Design and Optimization of Energy Systems

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Module No. # 01

Lecture No. # 36

Linear Programming

So, more or less we have, more or less we have looked at the techniques which we will frequently encounter in thermal sciences, that is, single variable and multi variable, both constrained and unconstrained. But in the next two weeks, we look at some special techniques, which generally, or, either not frequently used in thermal sciences, or, are not called, or cannot be called as classical optimization technique or non-traditional optimization technique. But, we do not have much time. So, we have three classes, this week and three, next week; out of which, one class will go for the assignment test. So, in five classes, we will achieve, what best we can do.

But, linear programming is a very important technique, because, operations research people, management people, people in sociology and management science, and all these fields, it is extensively used. There are some, also there are some problems in engineering, which can be eminently handled using linear programming. So, today's class, we look at linear programming, so it is also called LP.

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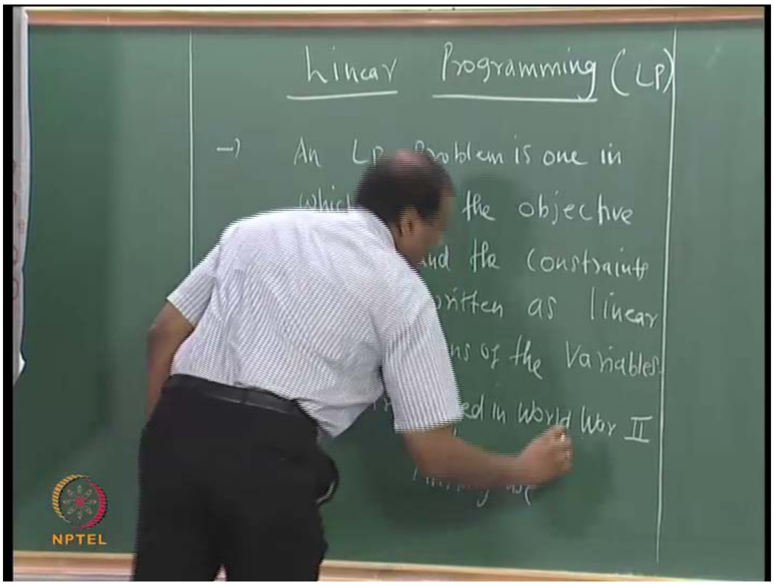

The LP problem is one in which both the constraints and the objective functions can be written as linear combination of the variables; I come again, a linear programming model or LP problem is one in which, both the objective function and all the constraints can be written as linear combination of the variables. So, you will not encounter sine h x, e to the power of m x, e to the power of x 1 and hyperbolic, all those things will not come; simply represented as linear combination of the variables, in which both the objectives function; the very colloquial terminology can be represented as, can be written as.

Obviously, now, you can see, why it is not frequently use in fluid mechanics, heat transfer and thermal science, because there are hardly situations or hardly any situation in which both the objective function, all the constraints can be written as a linear combination of variables. But, if there arise a situation in which it is so, then there is a body of knowledge which is available, which we can use instead of always to Lagrange multiplier method or Penalty function method and so on.

So, what could be the origin of this? Lot of people are taking management science, so you should not able to ask this question. When it is originate, the use of LP? World War II, obviously; so, first used in World War II. What should have been the possible problem for which LP must have been used? Optimal allocation of men, optimal allocation of ammunition; so, optimal allocation of aircrafts, optimal allocation of all these artillery, this thing, so that you, maximization of your benefits, I mean, strategy, to defeat the enemy and all that. So, obviously, these was the first flight they had, the alled forces.

So, it originated in United Kingdom or something, so the 30's and 40's; subsequently, war is very much of feel that the department in the industrial engineering, operation, research, your financial engineering, financial mathematics, engineering finance, whatever. So, it is highly developed. It is one of the well-studied or over studied subjects in management science. So, first used in World War II, so basically catered up with the military use. Also, used in sociology.

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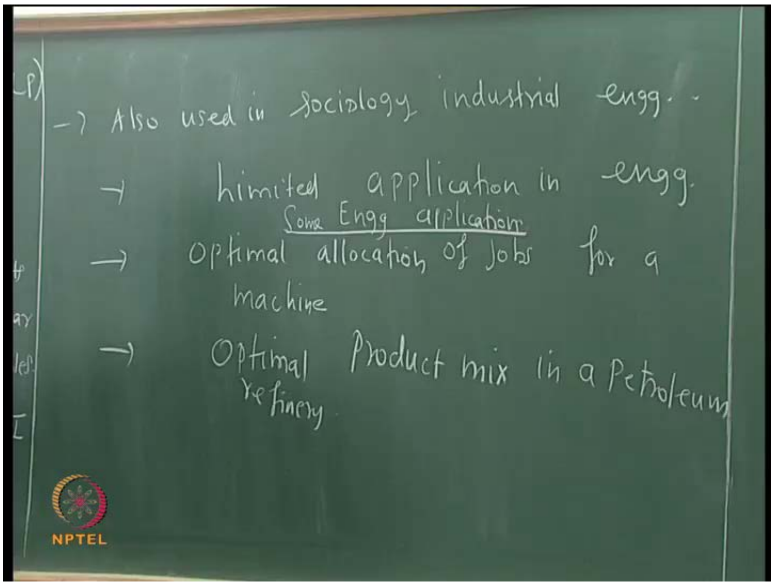

So, limited application in engineering. Can you give some examples in mechanical engineering, where you get linear combination of; optimal allocation of job to a special purpose machine; so, you have a CNC lathe, we can do many things- turning, boring, milling, whatever, shaping, whatever; and then, there are cost and time labor associated with each of this. There is a total machine availability. Now, there is a problem, optimal allocation of jobs for a special machine.

We also constantly do this linear programming in our mind, right. So, the objective may be different for different people; maximize the pleasure subject to the condition the CGPA should not be below 7.5, but good when CGPA is written as a constraint. Maximize CGPA is subjected to minimum pleasure; there are several maximizing both is not possible, they are orthogonal to each other. So, constantly we are doing, dynamic programming, all certain non-linear programming, everything is going up; sometimes when you get vigorous only, we sort of other means. So, by nature, everybody wants to optimize, you want to keep optimizing all the time. So, the objective function be different. Optimum allocation of job for a machine.

Optimal product mix in a petroleum refinery, right; are you getting the point? You can sell diesel, petrol, kerosene or naphtha, all those things; then, each will come with the cost of refining, the profit; whether you want everything should be petrol, everything should be kerosene, everything should be diesel, everything should be paraffin, and all that.

Optimal product mix, we call it as product mix; so, some engineering application. In operation, management operation research, they will always talk about this; and, those people of develop all the genetic algorithm, evaluation, optimization techniques, all those problems, where it is impossible for them to get derivatives and second derivatives and evaluate (( )) and so on. But, they are able to evaluate the objective function for any combination; are you getting the point?

Whereas, when you are approaching it from a thermal science perspective, finally we get some function; it may not be simple function, we may get sine h, cos h, h a to the power of m x, whatever. Generally, it possible for us retrieve first derivative, second derivative and all that, but in OR in other field, it is very difficult. But, you tell them, if x 1, x 2, x 3, x 4 and so on, then you can get y; but, do not tell me, what is dou by dou x 1 dou square by, dou square y by dou x 1 dou x 2, that is not possible. So, they are try to wean off or wean away from calculus based optimization; so, non-traditional optimization, they are also advanced the field. So, these are the broad preamble or prelude study of linear programming.

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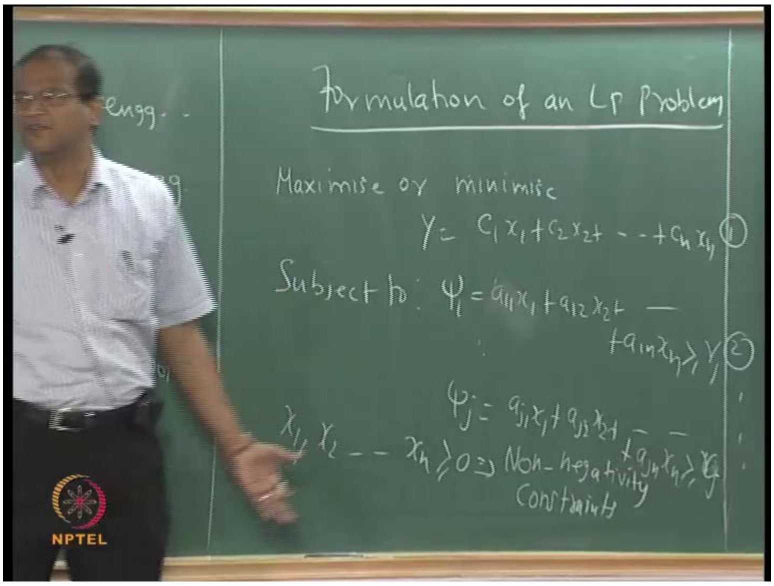

So, the formulation of an LP problem, it could minimize cost or maximize profit, maximize or minimize, Y is equal to; there is no problem as far this thing is concerned. But, you are now reading it as linear combination variables, x 1 to x n, are the independent variables; x 1 to x n could represent the product your company manufactures. Each one could be different. For example, a furniture company, x 1 and x 2, could be table and chair; for a petroleum refinery, x 1 could be petrol, x 2 could be diesel, and so on, ok, the quantity; c 1 could be the profit associated with this, or c 1 could be cost of refining. Your objective functions could be the minimization of cost or maximization of profit.

So, subject to, there are psi 1 is equal to; shall we use it as, a 1 1; is greater than equal to some r 1. Now, some books will say that, that can be replaced as equality, and then, you have additional variables, slack variables and so on; I do not want to introduce slack variables at early stage; it is possible to write it as need quality also, by introducing some artificial variables or slack variables, which represent something else.

So, psi 1; like that, you can say, psi j equal to a j 1 x 1 plus; greater than r j. Now, x 1, x 2 to x j are all physical variables; therefore, we have to assign or we have to write out extra constraint, that none of these variables can become negative; these are called non-negativity constraints; are you getting the point? Because, they are physical variables, they cannot take on negative values.

Now, this is our standard; you can write 1, you can put 1, 2, whatever. Now, this has to be solved. If you solve this, you will get the optimal values of, x 1 to x n, at which your y is maximize. We substitute these values; c 1 to c n and a 1 1, a 1 2 all these are coefficient, which are known afferent, that is, which are known before you start working on the problem. Once you get the optimal values of, x 1 to x n, you can substitute into your expression for y, and get the value of y star or y plus or y max or y min.

Now, you feel that you can use Lagrange multiplier method to solve this, right; equality you can change, you can use the Kuhn Tucker condition, what is the catch? If it was possible to solve the problem of this nature using Lagrange multiplier method, first of all special techniques to develop LP would not have been developed; so, there are some key differences between this problems and Lagrange multiplier formulation, problem, which can be solved using Lagrange multiplier; can you identify?

Please remember, even though we handle inequalities, we handle one or two inequalities, most where equalities; if everything was inequality, then your life become miserable, Kuhn Tucker condition, then you will go mad. So, basically, essentially the Lagrange multiplier method is used for solving problems in which of equality constraints; but, all of you, one or all of you inequality constraints can be handle this, does not matter; it is basically designed for handling inequality constraints.

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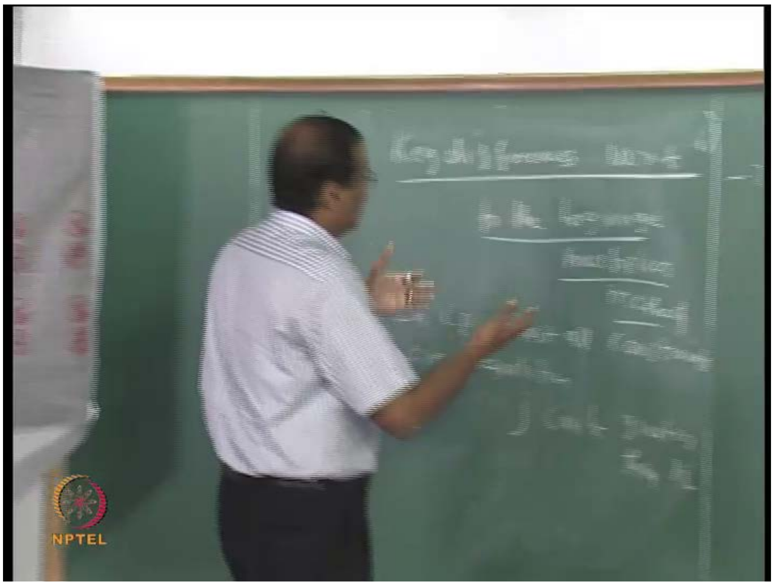

So, key differences with respect to the Lagrange multiplier method; in LP, almost all constraints are inequalities. Again, j can be very important; j can be greater than n, because is one in inequality; and, inequality only reduces your feasible domain, it prohibits certain solutions, that is all. Equality constraints have to be exactly great, that is why, an equality constraints is a lot more restrictive than the inequality constraints.

Therefore, in the Lagrange multiplier method, if m, the number of equality constraints and n was the number of variables, m has to be necessarily less than equal to n, where j can be greater than equal to; so, and of course, Lagrange multiplier method has other, though it has the disadvantages, it has other advantages that all these, it does not had to be such a simple and silly function like this; any complicated non-linear function it can handle.

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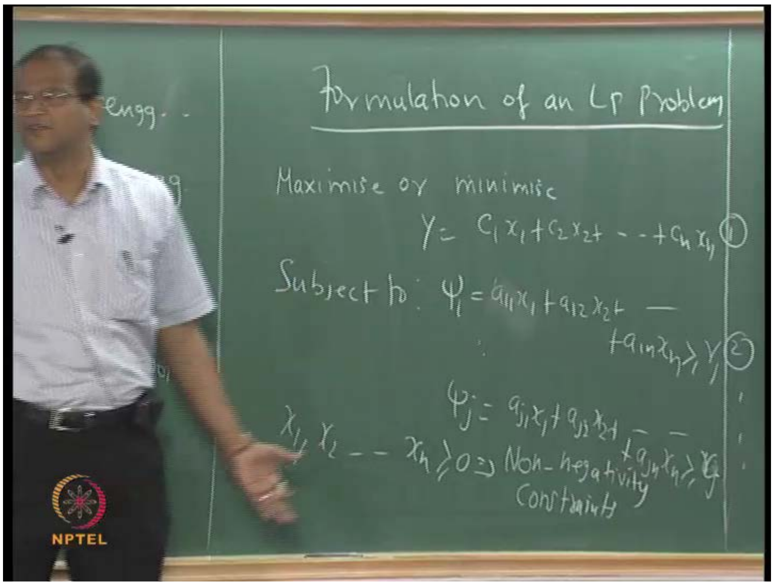

Why are these LP models? So, why is everybody so fond of linear programming? Because, just like we have console, fluent, CFX and so on, the industrial engineering community has developed very advanced software and commercial software available for solving large LP problems. And, business people use it, because they are always interested in, what is the profit and this thing; and, it is used by lot of analyze and so on. Therefore, the LP is widely patronized.

Also, they are always looking at, if the cost of this thing changes, if the price of this thing changes, if the demand goes up, if the demand goes down, what is the sensitivity of my final solution, they are always interested in such kinds of things; stability is not the effort. They are always expecting something to go up, something to go down. Sensitivity analysis can be eminently and easily handled using LP.

So, largely because of these two reasons, that commercial software is available; sometimes freely downloadable software available for handling large linear programming problems. Sensitivity analysis, which is also called as post optimality analysis; the Lagrange multiplier is like sensitivity coefficient, right; post optimality analysis can be easily done using LP models. Therefore, LP models are very useful.

Even if there is a slight degree of nonlinearity, if the variables come as x 3 to the power of 2, some people will just put it as x 3 square is equal to x 24 or something, and keep working with x 24. So, slight degree of nonlinearity can be in, handled using the LP method; of course, all the things are highly non-linear, it is difficult; little bit of nonlinearity, it can handle.

Because of lack of time, we will be handling only, we will be looking at only two variable problem; and, I do not want to get into the simplex method analysis; so, we will do the graphical method of solution. If we recall, in one of the earlier lectures on formulations of optimization problems, we did take up a problem, we did not solve it though, we just brought it the constraints, we identified how to formulate an optimization problem. I will give a different example. So, we learn the graphical method of solution to a linear programming problem through an example. So, this will be problem number?

Student: 39.

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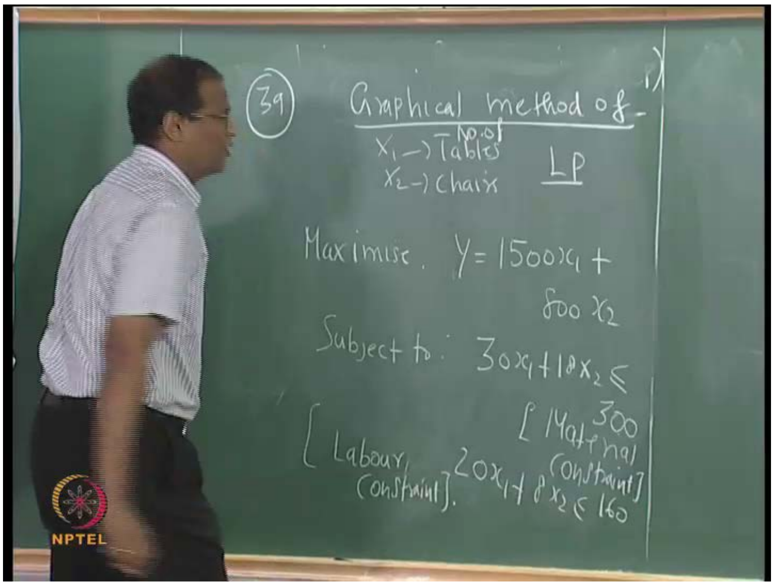

39. Though the graphical method has limited appeal, has limited scope, in the sense that it has only two variable problem can be handled; all the features on LP problem can be from very nicely, very elegantly brought out, by looking at the graphical solution itself. Therefore, we will take up a problem that concerns, which can be handled using the graphical method.

Problem number 39, a furniture company, a furniture company can make tables or chairs or both. A furniture company can make tables or chairs or both. The amount of wood required for making 1 table, the amount of wood required for making 1 table, the amount of wood required for making 1 table is 30 kilogram, 30. The amount of wood required for making 1 table is 30 kilogram, while that for a chair is 18 kilogram, while that for a chair is 18 kilogram. The total quantity of wood is available is 300 kg, the total quantity of wood is available is 300 kg. The labor required for making a table is 20 man hours, the labor required for making a table is 20 man hours, the labor required for making a table is 20 man hours, while that required for a chair is 8 man hours, so the labor for table 20, for chair 8 man hours. The total number of man hours available is 160, the total number of man hours available is 160. The profit from a table is rupees 1500, the profit from a table is rupees 1500, it is not a selling price, the profit. The profit from a table is rupees 1500 and that from a chair is rupees 800. Determine the optimal product makes of the company, determine the optimal product makes of the company using the graphical method of LP, determine the optimal product makes of the company using the graphical method of LP; so, this 300 kg and 160 it could be per shift basis, per day basis, do not think how can a company survival it makes only 10 chairs or 20 chairs, it could be a hourly basis or daily basis or whatever.

So, volunteers take one sheet each, I will start formulating this on the board. So, you have to plot the constraints, x 1 - number of tables, x 2 - number of chairs. Before you begin use the graph sheet, just formulate the problem- maximize, profit, subject to, and try to see if you can simplify the constraints. So, maximize Y equal to; let x 1 be the number of tables. So, Y will be; yeah, please look at the board and tell me, maximize Y equal to,

Student: 1500 x 1 plus 800 x 2, yeah; then, you have to put all the subject to, 30 x 1; Less than or equal to, less than or equal to, right.

You can still have some wood left; you cannot, if you make so many tables and chairs, you cannot be sure that exactly all the 300 kgs will be satisfied, but it cannot be more than 300. It can be equal to or less than, right; that is why, you have a less than equal to constraints; everybody able to follow this? It is not like satisfying the moment balance or energy balance or, is not like the continuity equation which has to be exactly satisfied, mass flowing out must be equal to mass flowing in, it is not like that.

At the end of the day, some residual wood can be there, that is, wastage, there will be a wastage know, anyway. I mean, when you finally, in this problem it may not be wood; generally, every constraints, there will be some surplus, that is called slackness. So, this is what constraints? Material constraints. Then, yeah, labor cost, what is that? 20 x 1. So, this is the constraints of the labor

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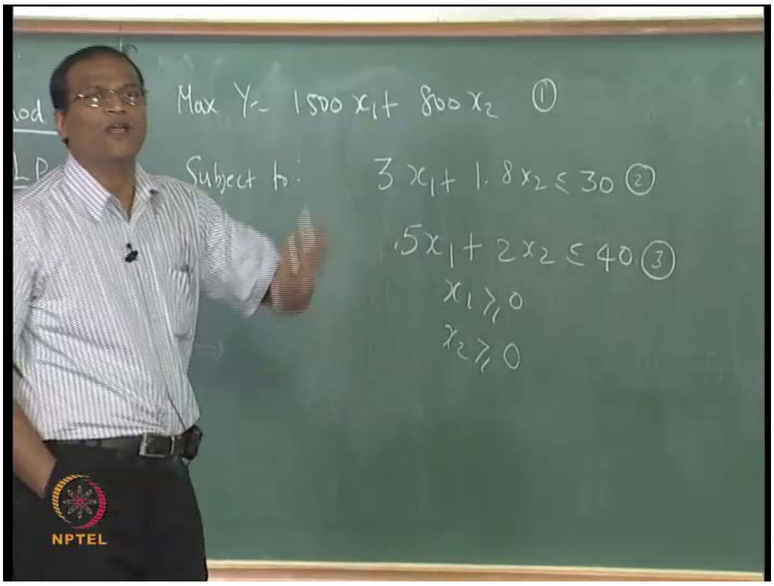

So, I will say, to make it easier, I want to make it easier for me; 20 x 1 plus 8 x 2 is; therefore, I just simplified a little bit. So, this is the formulation for this problem. Please do not forget the non-negativity constraints, right. I have to get the values of x 1 and x 2.

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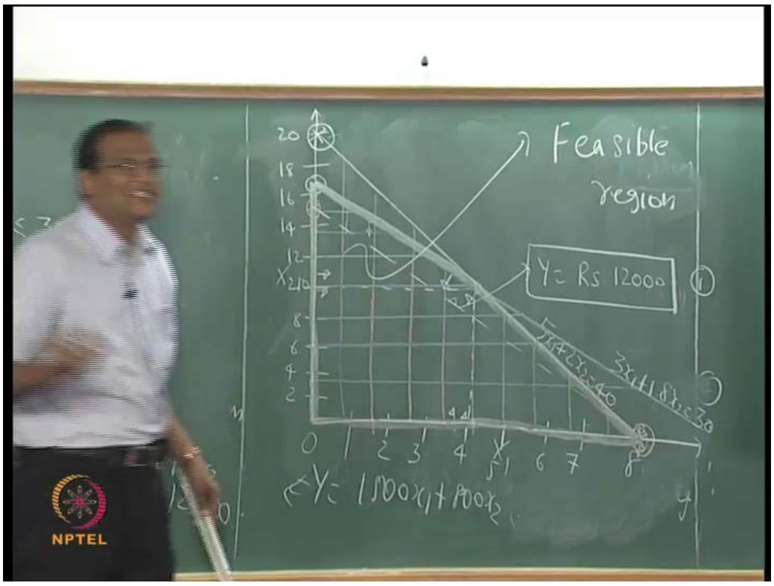

So, what you have to do is, graphical method. You have to first plot the various constraints and identify the feasible region. Feasible region is region in which the optimum can lie; outside the feasible solution, the optimum cannot lie, because, first of all, it is not a feasible solution at all. So, what is the feasible solution? A feasible solution is one which satisfies all the constraints. So, an optimal solution has to be a feasible solution first. First, you should have a working pump and then, you will decide on whether the pump is optimum or not. So, x 1 and x 2, just plot; has everybody got the graph sheet? So, when x 2 is 8, this fellow we want up to 20, X 2 we want up to 20? No need, but 20 is better.

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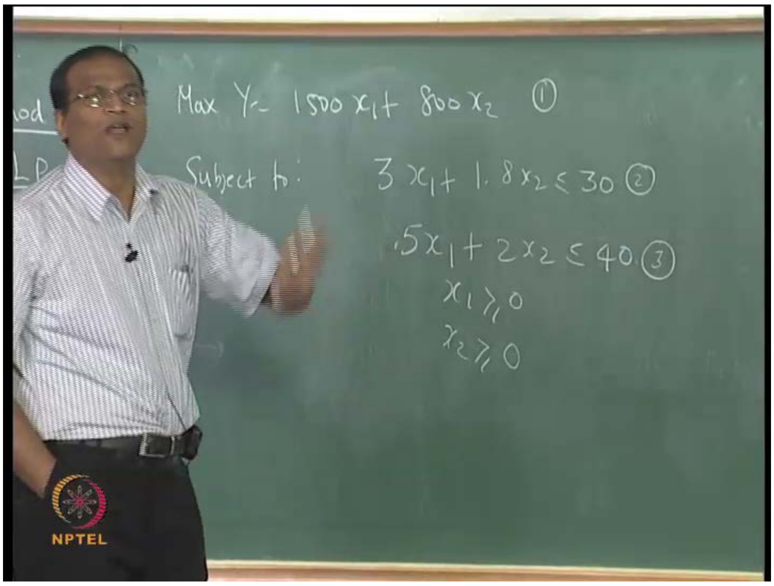

But, each of the constraints you have to get two points, because, you want to fit a straight line. So, let us plot the constraint 2; so, that is difficult now. I got x 1 only up to 30; does not matter.

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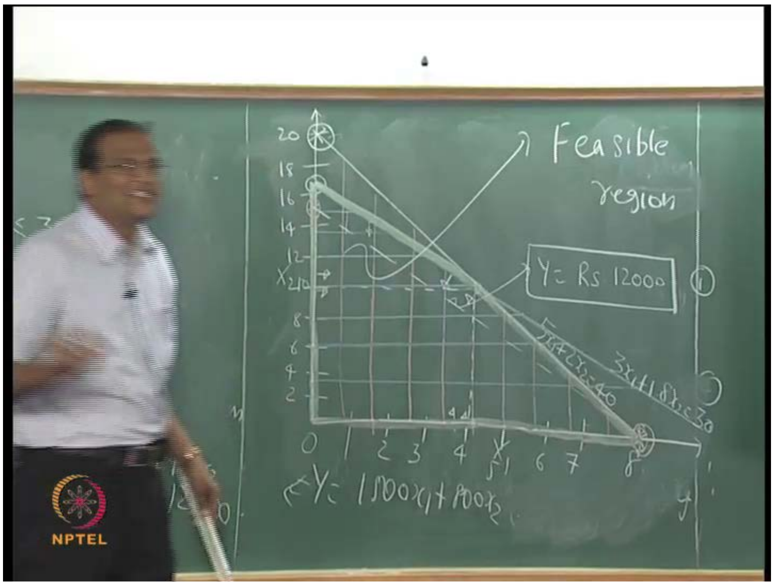

Let me plot the constraint 3 first; x 1 is 8, right? Then, x 2 is 20; you had a long scale, so this is multi objective optimization. Today, finally, I will draw straight line. That is, what is this fellow? 5 x 1, equal to 40, right? That line is 5 x 1 plus 2 x 2 equal to 40. So, we want, 5 x 1 plus 2 x 2 less than equal to 40; so, you have to put the direction like this, please put the direction like this, that is the feasible region.

And then, x 1 must be greater than 0; please look at the board; x 1 must be greater than 0, so, this is the feasible region; x 2 must be greater than 0, this is the feasible region. So, the solution can first be only in the first quadrant, and then all these constraints will reduce the feasible region, and then, finally you will identify the feasible region, which satisfies all the constraints.

Now, the second one is, what do you get, x 2 equal to, when x 1 is 0, constraint number 2, when x 1 is 0,

Student: 16; 16 point; and then, one more point you tell me?

Student: 10. 10, I do not have, tell some other points? 4 and, 4 and 10, right.

Student: 4 and 10.

Shall we use different colors now? So, this one is; I want here, so, let me manage; yeah, I have to get this, know. The feasible region is one which is indicated within in the checks.

Now, some other color will be good, let us use the green, so the feasible region is, that is the feasible region. So, any point within the feasible region will satisfy all the constraints, but each point will result in a different value of y. So, first we identify the region in which y max is expected to lie, because y max has to be first feasible solution, agreed.

Now, getting the y max is not shows a straight forward, what do you do now? You have to assume some value of y, draw the straight line and keep moving the straight line, till it just moves out of the feasible region. So, we find out when it just moving out of the feasible region, the point or the points it cuts in the feasible region is, or the optimal solution to this problem.

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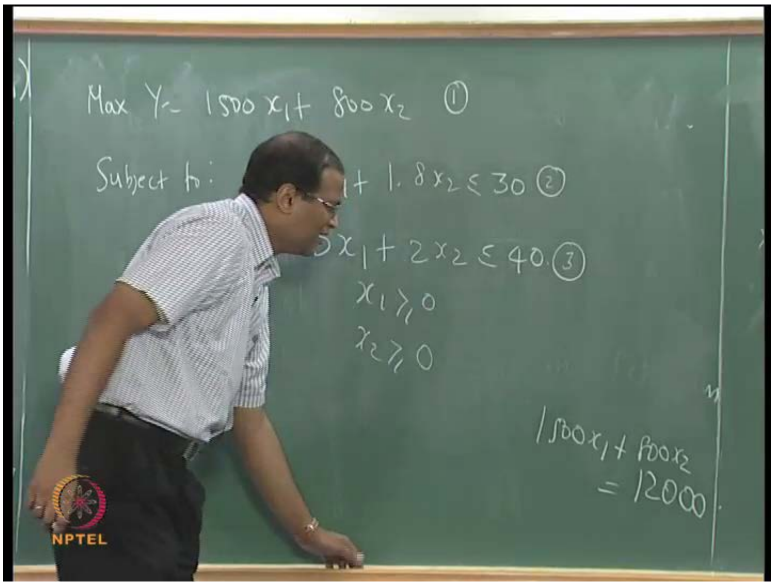

Now, let us plot this. Let us take one ISO objective line, y equal to 1500, so, let us say, shall we say 6000 or 8000?

Student: 12000.

12000 is not the answer, know.

6000 cannot be the answer, what is it, Sampath.

We will put 8000, we should not put 14000.

Too small; the 1500, it cannot be divided.

Student: 6000 sir.

Student: 12000 sir.

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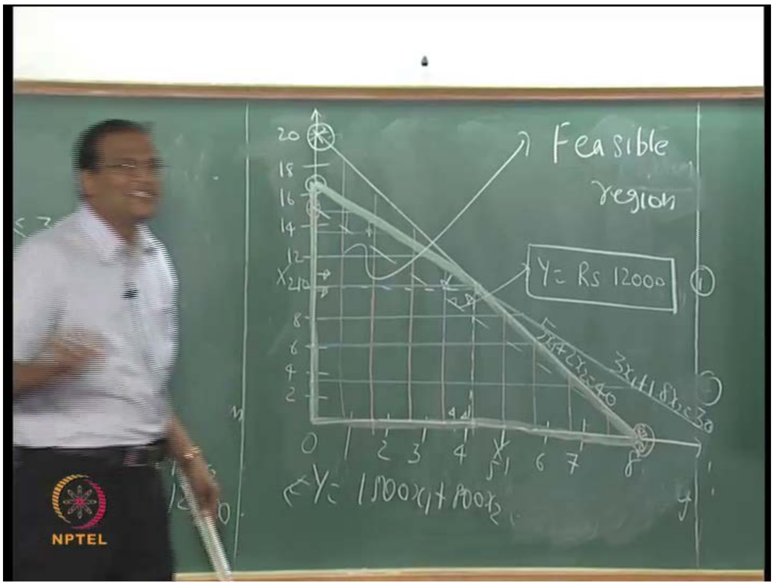

12000, I will put 12000. Now, find out the points of this? So, this is my Y, so x 1 is 8, x 2 is 15. Whatever I have indicated with dash line, is basically; Y is equal to rupees 12000. So, this dash line is the Y equal to 12000 rupees line; that means, any point on that line will give combinations of x 1 and x 2, which will result in a profit of 12000.

So, which mean, if 12000 line is here, if I move here; you have to draw series of parallel lines. If I move here, I will get combination of, x 1 and x 2, which having profit equal to may be 10000, 9000, 8000 like this. You can start with here 0, 0; not do anything at all, have all the material and all the labor with you, that is trivial solution. So, you can also have like this; you can keep on having like this, and you can get out of the room and all that, then you can have a unbounded optimum, you will put x 1 equal to infinity, x 2 is equal to infinity, Y is equal to infinity, great.

But, then, what does it tell? It says, you have not put the constraints, because all economical departments have to be closed, if that is the case. Life is not about maximizing profit; maximizing profit of the constraints are always chasing. So, within the feasible region, we have to find the solution. Now, there is a theorem which says, that if once you identify the feasible region, though there are infinite number of feasible solution, the corners of the polygon which are formed by the coordinate axis, as well as the various constraints, qualify as a candidates to the optimum solutions; that these are the properties which can be proved in a row o r, if you take o r those o r 1 or o r 2, they will prove this. Which means, it is easy for us, we have to identify; we have to get the value of y only at this 1, 2, 3 and 4 points. Anyway here, it is silly; x 1 equal to 0, x 2 equal to 0. So, now, here you have to take a column making all chairs, here it is making all tables, here it is making tables and chairs, each of this will result in a value of Y and, now tell me, which is the maximum value of Y?

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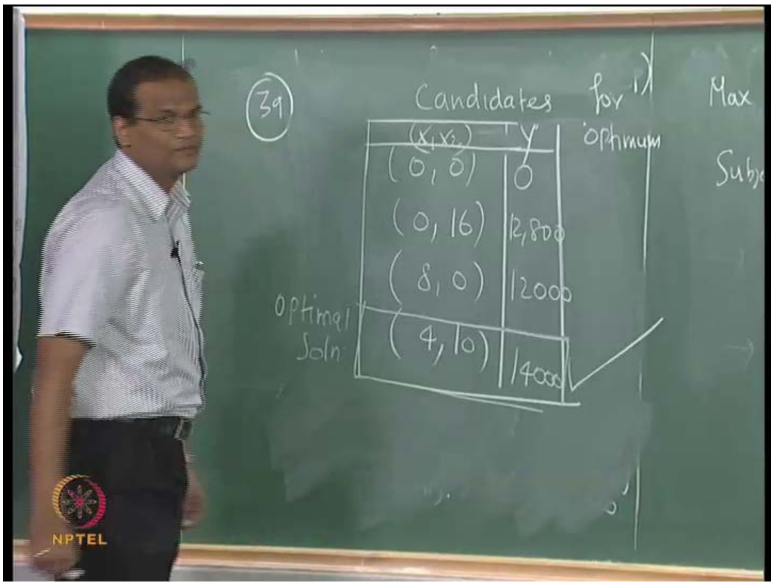

So, we take candidates for optima; the other one is 20 and? No, no, what is this?

Student: 16.67.

16.67. And, we will make it 16; 17 will violate, Then?

Student: (8, 0), (4, 10).

0; here, 12,800? How much is it?

Student: 12,000.

Student: 14,000.

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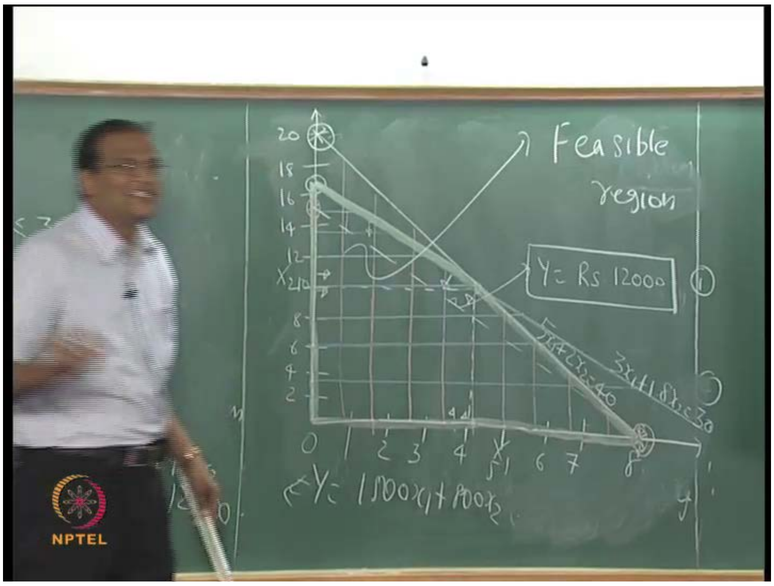

However, if it so happens, that the ISO objective line which is like 1500 x 1; watch carefully; if the ISO objective line, which happens to be 1500 x 1 plus 800 x 2, such that, it becomes parallel to one of the constraints; then, instead of these two lines intersecting at a point, this whole thing will be an optimal line, instead of an optimal point; so, these are all alternate optima. Any point on this line will be an alternate optima, which will give the same value of Y.

Suppose, we look at the solution, where Y is equal to infinity, x 1 is equal to infinity, x 2 is equal to infinity, that is called an unbounded optimum. It is a warning; it tells you that you miss one or more constraints, because it is unphysical. So, this is how you solve two variable problem, using the, LP problem, using the graphical method. This can also be solved by introducing, what are called slack variables.

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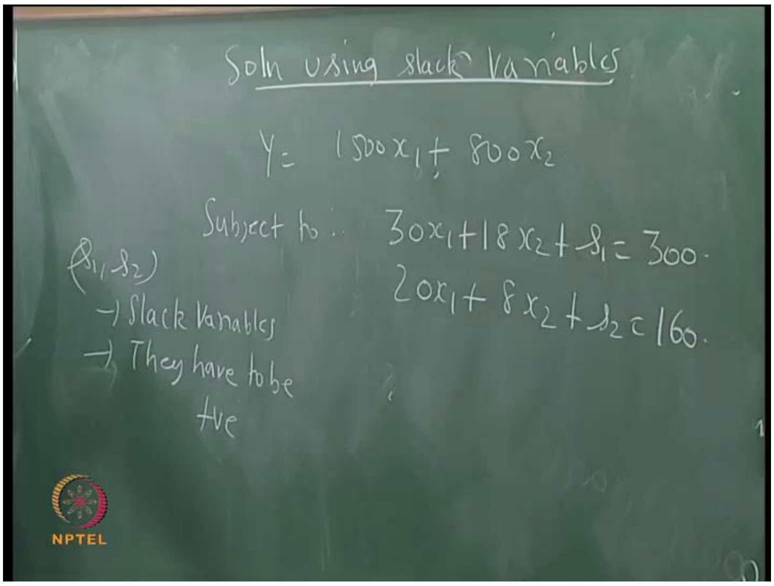

The slack variables are introduced to convert the inequalities to equalities; subject to, what are the first constraints 30 x 1, is it?

Student: 3 x 1. 3 x 1 plus, no, no, now I retain the original constraints itself, 30 x 1 plus; other one was?

Student: plus 8 x 2. So, s 1, s 2 are called slack variables; they have to be positive; why should they be positive? Because, after the optimal solution is determined, s 1 tells you the amount of wood which is available, which has been wasted in the process; s 2 is the number of man hours which has to be available; they cannot become negative; they can at least be 0, but they are to be non-negative; are you getting the point?

Suppose, you have greater than equal to constraints, you can put minus of s 1, minus of s 2. The FUNDA is, they are not unrestricted in sign; they are restricted in sign; they have to be positive. Now, already we had 2 constraints, we had 2 variables; now we converted the 2 variable problem into 4 variable problem, and we shot ourselves in the foot, how do we solve this?

I mean, cannot, it is, if 2 equation and 4 are unknown, you cannot hope to solve this; however there is way out. We make 2 variables 0 at a time, you can make 2 variable 0 at a time and find out the resulting y; because, since s 1, s 2 are also not unrestricted in sign, they have to be only positive. So, it is like, it is equivalent to representing x 1, x 2, s

1 and s 2, on a 4 dimensional plane, and then, trying to find out the feasible region. So, the same FUNDA of this ISO objective line, cutting this polygon at one of the point, emerging as the optimal solution; that FUNDA holds good here also. So, if you do that, how many combinations are there? If 2 are made 0 at a time, how many combinations are there?

Student: 6. 6, ok

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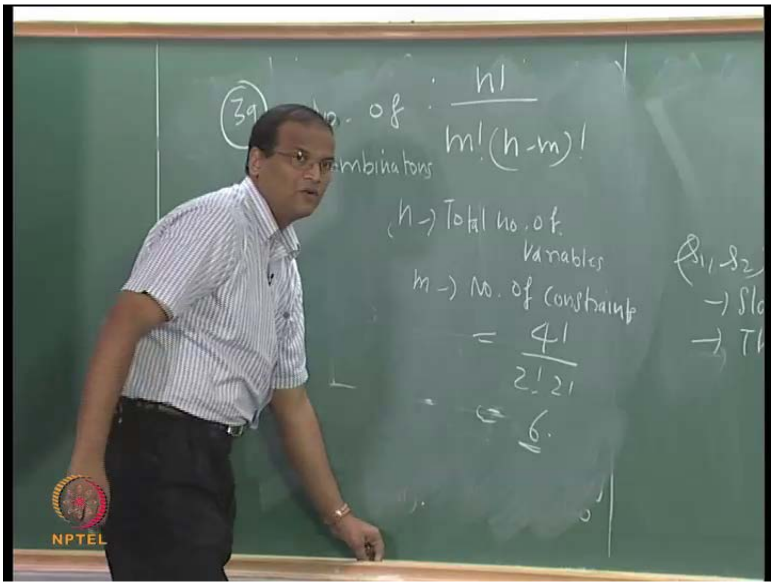

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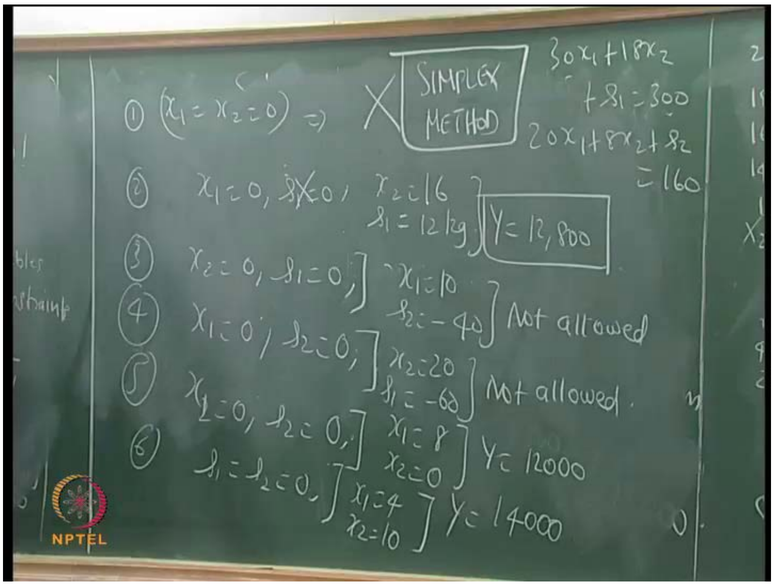

So, if n, total number of variables, this total number of variables including the slack variables; what is n now; x 1 x 2, s 1 s 2, n factorial will be, how much is, 6. So, we need to evaluate 6 combinations; silly solution, we do not have to worry about that. Now, please solve the system; quick, it just take 2 or 3 minutes.

What was that, the other one?

Student: 20.

Now, x 1 equal to 0, so what is x 2?

Student: 16.

16; 300 divided by 18 is not exactly 16, how much you get?

Student: 16.

Therefore, x 2 is 16; therefore, s 1 cannot be 0, so s 1 will be?

Student: 0.6.

Just 0.6 kg, how many of you follow what we are doing? In the first case, I am putting x 1 equal to 0, and s 1 equal to 0; that means, 18 x 2 is equal to 300, x 2 is 300 by 18, but you cannot have 18.67 tables or 16.4 chairs. So, I am rounding off to the lowest. If you do that, automatically, s 1, has to become positive, otherwise the 300 will be violated.

Student: 12.

Yeah, so you will have 12 kg is left. Now, what is the y? Straight away, 12800? No?

Student: 12,800.

Yes, 12800; x 2 equal to 0, same problem x 2 is equal to 0, s 1 is equal to 0; fine, x 1 is equal to 10, y equal to? Is it correct? That is not possible; why it is not possible? It is violating the other thing, yes. So, x 2 equal to 0; so, what will happen, the s 2? Let us put, x 2 equal to 0 here, s 1 is equal to 0, so x 1 equal to 10; if you put x 1 equal to 10, x 2 equal to?

Student: Minus 40.

Minus 40, not allowed, you cannot have our surplus of minus 40 man hours, not allowed; x 1 equal to 0, x 2 equal to 0, x 2 equal to 20, again, right; x 2 equal to 20, what happen?

Student: s 1 is minus 60.

S 1 is minus 60, not allowed. So, x 2 equal to 0, s 2 equal to 0; is that correct? This is 12000. This is x 1 equal to 4, x 2 equal to 10, correct? 4 and 10, yes. If you do not like the graphical method, you do not have the pen, graph sheet, whatever, you can use this. This has been modified under powerful technique called, the simplex method, has been developed based on this. Simplex method is an algorithm of this; I demonstrated how this algorithm works for a two variable problem; you can extend it to several variables, and you can draw table; they call tabular or something, and then they will proceed and then they will. So, it is only table, so that is all the o r peoples solve the. And then, you can introduce artificial variables and this thing method and all that, all this will be taught in operation research.

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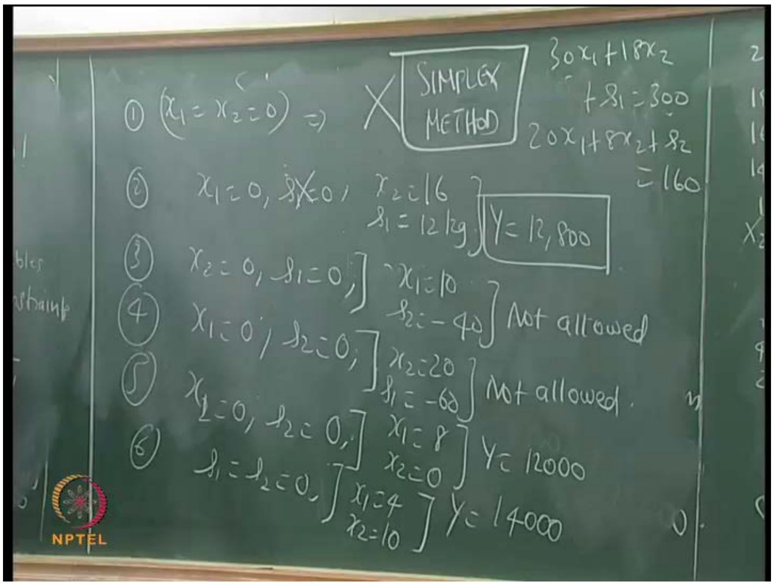

So, to cut a long variable short, it is possible to introduce slack variables. Slack represents the difference between the left hand side and right hand side of the constraints, that represent the amount of excess material available. If it is a material constraint, s 1 is amount of wood left, s 2 is amount of labor left; in this case it is so happen that, s 1 and s 2, are both 0 for the optimal solution; that means, s 1 and s 2 are active or binding, the labor constraint and material constraint are both active and binding constraint in this problem. Because, they operate as equality, right.

You make at an optimal solution, where one of this thing is not binding, it is possible, it will happen if the table, profit from a table is 3000 rupees and profit from a chair is only, you do not make, you do not make chairs; but, where will people sit, that is a philosophical question, which cannot be factored in. So, if the, from pure economics, that is a solution.

So, this is the best, I could possibly teach you in 1 hour. I have taught it in NIT Trichy 16 years back, some 10 hours of Bingham method and all that, but that is not, but the course was operations research; can you believe that I taught operation research? But, I think you got a good idea of this. Whatever, we have done for two variables, can be extended any number of variables, and freely downloadable programs are available. So, if you encounter a multi variable problem, where both the objective functions and constraint are linear combination, you can use some other powerful techniques like this; study the solution and get the solution and do the post optimality analysis.