# Advanced Finite Element Analysis
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## Lecture – 1

In this course, we are going to talk about basically non-linear finite element analysis.

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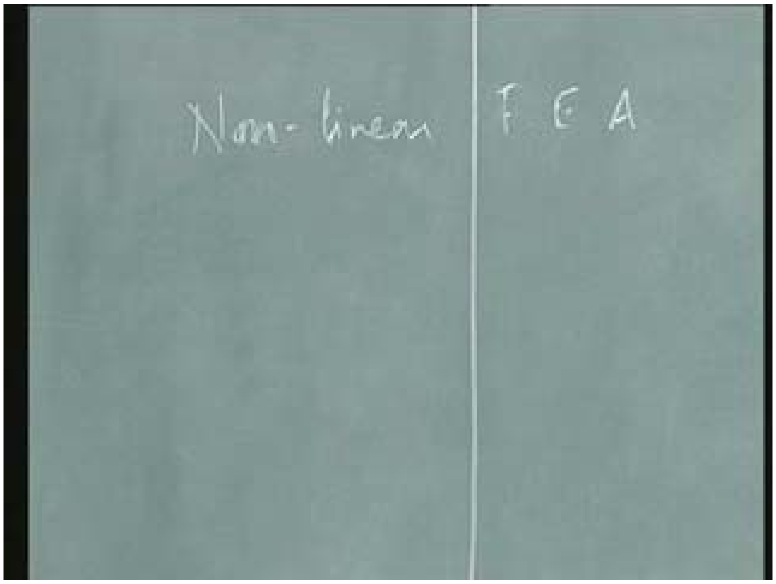

In the last course we have done finite element analysis, which is basically applicable for linear problems. The concepts that we developed in the previous course will no doubt be very useful for us in this course, but we have to learn lot more in this course in order that we can apply the concepts to non-linear problems. Non-linear problems are very important especially in mechanical engineering, where you come across situations of non-linearity because of contact, because of material behaviour, as well as because of change in the displacement constraints and so on. Now, let us look at the type of problems that we can do using finite element analysis and then we can understand what the importance of non-linearity is, very clearly.

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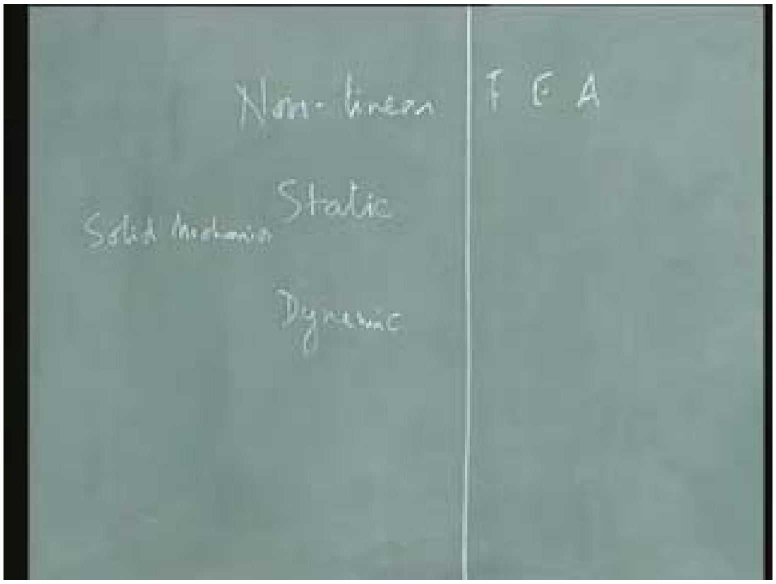

If you look at the broad class of problems that we can solve using FEA, you will agree that it can be broadly classified into static and dynamic problems. Of course, we are talking only about solid mechanics problems; we will come to other problems which can be dealt with using finite element analysis such as heat transfer problems, fluid flow and so on. In fact, today finite element is applied to problems of electromagnetics; things which you might have thought not possible from the earlier course on linear finite element analysis itself, because basically that course was targeted to solid mechanics and to a certain extent heat transfer. But, remember that finite element analysis is a general technique to solve partial differential equations and once you understand the type of partial differential equations which govern a particular problem, it is possible to evolve or to formulate a finite element analysis algorithm and implement that in order to solve this kind of issues and problems. Though, in this course we will not be dealing with electromagnetics or fluid flow and so on because that is a topic by itself, we will be concentrating in this course on solids and solid mechanics problems, especially solid mechanics problems which can be classified as static analysis cases.

Of course, many of the techniques that we are going to develop for static analysis are applicable straight away to the dynamic problems and we may have to resort to certain simple special means to solve them using dynamic analysis; may be towards the end of the course, we will indicate how for example explicit finite element analysis works in the dynamic case. But, basically the course will concentrate on non-linear static analysis using or for application with an aim to apply them to solid mechanics problems. The advantage here is again that once we see that the techniques are developed for static solid mechanics analysis, it can be extended very easily not only to dynamic analysis, but many of the techniques are common to other problems like say, heat transfer and so on, as well. We need lot of, of course, background theory in order to understand the application of the non-linear techniques to static problems. But, even before we go to that level, let us see what the type of problems are that we can do in these two classifications.

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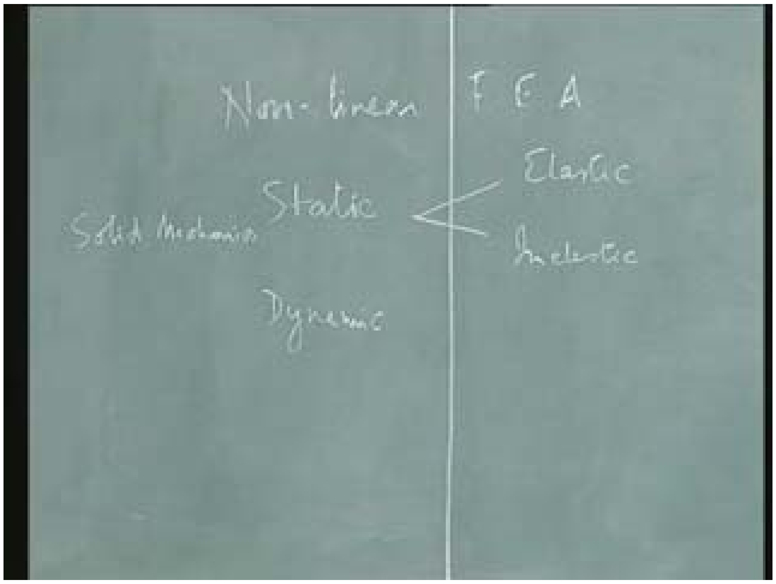

Now if we look at static analysis, we can do of course an elastic problem or elastic analysis or what I would call as inelastic analysis. In elastic analysis itself this is, this is as far as material behaviour is concerned, we will come to other geometric issues later.

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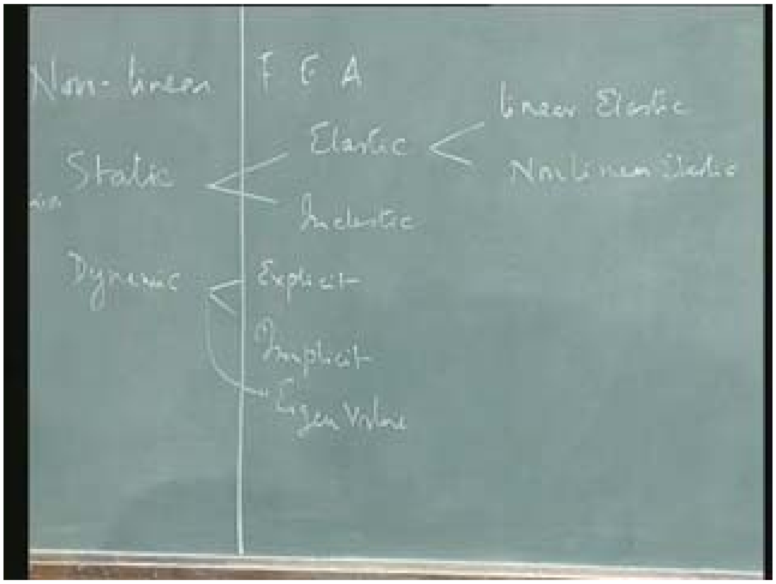

As far as the elastic analysis itself, you would see that this can be classified again into linear elastic and non-linear elastic. We will define very carefully what this non-linear elasticity means, later in the course. It is very important that we understand lot of continuum mechanics in order to appreciate what this non-linearity means and what it involves. So, in this course we are going to cover lot of continuum mechanics portions as well, so that we will get the overall picture. What we have essentially done is static elastic, linear elastic that is what the route is, which we have taken. Please note that, this is not very separate or can be separated from say dynamic analysis. In other words, if you want to understand the whole picture, you should realize that dynamic analysis for example if I do an explicit analysis or an implicit analysis or for that matter an Eigen value problem, whatever I do in dynamic analysis as far as the material behaviour is concerned we can pluck from this route as well.

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When we talk about inelasticity, of course, one of the most important things is plasticity and viscoplasticity. We can add if you want, may be one more case which we call as viscoelastic materials as well. Somewhere between the two, we can say that we have what is called as viscoelastic materials. This is as far as material behaviour is concerned. Of course, there are lot more models under say, for example non-linear elasticity. We will not worry about that right now, but these things can again be classified from the point of view of the type of deformation that the body would undergo. The body would undergo either what we call as small deformation or finite or large deformation.

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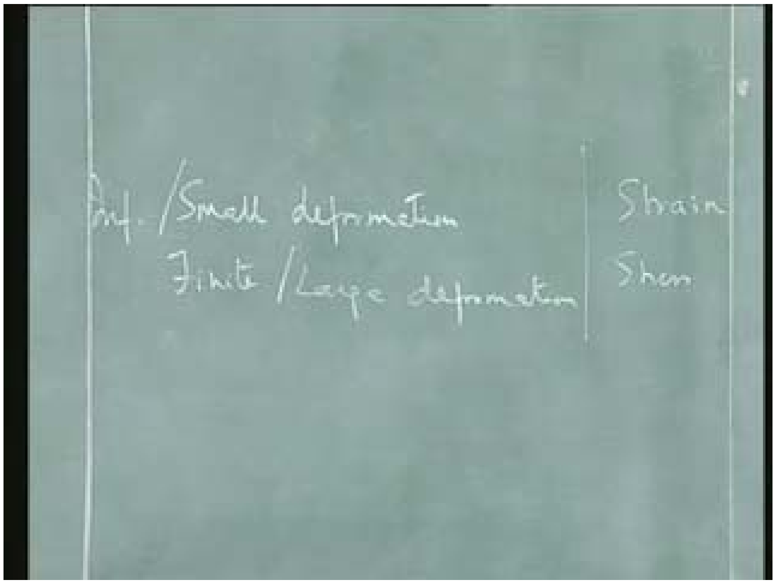

Please note that the small deformations and finite or large deformations, may be first thing is that you may be worried what this finite deformation is. You may worry or you may be worried or you may be wondering small deformation is finite, why is that I put this finite and large in one single bracket? Note that this is finite versus infinitesimal. So, small deformation if I have to call, they are called as infinitesimal deformation. It is not that it is finite deformation and infinite deformation, it is infinitesimal; that means very small deformations versus finite or large deformation.

These two which define the kinematics and motion and deformation have important implications as far as definition of strain or deformation and stress is concerned. The way we have done so far the analysis for linear elastic case, was one which would address the small deformation problem or infinitesimal deformation problems, where we are not worried about very rigorous definition of stress and strain. In order to give a small background even before we proceed and do lot more mathematics, we should understand from our earlier classes that for example the area of cross section of a piece, say for example, a bar is important for the definition of stress as well as the length.

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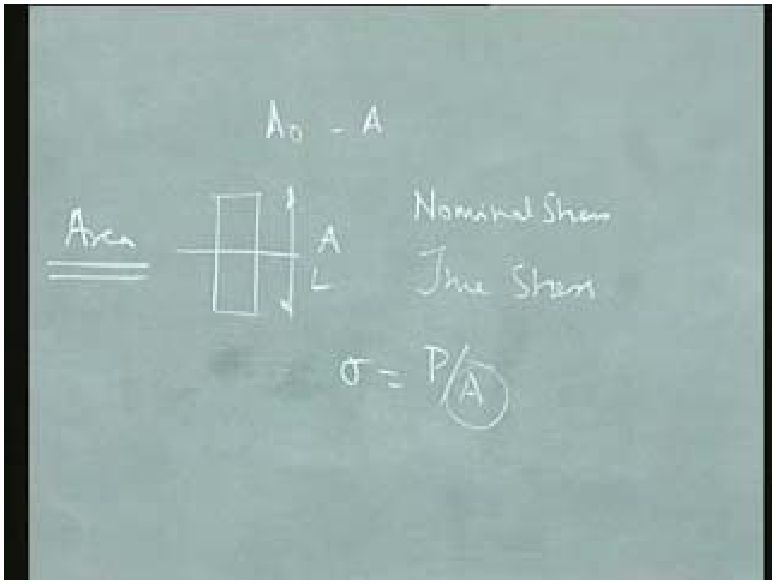

Probably all of you would remember that when you did, say for example, metal forming problems you would have defined two types of stresses, which you would have called as engineering stress or nominal stress and true stress. Do you remember what the difference between the two is? Yeah, the difference is that when I define the stress to be the load per unit area, it is important that the area is properly addressed; this concept of area is properly addressed. If the deformations are small or in other words if the area change of this body is small, then we need not bother or we need not worry whether the initial area say let me call that as say $A_0$ and the final area after deformations have taken place after the body has come to equilibrium, because of the application of load are they the same or different. That is the question.

If the deformations are small, the initial area and the final area say A will not be very different. We can approximate $A_0$ to be the same as A. So, here we can either put A or $A_0$, so, that is fine. But, if the deformations are large the body is going to say for example, you are testing rubber piece. Take a bar of rubber piece and then extend it. Then definitely you will know, you would have seen it that the area or the cross sectional area changes drastically; you would have seen that. In fact, the area may reduce 20, 30, 40%, may be even more, much more, in which case what is the area that you put would decide the value of the stress.

So, we distinguish between two stress quantities by saying that if you put the current area of cross section we call that as true stress and when you put or substitute the original area you call this as nominal or engineering stress. In the same fashion it is important to realize that this length also plays a role because you would have defined strain in your earlier classes to be, how did you define?

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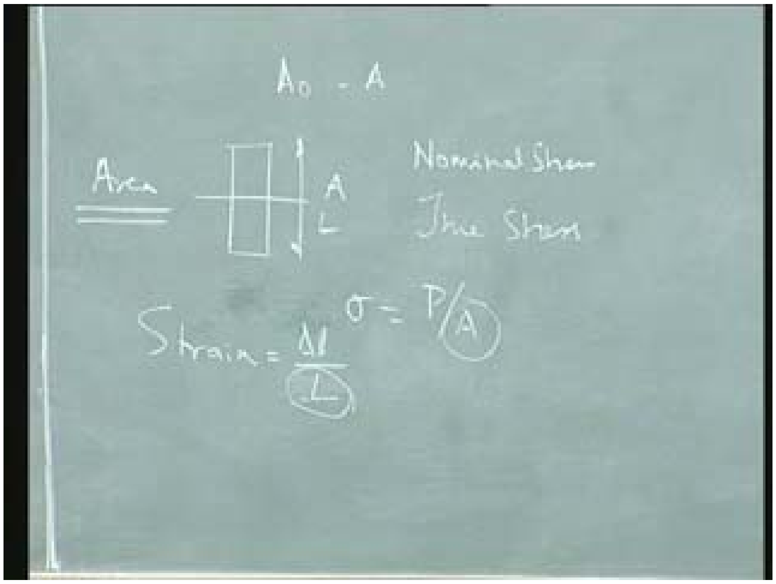

For example for this simple bar, change in length by original length very well known formula, so that we call that as delta l by L and it is important to realize that here again we have a very similar issue whether you put the original length or the current length, current length which changes with time. Again, we get into two different strain measures which you would call as engineering strain or true strain.

Of course, this will result in a measure of log to the base e, natural log and so on. So, you realize that this is going to have a major role play the measure that we use for defining strains. So, both the stress measure and the strain measure they not only tell you or measure the deformation, but their definition itself depend upon the deformation.

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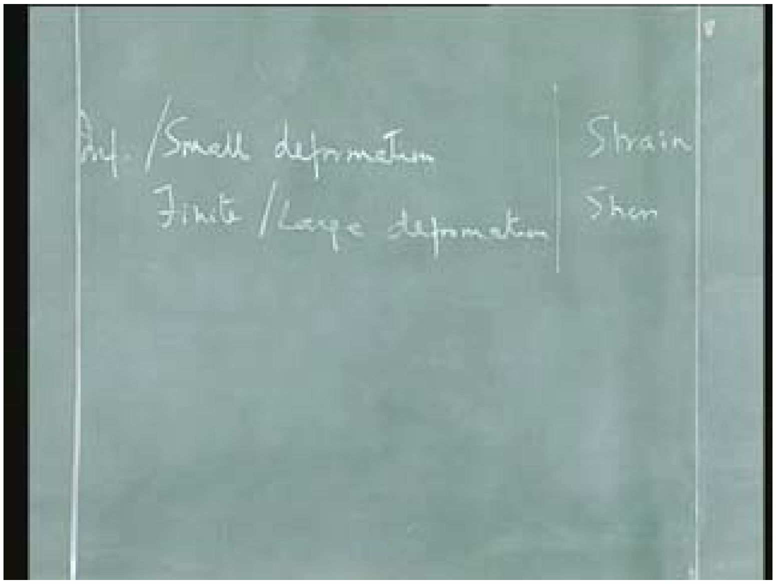

Inf. / Small deformation
Finite / Large deformation

Strain
Stress

So, you have to really look at these two guys here, whether the deformation is going to be small or infinitesimal or the deformation is going to be finite or large. You now have an idea, what we mean by small deformation and what we mean by finite or large deformation? We have to study really what are, these are okay; I mean these are fine, for unidirectional case.

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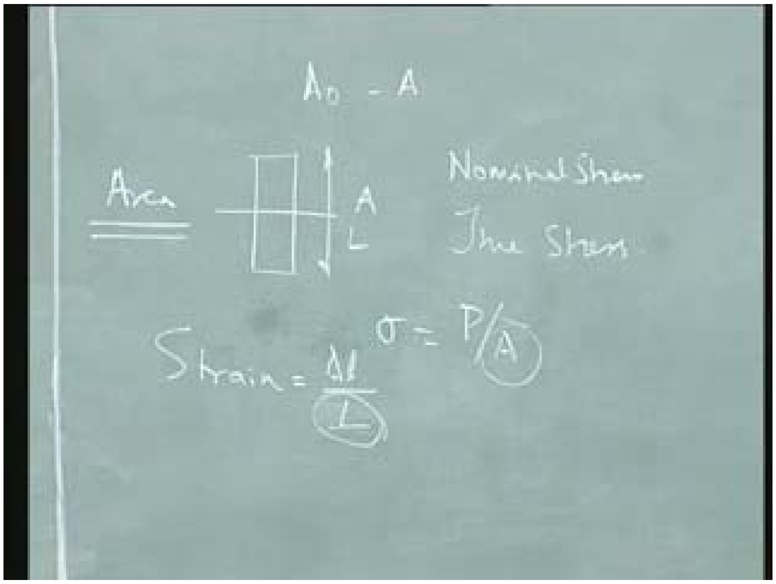

Yes; please note, yes, the question is what is the relationship between stress and strain? Is it that they are functions of time and so on? There are two different types of behaviour, we will come to that in a minute, whether they are functions of time. But, please note that we have not yet come into the region of stress strain relationship. We are only talking about stress measure and strain measure. The relationship between stress and strain are defined through what are called as, Young’s modulus is only a very specific case, but they are defined through what are called as constitutive relationships, constitutive equations which would define that is constitutive relationships or equations which in a very strict term gives us the behaviour of the body or tells us how the body behaves when the body is subjected to external loads or forces or influences.

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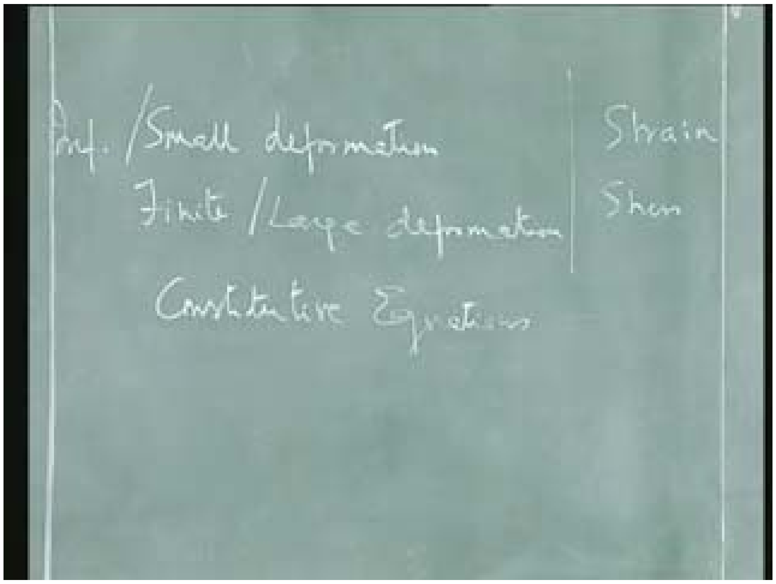

Inf. /Small deformation
Finite /Large deformation
Constitutive Equations

Strain
Stress

The stress strain relationships are given by constitutive equations, which actually result in the body being classified as linear elastic, non-linear elastic, viscoelastic, plastic, viscoplastic and so on. Please also note that your time factor comes into picture in the constitutive relations as well. For example, plasticity basically deals with rate independent that means the time is not a factor and viscoplasticity deals with time dependent behaviour, when the material becomes plastic. We will see in a minute where these are applied, but first thing I want to distinguish, I want you to distinguish is the fact that material behaviour is one aspect and the deformations and the ensuing definitions of stress and strain are another aspect. Is that clear? So they are two different aspects.

For example, I can have small deformation; I can have plasticity along with it. It is not that when material becomes plastic, you need to all the time do only a finite or large deformation. I can do plasticity with small deformation. Is that clear? So, they are two different things. Yes, may be when I do non-linear elastic say, for example rubber, the deformations are usually large. So, I may not be in a position to do small deformation, I may have to do only large deformation; but nevertheless, they are independent. In fact, in other words, what we mean to say is the stress measures that we use, stress measures in the sense that the definition of stress, one of these guys here, the stress measures that I use, are to a great extent independent of the type of material behaviour. Is that clear?

I am going to give special names to that nominal stress, when I go to multiaxial situations. I would call it as say, first …… of your ….. stress. I am going to define them very carefully later, but once I define them from this point of view, from deformation point of view, I will use the same stress measure whether I am looking at a problem here or a problem here. So, they are two different things, because one we would look at the kinematics and the other is the material behaviour or the constitutive relationships between stress and strain. That is the next thing. We will talk about that later. These are the material behaviours and those are the kinematic relationships. Note that these two can also be combined with an implicit or an explicit dynamic analysis as well. In other words, I can do a dynamic analysis, an explicit analysis, plasticity, finite deformations.

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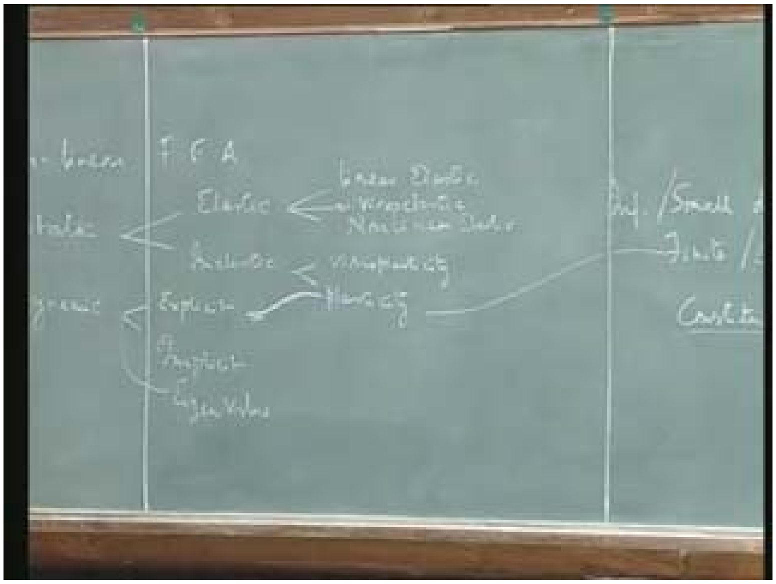

I can go, I can join them together; I can join them together. In fact, if you want to do crash worthiness of a vehicle for example, you would do an explicit plastic analysis with finite or large deformation. There, they can come together or I can do, if I want an implicit plastic small deformation. I can, I can pluck one of them from here, one of them from here, one of them from here and then combine them. The most important decision in finite element analysis, especially if you are doing practical finite element modeling is to pluck the proper leaf from here, from here and here. You have to know whether you are going to do a static analysis or a dynamic analysis, you are going to do whether an elastic or an inelastic case, whether you are going to do small deformation or large deformation and so on. Then only, a picture would be complete.

For example, many of the metal forming problems, you are looking at metal forming problems, many of the problems can be considered as a static problem; can be considered as a static problem, which is basically inelastic which can be looked at as, in other words as plasticity being important or the constitutive equations being governed by the plastic type of equations or plasticity type of equations along with large deformations. So if I want to do a metal forming problem, I will pluck static, I will pluck plastic, I will pluck finite deformations. Is that clear? So, this is possible that we do.

Now, we can extend this further by putting lot more models. But before we do that, let us realize or it is very important to realize that non-linearity also comes into picture through what is called as contact; through contact.

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I am sure all of you know that contact is very important say, for metal forming problems, which we have discussed; most of the time the contact between the punch and the material, material and the dye really gives that shape for the body. So, contact introduces non-linearity through an inequality relationship. So, again contact is a separate entity. It is not that contact is there only for contact with large deformation, with plasticity alone. We can pluck another leaf from contact. In fact contact can be classified in a very broad scale as say frictionless or with friction. We will go into more details later, just say for example frictionless and friction depending upon what is important. Most of the time, we would look at friction to be important.

For example, I am going to do analysis of a tyre. Then what would I do? I would pluck depending upon what I want. I want to know what would be the pressure distribution or foot print loading of the tyre depending upon whether I want to do that or I want to find out how the tyre behaves during cornering or during breaking. Depending upon the problem which I want to do, I will either pluck a static or dynamic load. Then, I will go ahead and see what the type of behaviour is. I know that it is an elastic behaviour and I know that it is non-linear elastic. So, I would pluck this or this, then this, this and I know that rubber is subjected to very high strains. So, I would pluck this and I know that there is a contact between the tyre and the rim and the tyre and the road; so, I would go and pluck contact. So, the problem is complete when I model using a contact algorithm, when I model the problem using finite deformation, when I model the problem using inelastic material behaviour, model the problem using static or dynamic analysis. So, the route is complete. Any question?

Yeah, so, if there are no questions, let us look at contact in slightly more detailed fashion. Of course, as I told you contact introduces the non-linearity through what is called as inequality constraints. We are going to have, you know, separate lesson on contact, but I just want to classify again contact to be in a slightly more broader sense.

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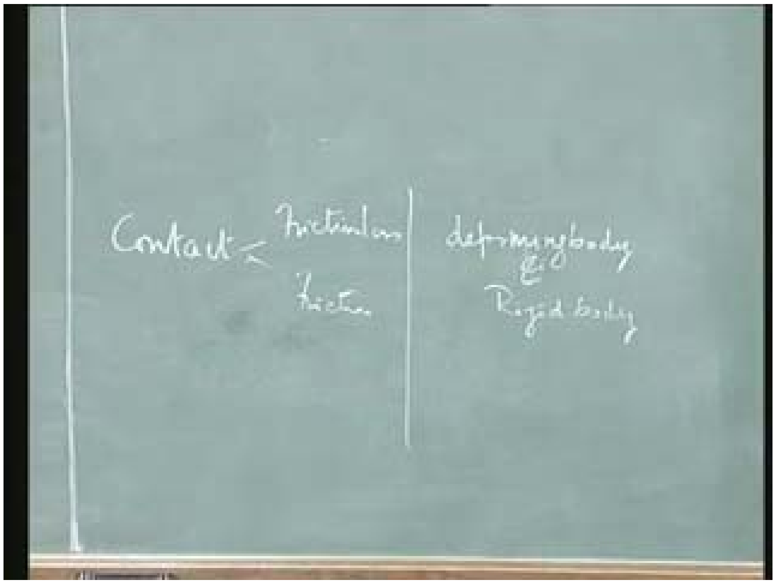

The contact can be between a deforming body and what is considered to be rigid body. Please note that any body can be modeled as a rigid body. That means that we make an assumption that the deformations are very, very small, so small, that the body is considered to be rigid for all practical purposes. The tyre example which I said and I said that there is a contact between the rim and the tyre. Immediately that would prompt you to tell me that the rim can be modeled as a rigid body when compared to the tyre which would deform to very great extent. Note that this concept of which is deforming and which is rigid requires an engineering, a good engineering sense and understanding of the problem.

For example, I have a very, very thin sheet of steel, extremely thin sheet of steel, which is deformed by a thick rubber block, thick rubber block. Then, which would be rigid and which would be deformable? Say, we are talking about 1 mm thin sheet, which is deformed by a rubber block like this or this duster. So, obviously that thin sheet of steel is going to deform much more than this block, rubber block. So you can, for all practical purposes consider that block to be rigid and that sheet to be deformable. Yes, that is a good question. Can we consider both of them to be deformable? Yes. That is the next thing or next classification that both of them are deformable. Yes? Yes, how do you decide in other words, the question is how do you decide whether a body is rigid or deformable?

As I told you, the first thing is that there is an engineering judgement that is involved. I will come to that answer in a minute, but before that let me tell you what you should not do. It is not that you compare say the stiffness, E values of these two and take a decision based on E; which is deformable and which is, because most of you would hear that when E is very large, the body is very rigid. True, absolutely true; but please note that, the stiffness of the body depends not only on the material property called E or the Young's modulus, but also on the geometry. So, both of them go together to define the stiffness.

What is the type of engineering judgement we are talking about? Suppose, let us look at two gears mating. You want to analyze the contact stresses of two gears that are mating, whatever be the type of gears. Now, can you classify this under a deforming body and rigid body under this category? Both are deforming, you cannot say gear 1 is not deforming and gear 2 is deforming and so on. You have another set of contact conditions called deformable-deformable contact or deformable body, two deformable bodies contacting.

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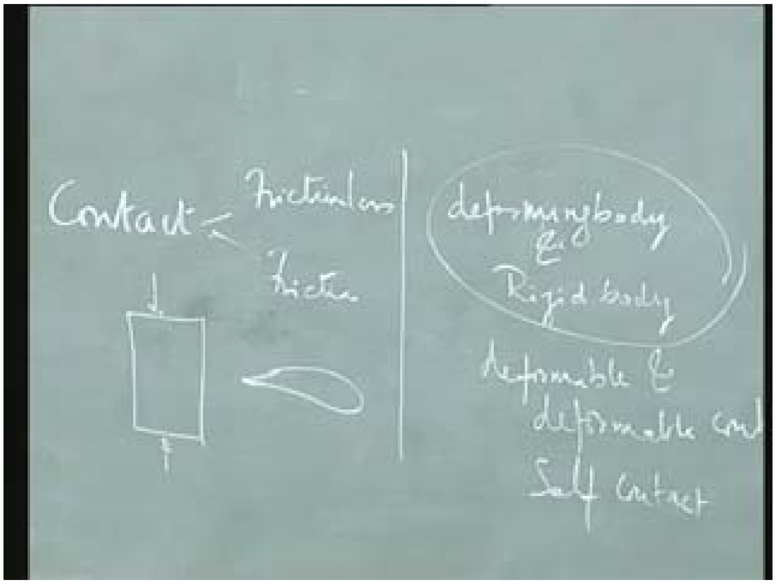

We usually call this as deformable and deformable contact; loosely defined like that or you can say, in more correct sense contact between two deformable bodies. The third category is what is called as self contact. Body can deform and come into contact with itself. Suppose I have, say for example I have a very thin sheet and I apply load. It can buckle and when you look at it in the side view, it may start touching. If I start applying load like that, it may start touching or during impact, say you are testing crash worthiness of a car; may be the hood of the car may crumble or may collapse and may start contacting itself at different places. So, that is what we call as self contact. It is very important to realize that there are many packages which do not support self contact for static analysis, there are reasons to it. So these are the three categories. You have to be very careful what you choose.

Why is that we are classifying it like this, because the ease of doing analysis depends also on your proper choice of the type of I would say, modeling that you do. In other words, all components in this world are 3D; all components in this world deform. If you look at it from that philosophy, then you have to really go for a deformable-deformable contact, all the time 3D analysis and so on. Then it would not be possible for us to solve very complicated problems. So, we have to make some simplifying assumptions in such a fashion that it does not affect the results. We have to be as close to reality as possible, at the same time make good engineering model out of it or a mathematical model out of it.

So, again we can pluck say for example I am going to do my old problem, metal forming. Then, I can do what is called as a deforming body and a rigid body to be the contact algorithm or the choice of contact is now between rigid body. Which is going to be rigid? The dye. For example, sheet metal I am doing; I can use this dye to be or the punch to be the rigid body and that sheet, steel which is deforming is going to be my deforming body. But please note one thing very carefully that, these are or these define type of analysis. But, these are also linked to the elements. They are again independent of this kind of analysis. You know the type of elements.

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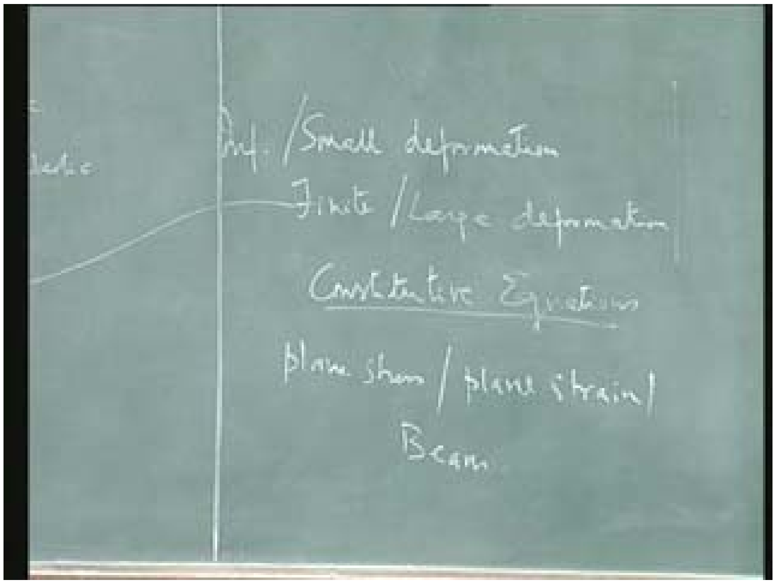

Probably you would have studied about or yes, we did that on plane stress or plane strain or beam, axi-symmetric, shell, whatever it is. Again, one more leaf you have to pluck from that list. So, I can do for example, a shell analysis; a shell analysis which is or I can choose an element called shell. I can do an inelastic, in other words, a plastic analysis or elastoplastic analysis for sheet metal forming, with finite deformation along with contact for deforming and rigid body. So, for sheet metal forming I do that or membrane elements. Many times we use membrane elements, where the stiffness in the bending directions is low. So we choose membrane and then go through this whole thing or I can choose 3D; 3D element from here, pick that up, then I can go through this whole route.

So, the type of elements, of course, mimics the structural behavior. Yes, this selection, this selection depends upon or in other words how do I select an element? That selection depends upon the type of structural behaviour. I know that the structure is going to behave in a three dimensional fashion. I know that this is a very long bridge. So, I know that the behaviour is going to be plane strain or I know that this is going to be an extremely thin sheet subjected to in plane loading; no, it is going to be plane stress and so on. So, that is different; the structural behaviour is different, the structural behaviour.

What are the things you have to consider? The structural behaviour, the material behaviour, the type of deformation and whether you have contact or not. The structural behaviour you can include also whether it is static or dynamic. So, when you have a problem in hand, you look at all these things. When you have a problem in hand and choose say one of the elements which is independent of the further choices that you are going to make, when you choose say a static analysis, elastic analysis, linear elastic analysis, small deformation with no contact, you are doing a simple linear elastic problem and all the concepts that we did in the last course are valid for that.

But, look at that, choices. If you remove that and choose any other option from here, any other option, you move away from this first category and choose any other option here then, you cannot do a linear analysis. Then, you have to do a non-linear analysis. It is unfortunate that there is a general tendency to believe that it is possible to reduce most of the problems to linear case. Please understand that especially in mechanical engineering, you do not use a component independent of others or in other words most of the components are in assembly, they are in contact. So, contact is very important, because many times we may not be able to judge how forces are transmitted from one component to another component.

We are also interested to find out what would be the contact stress between the components, because many times failure originates at these contact locations. All of you know about this, say for example in gears. So, contact is a very important thing in mechanical engineering. You may, you may be surprised to look at the results when you remove contact and make some very simplifying assumptions. No doubt, I said that simplifying assumptions have to be made, but again you have to know the consequence of such an assumption. It is not that I am making a simplifying assumption, I will leave this out and do the problem as a simple case, because I have, I have only this kind of package or is easier to do and so on, may not fetch you good results. That is the first thing that if it is a linear elastic case, you will be in that first ring of things. Is that clear?

This is only one part of the story that I can pluck here, do this and so on. But, we may not be or it may not be enough or we may not be satisfied doing only a solid mechanics problem. Many problems warrant what we call as coupled problem as well or coupled analysis as well; many problems warrant a coupled analysis.

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What is a coupled analysis? For example, let us look at a welding problem. Of course, we are interested in welding problem. What is that we are interested in? Temperature distribution and we are also interested in many places, stress or deformation, because the deformations that result from this temperature distribution becomes very important; in fact, may be much more important than just the temperature distribution. So, we do a coupled problem where one field values affect another. On the other hand, look at say for example machining, a machining problem. I want to analyze a machining problem. Now, what is the situation here? In the first case, it was the heat or the temperature which had an effect on the deformation, basically through co-efficient of expansion and so on. But, in the second problem, the second example which I am telling you is a machining problem and what is the problem here?

Yeah, deformation; agreed, large deformation. Yes; how do you get the heat? All of us know that when you machine it, after you machine it you touch it, it is going to be very hot. So, what has happened there? Heat produced due to so many effects; one of the major effects is the conversion of plastic work into heat; conversion of plastic work into heat. In fact, 90% of the plastic work that is done on the body is converted into heat. Is that clear? So, here again there is a coupled problem. But, in both the cases the coupling is one way. It is a one way coupling. I am interested in temperature, because I want to do stresses. But, in the first problem my stress effects may not be very important from the point of view of increasing the temperature. In other words, the temperature in that welding problem is totally controlled by my weld rod, its movement, how much current I am giving or in other words how much energy I am inputting and so on.

In the second problem, I am interested in the temperature, basically from the other end. I have plastic work and the temperature is the result of this plastic work. But, I am not very much worried whether the temperature would increase my stresses further and so on. These are, I would say, I would classify them as loosely coupled problems, where one has an effect on the other. But, there are problems in which you can say that one has an effect on the other and vice versa. Then, the strategies have to be very, very different. This may involve say, for example rolling tyres with viscoelasticity; it may involve temperature, as well as the stresses and so on.

The next category of problems are coupled analysis problems, where again non-linearity may play a great role. It is not that non-linearity is present only in the deformation, but non-linearity may also be present in temperature. How? In the thermal problem. How can non-linearity be present in the thermal problem? Yeah, so the properties or in other words the thermal properties may also be a function of temperature; can also be a function of temperature. This is again, this may again come under region of non-linear analysis. Of course, one of the major, one of the most important problems today which has impact on so many fields is what is called as fluid structure interaction problems. Here we saw the effect of thermal loads acting on the body and giving rise to stresses. On the other hand we can have a situation, where fluid flow has an effect on the body and would result in increase in stresses.

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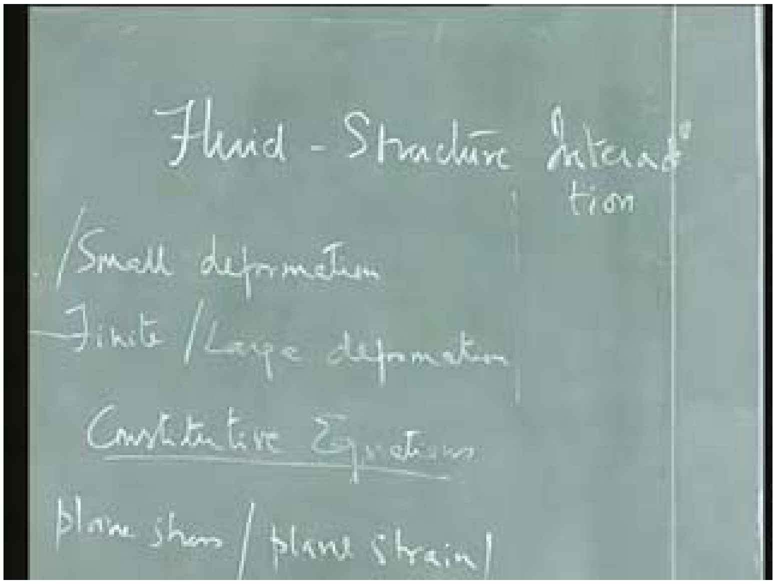

Fluid - Structure Interaction

/Small deformation

Finite /Large deformation

Constitutive Equations

plane stress / plane strain

In other words, fluid structure interaction, I will write that here, fluid structure interaction also becomes a problem. When we develop finite element analysis, we have to take into account so many things. But, this list does not stop here. Apart from all these things, certain conditions may have to be met.

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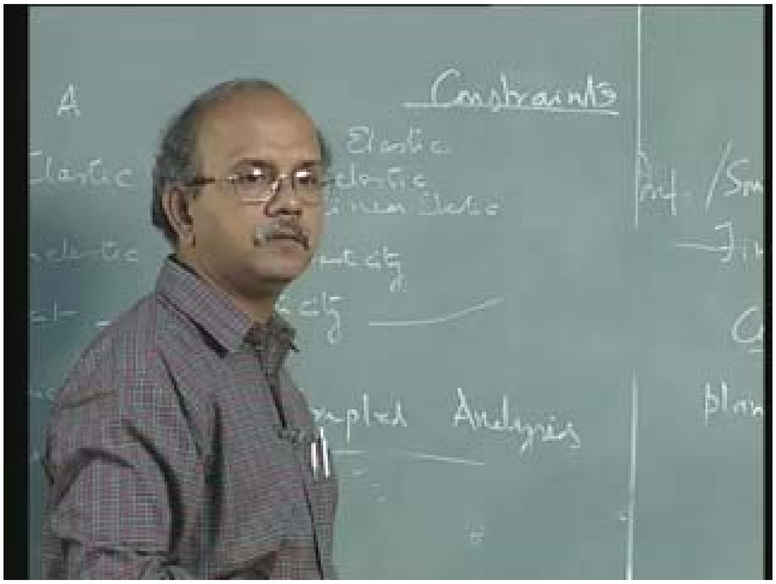

These conditions are broadly classified as constraint conditions or constraints. What are they? The body because of its material behaviour may not only be, say for example non-linear elastic, but also may become incompressible; may become incompressible, which means that there will not be any volume change. Not that every body, every non-linear elastic body is incompressible, but in many situations you may have to deal with certain what we call as internal constraints; constraints with your material behaviour that the body becomes incompressible. In the plastic case also the body becomes incompressible. In other words, plastic part of the deformation results in incompressibility. All of you would have known that the Poisson's ratio approaches 0.5 when body goes into the plastic region and the result is due to the definition or due to the constraint called incompressibility. Is that clear?

Now, the challenge is how are we going to solve set of these problems? How are we going to develop algorithms to solve them? What is the strategy which normally is adopted in non-linear analysis? The first issue; second issue, of course, is how am I going to define stress, strain, how am I going to characterize deformation? How am I going to say that it is a finite deformation, large deformation or finite or large deformation, small deformation, how am I going to characterize them? That is the next major issue. So, these challenges are the ones which make this whole study of non-linear finite element analysis very interesting. As I told you, in this course we are not going to take independent cases and develop. We may do that for certain cases, but we are going to, you are going to see from next class onwards that we are going to develop some general strategies, this plug and play we are going to do that even for algorithms. How we are going to do that we will see it in the next class.