value? Delta epsilon which I have to use now happens to be 1 minus 0.444. So, what is that value? It happens to be around 0.0005 comma 0.0005 comma 0.0005. That is the delta epsilon value which I have to now use; that 4 naught naught naught 4 has been consumed into the elastic part. That is correct.

Now, we have to calculate what is a transpose or a? a is dow F by dow sigma. So, I have to calculate dow F by dow sigma first and this is what we were talking about as the explicit scheme and dow F by dow sigma happens to be, I can give you an explicit formula for this and that happens to be sigma, I will just write that down here; you know all these things. This is a standard derivation, you can derive it later.

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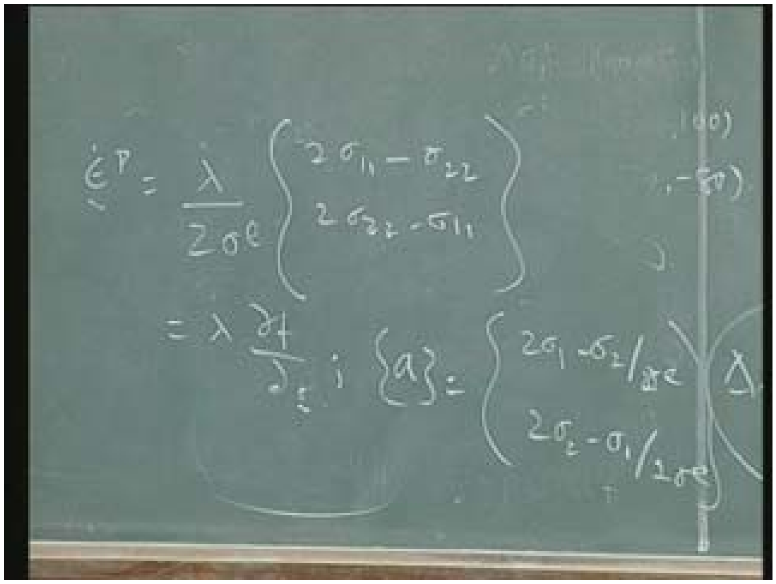

That happens to be 2 $sigma_{11}$ minus $sigma_{22}$ divided by the, in other words, let me write down the full equation, then you will know it. The full equation for epsilon dot P is equal to lambda dot divided by 2 sigma e into 2 $sigma_{11}$ minus $sigma_{22}$, in this case it happens to be $sigma_1$ and $sigma_2$ and then the second case, it happens to be 2 $sigma_{22}$ minus $sigma_{11}$ and if there is a dow, of course, you have one more term. Let us stop with this, so, these are the two terms.

What is my, what is my dow F by dow sigma? Remember that this is equal to lambda dot dow F by dow sigma. That dow F by dow sigma happens to be a. So, a is equal to 2 $sigma_1$ minus $sigma_2$ divided by 2 sigma e or in other words, from these two I get, a to be 2 $sigma_1$ minus $sigma_2$ divided by 2 sigma e, then 2 $sigma_2$ minus $sigma_1$ divided by 2 sigma e. This is my, what is this sigma e? Equivalent, which is $sigma_y$; equivalent stress which is at this case happens to be 200 where it cuts the yield point at that point. So, that is the a. Please calculate what a is and then substitute it. Once I calculate a, substitute it into this expression and calculate lambda delta.

a is very simple. What is 2 $sigma_1$, what is 2 $sigma_1$? Yes, this is at the point where it hits the yield surface. What is the point at which that is sigma contact, what was that? What is that sigma contact? Look at that. What we got when we hit the, 200 comma zero. This is the values which we got; 200 comma zero. So, this happens to be 2 into 200 minus zero divided by 2 into 200.

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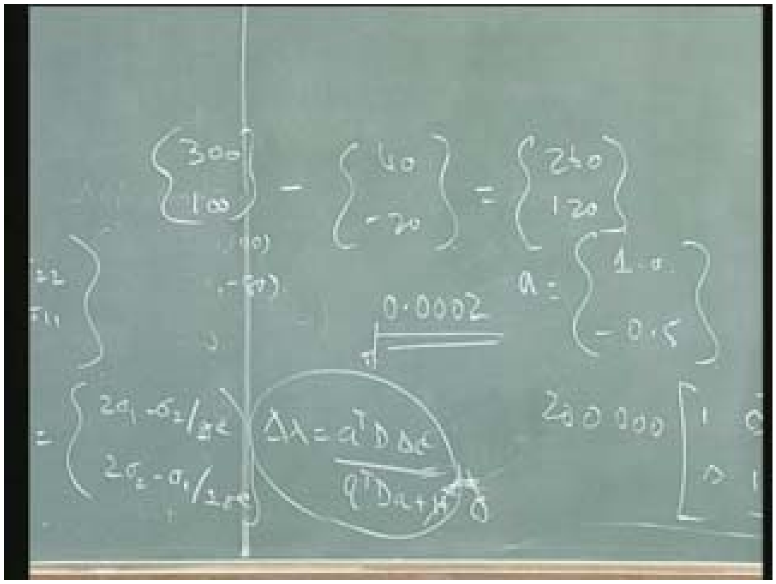

So, a happens to be 1.That is the first matrix and the second one, the second one just look at that. $Sigma_2$ is equal to zero, so minus 0.5 and what is D? D happens to be 200 000 1 0 0 1. So, you know all things here. H is equal to zero. So, substitute that and find out lambda delta. It is very straight forward; substitute these things, so, lambda delta happens to be 0.0002. You can verify it, lambda delta; it is a big multiplication, that is all. Lambda delta happens to be 0.0002. Once I calculate lambda delta, I have to substitute it into this expression to find out what sigma T is. Please do that. Let me see what you get; it is very simple.

D into a; you have to multiply D into a into D lambda. Please do that. D, I have already given, lambda I have already given, a I have already given; multiply it and tell me what should be the correction term. This term, what should be the correction term there and what should be the final term? Yeah, please, is there any question? Please do that; it is a good exercise. Please work it out and tell me what should be that term. Yes, 40 and is it

...? 40 and minus 20. Is that correct, does everyone agree? 40 and minus 20 and what is now the sigma T, sigma trial? Yeah, sigma E minus this; what is sigma E? Sigma E is 300 100. So, 300 100 minus of this; so, this is 300 100 minus of this. So, what would be the value here? Obviously, it is 260 120; 260 120 and so, that is the point where we have come back after correction and so, if I plot that yield surface after going out I have come back to that point.

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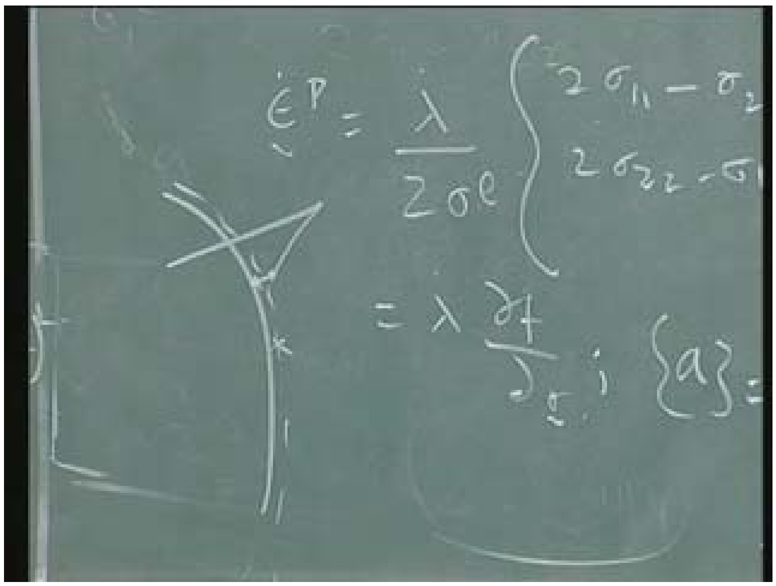

But, unfortunately this step of expanding the yield surface is not there, because I have not given a H value. I have not given a H value. So, assume that it is at the same point. What I have to do now is to pull this back on to this yield surface, by the correction term which I had given you. So, that correction term will take this closer to the yield surface. This, in a nutshell, is the complete stress update algorithm. That is the first thing, explicit scheme.

Now, I said that a second scheme is the variation of explicit scheme called sub incrementation. What is meant by sub incrementation? The total delta epsilon is now split into say 5 parts. The total delta epsilon we had is what? Naught naught naught 9; so that is split into 5 parts say, naught naught naught 9 divided by 5 say, 5 parts. It is not that 5 exactly; it may be 10 parts, people use it even 50 parts and then for each of these say, let us say that I am dividing this into 9 parts. So, I have naught naught naught 1, for every sub increment. So, using this naught naught naught 1, I repeat all these steps. Then again, it is in a loop now. Again, another naught naught naught 1, then I go to the second step and then update it. When I do like that, because I will update my yield surface periodically, many times or even most times, it would happen that the drift away from the yield surface is much less. In fact, explicit plus sub incrementation is an excellent technique by which you can solve stress update part. Is that clear? So, these whole calculations are repeated for every sub increment. So, that is the explicit part. May be, quickly we will run through the implicit part in the next class and then take up the large deformation later.

Is there any question? Stop now, we will look at, I am not going to the details of implicit scheme; again it is a very vast topic. Just indicate how we do it, come out, proceed with the large deformation.