Design and Optimization of Energy Systems

Prof. C. Balaji
Department of Mechanical Engineering
Indian Institute of Technology Madras

Lecture No. # 32

Reduction ratio of Fibonacci search method

So, yesterday we were looking at the Fibonacci search method where we exploited the power of the Fibonacci sequence in eliminating one of the two points in the 2 point test at every iteration. So, we solved this cylindrical solar water storage heater problem and finally we got an uncertainty of about 0.173 meters, we reduced it from 3 meters. We started the radius starting from 0.5 to 3 and a half meters and finally came to something between 0.76 and 0.94 or something like that, right, 0.77 and 0.94 or something like that. So, though theoretically the RR is supposed to be 1 by F of n, we were never able to reach it because in the last iteration both the points were at the center and we were not able to proceed any further. So, we will have to revisit our reduction ratio I mean whatever we arrived at.

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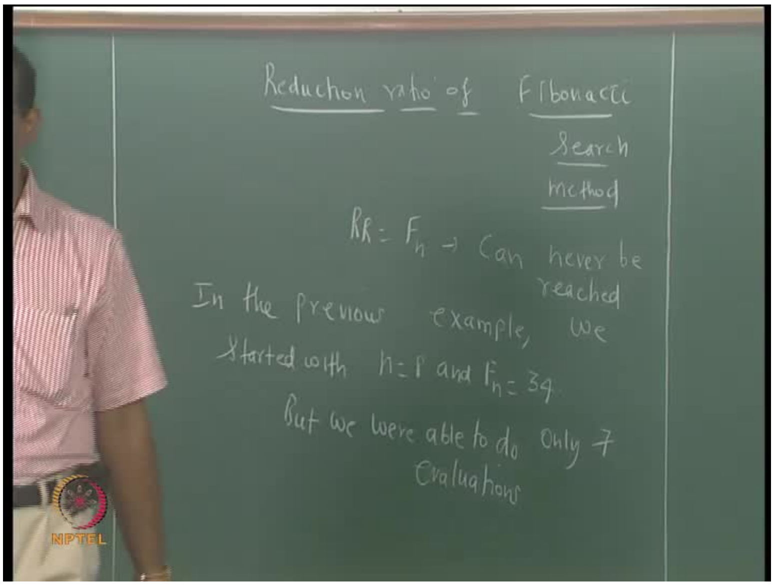

So, what is the actual reduction ratio? Because if we take F of 8 is equal to 34 and we started dividing the interval as 21 by 34 and then 13 by 21, we were not able to do the eighth evaluation at all, the eighth evolution was at the center; it does not help us. Because we are not able to get 2 points because it reduces to half, okay. But we were able to do only 7 evaluations, correct.

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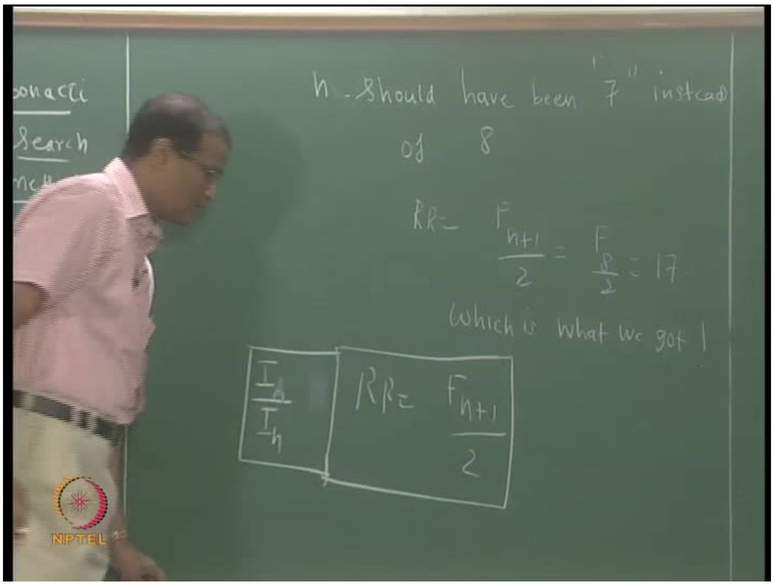

Therefore, so we have to revisit this. We have to pose the problem as n should have been, correct. This is what we got. 1 by F n theoretically there is nothing wrong in that formula but cannot be reached unless you do some small tricks; I mean you change that point slightly left of the center or something like that but then it does not become the Fibonacci. The basic point is the last two evaluations you are not able to do because you hit the center.

So, now we will say that the RR is equal to F of n plus 1 by 2. So, RR is basically I naught by I n, right. So, I say I want an accuracy of 0.18 meter. We could have started out with this problem as follows; minimize surface area of the cylindrical storage water heater with an initial interval of uncertainty of 0.5 to 3.5 meter, I want to get a final uncertainty of 0.17 or 0.18 meters. So, 3 divided by 0.18 reduction ratio is 17. If the reduction ratio is 17, F of n plus 1 must be equal to 34. So, 34 correspond to the eighth number in the Fibonacci series; therefore F of n, so therefore n should have been 7 and you will start with 7 and you will do 7 evaluations.

Anyway, 7 evolutions you do not have to worry about how many evaluations you are doing. Anyway at the end of 7 evaluations, it will come to a halt. This is the way you work out a problem, right. You have to work backwards from the final desired level of accuracy you need on your solution. Is that fine?

So, now, you will go to yet another powerful method in our quest for developing efficient single variable searches. So, thus far from whatever we have seen, there are some basic cardinal principles, right. The 2 points must be close to the center of the interval and the 2 points must be equally spread from the both sides of the interval and common sense tells us that the 2 points must be at the same distance from the ends of the interval because a priori you do not know whether the left hand side will be cut or the right hand side will be cut. Any disturbance to this, that is the distance of the left side is not equal to the distance on the right side, sometimes it may be advantageous, sometimes it may be terrible, okay.

Therefore, in all these methods including the Fibonacci method, it is at the same distance from both the ends, right. So, the Fibonacci method actually obeys all this. Always we use a 2 point test; it is symmetrically placed about the center, it is symmetrically placed about the 2 ends and we get a good reduction every time but the only thing is there are two or three disadvantages as far as the Fibonacci search method is concerned. Basically because n is equal to 7 or n is equal to 8 is basically for trivial problems which is solved in the class. When you want to solve very complex optimization problem hundred variable problem, when you desire a high level of accuracy, n may be 200, 300, 400, whatever.

So, you have to calculate and store the Fibonacci numbers first in your program, then you have to recall those Fibonacci numbers every time, okay. That is one thing and the reduction in the interval of uncertainty is not the same in all the iteration because 13 by 21 is different from 8 by 13 is different from 3 by 5 and so on. So, can we think of some method which enjoys all the advantages of the Fibonacci search method; at the same time we are having the same ratio of we have the same interval reduction. That is when you go from I naught to I 1, whatever be the percentage of reduction in the interval of uncertainty when you proceed from I naught to I 1 is exactly the same as what you would get from I 1 to I 2, I 2 to I 3 and so on. Can we come out with such a thing? Let us see.

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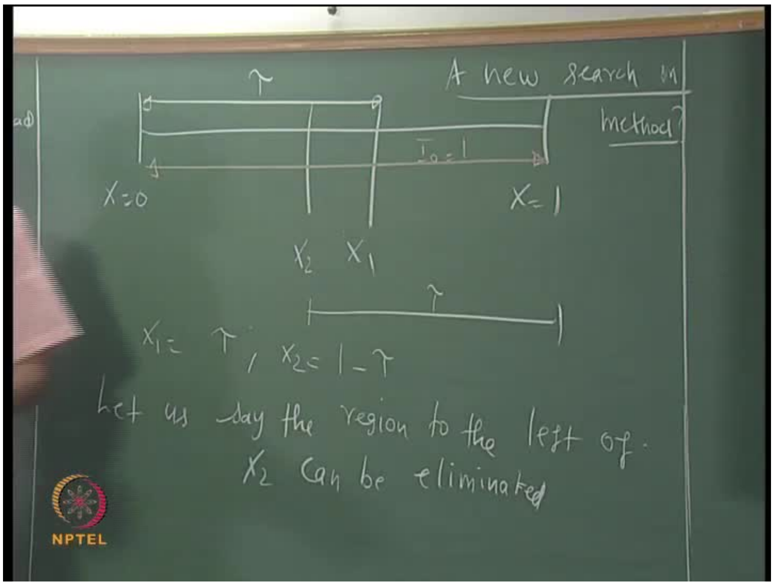

Can we try; can we try for a new search method? So, to make the problem tractable instead of 0.5 meter to 3.5 meter and a and b I can have it as 0 to 1; it is just for demonstration. So, I say that X equal to 0, X equal to 1 in appropriate unit meter, centimeter, kilometer, whatever. I take 2 points. I take 2 points X 1 and X 2 and I say that these are at distance of tau from both the ends. So, X 1 will be tau, X 2 is equal to?

Student: 1 minus tau.

1 minus tau that is it; we are still talking about a unimodal function based on the 2 point test. One portion of the interval can be eliminated. Let us say the region to the left of X 2 can be eliminated. Let us say, so what is my I naught? I naught is 1, okay. What is I 1 now? Good, I 1 is tau.

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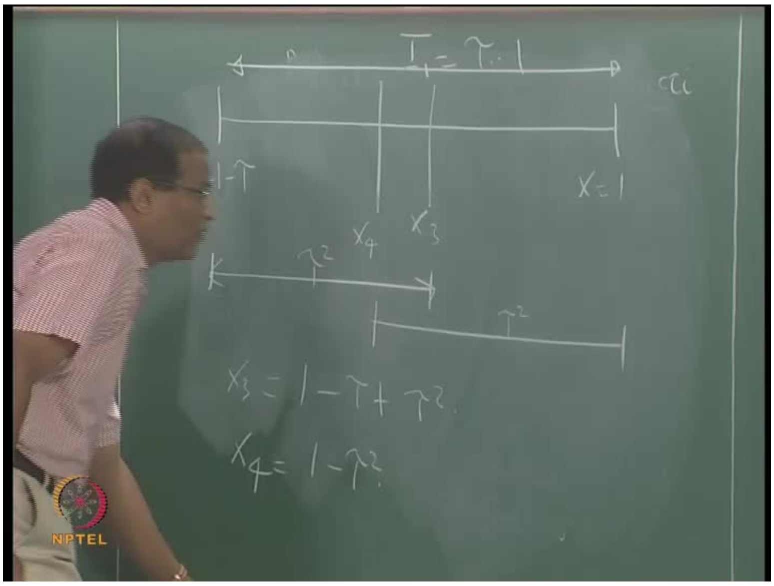

So, we are starting from X 2, right. We are starting from X 2.

Student: Yes sir.

Okay, this is X equal to 1, 2 points X 3, X 4. Sorry?

Student: It has to be 1 minus T.

Yes, correct. Okay two points X 3 and X 4. What is our goal, what is our objective? Our objective is to come out with an algorithm such that if I 1 is equal to tau into 1, I 2 is equal to tau into I 1 or it should be tau squared into 1; you are getting that point? We are trying to seek such a. So, this will be tau squared, this will be tau squared, correct. I do not know what the tau is; I am just trying to see if there is one tau.

Now X 3 equal to 1 minus tau plus tau square. Why is everybody blinking? What is X 3? 1 minus tau plus tau squared. X 4 is 1 minus tau square, nothing great so far. So, we started out with 2 points X 1, X 2 in the interval 0 to 1 and we are saying that in the first iteration the percentage of reduction is tau, then we are saying that this percentage reduction has to be maintained. So, in the second iteration we take the 2 new points X 3 and X 4 but now we want to retain all the advantages of the Fibonacci search method. Therefore, I want X 1 to be X 4.

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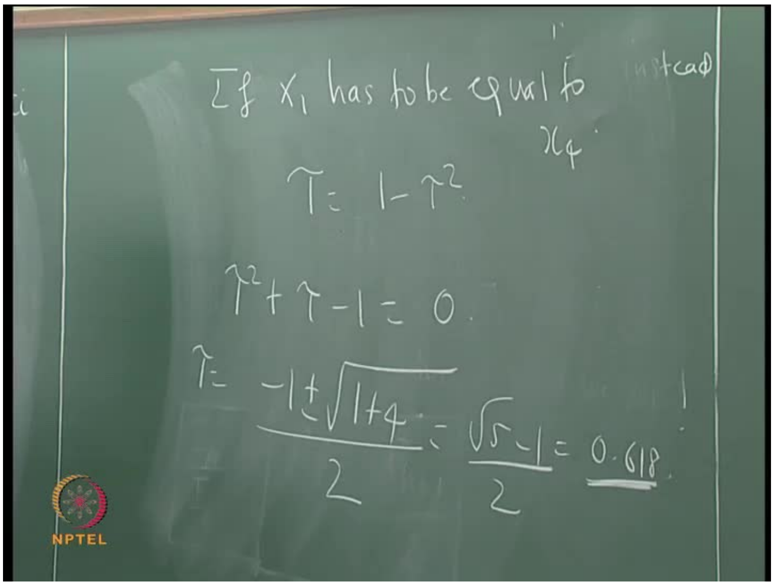

If X 1 has to be equal to X 4, so tau is equal to. What is this? 0.618. Yeah, what is the root 5? 2.236, it is 0.618, is this is clear so far, okay? Yeah.

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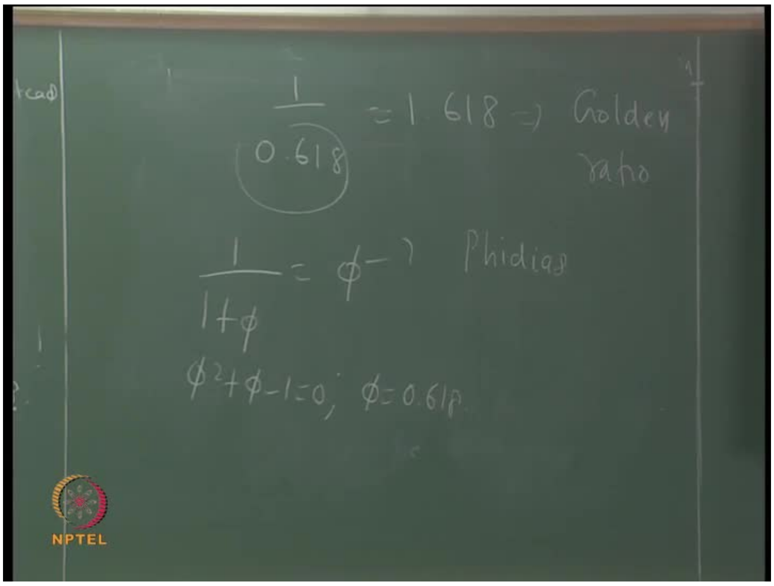

What is so great about 0.618? Check, check, check, that is why it is satisfying our property X 1 is equal to X 4. So, 0.618 or 1.618 is called the golden ratio; either this or this is called the golden ratio or the golden number. So, if phi is your golden ratio what is the property it satisfies? 1 by 1 plus phi is equal to phi. Again if we solve phi square minus phi minus 1 equal to 0, phi is equal to 0.618; that is the power of the golden ratio.

So, now why instead of struggling with Fibonacci every time you multiply 0.618 and 38.2 percent of the interval will be gone; that is the golden section search. Take 2 new points at a distance of 0.61 times I naught from both the ends. Next it will be 0.618 times I 1 from both the ends. So, you will get a very good reduction ratio. Now are there other properties which this fellow has? This phi itself, why this name phi itself? It was first used by some Greek guy, right. So, he took the first 3 letters of his, okay.

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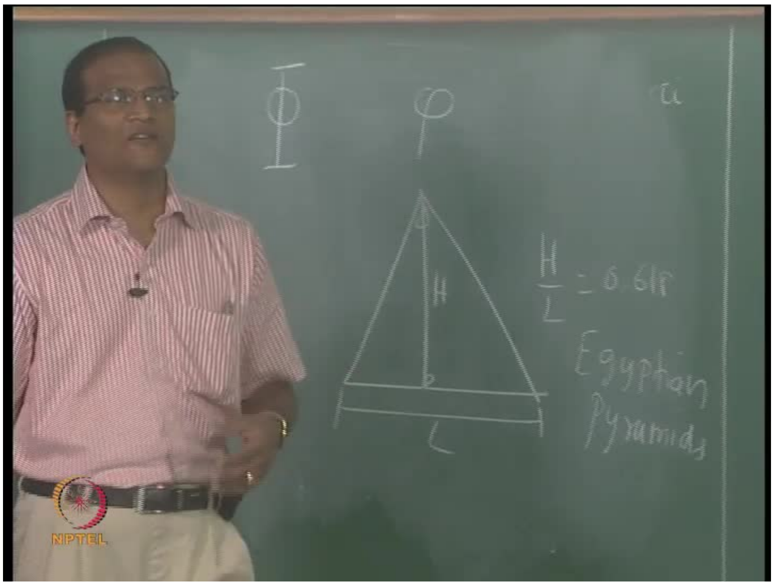

So, in fact if it is written like this, this by this is the golden ratio upper case, lower case you write like this that is also in golden ratio. So, the ratio of Egyptian pyramids is in the ratio of 0.618. There are also claims Leonardo da Vinci’s and all that, our beautiful face the ratio of the height to the width of the face must be 0.618. So, there are so many other things about it but the Holy cross of the Christians is not in 1.6. Does anybody have a cross now? I do not think it satisfies 1.618. I checked it once. So, it is some. I got it from some student; it does not appear to be and the A4 size is also not 1.6 and 297 by 210, what is the ratio A 4 size paper?

Student: 1.41.

1.4 is more aesthetic man. There should be something; everything is dry math solving equations minus b plus or minus what is the use? Whatever is good to the eyes. 1.4, the A 4 size is 1, why should it be A 4 size? Why cannot we have like this? So, for most people it is comfortable it seems. That is not unfortunately golden ratio that is silver ratio or something 1.4, okay.

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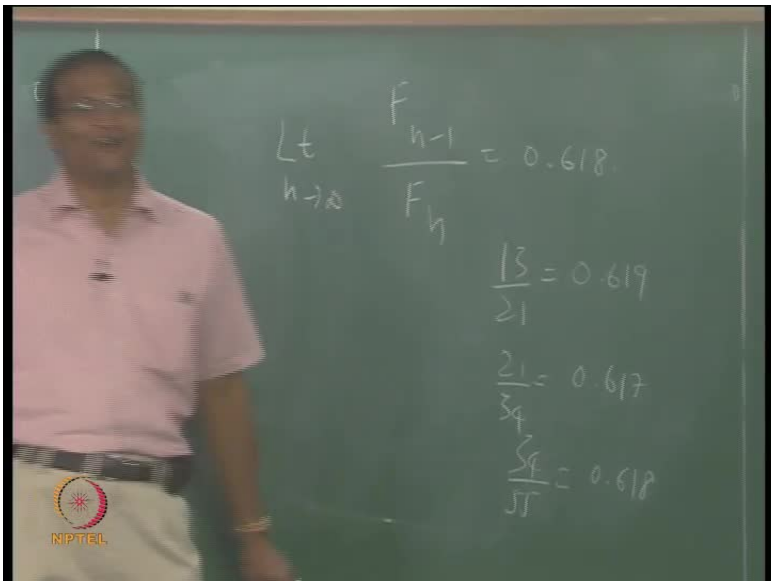

So, now what is the connection with the Fibonacci series? Please check this also. As you approach larger and larger numbers, the Fibonacci sequence introduces asymptotically to the limit 0.618. So, all these are interconnected. So, if you want to enjoy all the benefits of the Fibonacci search but you do not want to calculate and store the numbers, you do not want to have unequal intervals. So, use the golden section search, alright fine. This is true, 21 by 34 itself is 0.618, right. What is 13 by 21?

Student: 0.619.

21 by 34?

Student: 0.617.

Okay, 34 by 55?

Student: 0.618.

We will stop here, okay. And we are having trouble when we reach that 1 by 2; that is where we struggle, right, the algorithm struggles, fine.

Now we will revisit the cylindrical solar water heater storage problem but now with the golden ratio, we can do it very fast. You do not have to remember the Fibonacci number, okay. Now problem number?

Student: 35.

And then hopefully we will give a decent burial to the cylindrical storage because our single variables our quest for single variable search ends here and from the next class onwards we will do two variable search and then we will start a new problem, okay.

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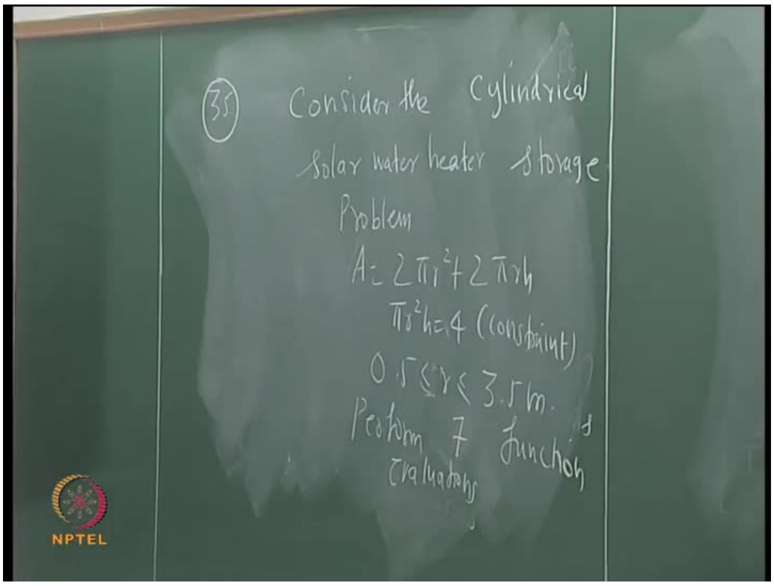

Problem number 34, 35, what is it?

Student: 35.

So, perform seven functional evaluations so that it is the same number as what we did for the Fibonacci search. Your problem reads like this, perform 7 functional evaluation and report the final interval of uncertainty. So, this is the problem.

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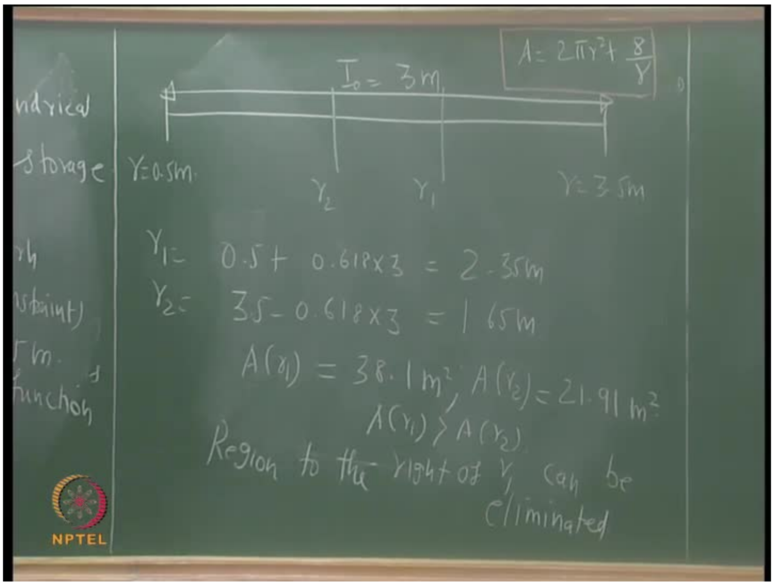

So, A equals 2 pi r squared plus 8 by r. We will start. I think the good news is most of these r 1, r 2, r 3, r 4 will be similar to what you got in yesterday's class. So, you can really do the problem pretty fast. r 1 is going to be. So r 1, 0.618 times 3. What is 3 is I naught. I naught is 3.5 meter minus 0.5 meter. 2.35, yesterday we got 2.35 know we will use that 2.35. Yeah, if you do not like it 2.44 you can use, what is the other one?

Student: 1.656.

1.65.

Student: Yeah.

If you have yesterday's notes, you already have A of r 1 and A of r 2. A of r 1 is?

Student: 31.3.

31.3, is it?

Student: A 2 was 38.1.

Correct. 38.1 meter square. A of r 2?

Student: 21.91.

21.91, A of r 1 is greater than A of r 2 and you are looking at minimization. So, the region to the right of r 1 can be eliminated, correct.

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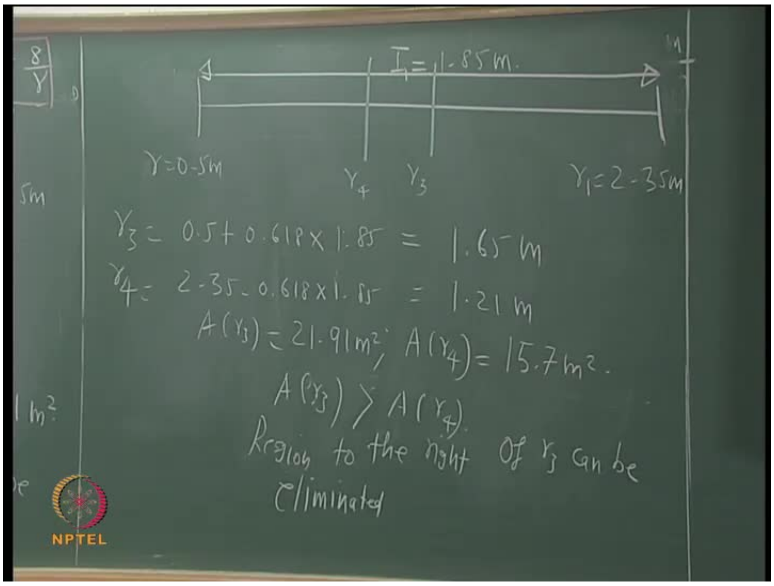

So, I1 is?

Student: 1.85.

Correct. How much is the interval reducing?

Student: 1.6 percent.

Interval is reducing by 38.2 percent, okay. From 3 meters it became, correct. So, 61.8 percent of the original interval remains. Now r 3, r 4, r 3 equal to how much is it Deepak?

Student: 1.645.

1.64 is it? So, we cannot use 1 point we will use the 1.65. What do you say Samaj, shall we use 1.65?

Student: As of now you can use it.

As of now we will use; towards the end some small problem will be there that we can manage, okay. r 4, that will be new.

Student: 1.21.

Yeah, that will be new.

Student: This also you can calculate as of now later.

Yesterday we got 1.20?

Student: Yeah. This is also 2067.

Yeah.

Student: And yesterday it was very close. Yeah, then we have to use 1.20 value itself.

You can use the 1.20 value itself, does not matter. So, A of r 3?

Student: 21.91.

21.91. A of r 4 think it will be lower.

Student: 15.7.

Fine, region to the right of r 3 can be eliminated. So, how many evaluations are over now? No, no, how many times you evaluated the function?

Student: 4 times.

Not 4 times.

Student: 3 times.

3 times, we did only 3 times. The fourth one was buy 3, get 1 free, correct. So offer, it is an offer. So, the fourth one is an offer. Now we will go to next iteration.

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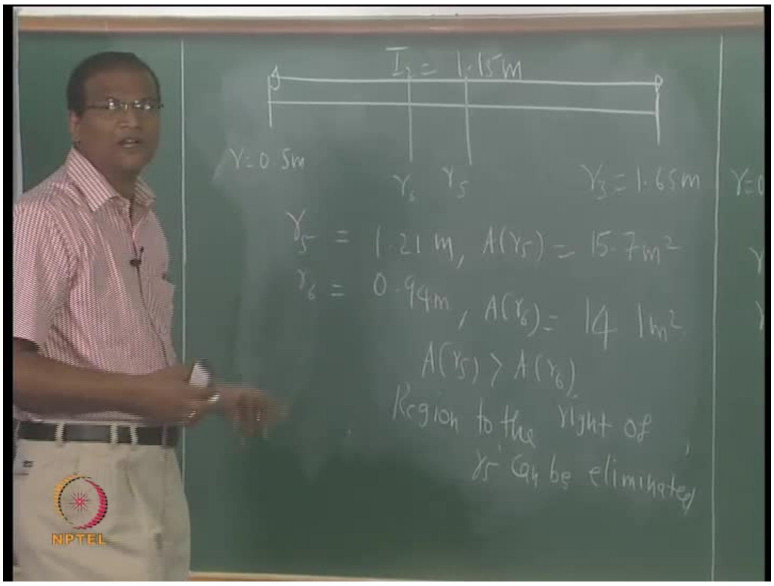

r equal to, it is not good. What is this now? r 3. r 3 know? I 2, green chalk, it is much faster than Fibonacci; it is much easier to work out, right. It is pretty rapid. Yeah, now you have to tell me, r 5 equal to?

Student: 1.2, 1.208.

r 5 is?

Student: 1.208.

1.20. r 6?

Student: 0.94.

Okay, A r 5? 15.7. r 6 also we have from yesterday’s notes.

Student: 14.1.

Okay, again we are eliminating to the right, is it? Yeah, so be patient. I asked you to do 7 evaluations. We have completed 4; it has just taken 10 minutes, not even taken 10 minutes, it has taken 8 minutes. So, if I asked you to do 7 or 8 evaluations, I expect you to finish between 12 and 15 minutes.

Student: Sir, are you giving the same problem?

No, no, not the same problem. Anyway, now you can guess exhaustive search I will not ask, right. So, it could be dichotomous, Fibonacci or golden section, just practice that is it. So, you have to bat on Monday. So, you will have to even if you know the algorithm, you should not make mistakes. By the way it is open notes as before. So, please come by 8.50; sorry 7.50, 8.50 everything is over, the game is over. So, please come by 7.50 and do not get involved in what is known as the run around scheme. That is you do not bring the id card, then you have to go and meet the HOD. That is called the run around scheme or the heat exchanger. So, do not get involved in all that; do not bring photocopied notes, okay. Bring your id card, calculators the simple rule. If you follow these simple rules you can write the quiz very peacefully. So, get hold of yourself, organize yourself; you are not in the first year, you have done so many times, right.

If you ensure that you are not indifferent, then you will do all this, okay. So, anyway my papers are all very computation intensive. So, be ready to receive the paper by 7.50. Whenever they give, just take the paper and start working out; 50 minutes is not sufficient, okay. If somebody has missed some class, do not bring photocopied notes and then somebody invigilator this thing and, okay. Please copy the notes today and tomorrow; that is a good way of learning the lessons also. If you do not have time to do that at least copy the algorithm; copy the algorithm and you should know where the algorithm lies in your notebook. If you start searching, if you try to be like a great scientist and oh Fibonacci; the notes is just to, suppose what is the 8th number or the 9th number it is only for that. What is Fibonacci sequence if you try to discover the method in that 8’o clock, it will not work.

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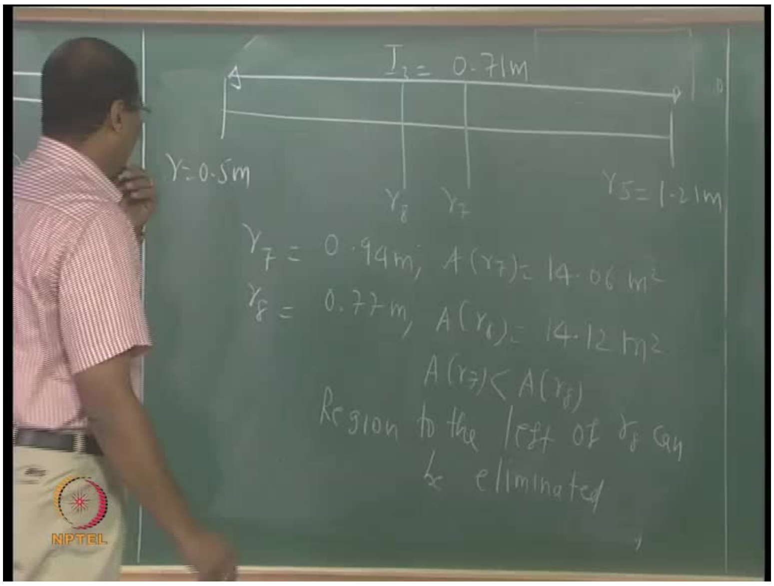

Now region to the right of r 5 can be eliminated. r 5 is what 1.21. So, r 7?

Student: 0.94.

0.94. Good.

Student: 0.77.

0.77. We are reaching the end of our Fibonacci search, right. A of r 7?

Student: 14.13.

13. Make it more than that man.

Student: 14.12.

14.12.

Student: Yeah, actually that is 14.06.

Then we will say 14.06. Now you have to get?

Student: 14.12.

So, A r 7 is less than A r 8 for a change. So, the region to the left of r 7 can be eliminated. No the region to the left of?

Student: r 8.

r 8, correct.

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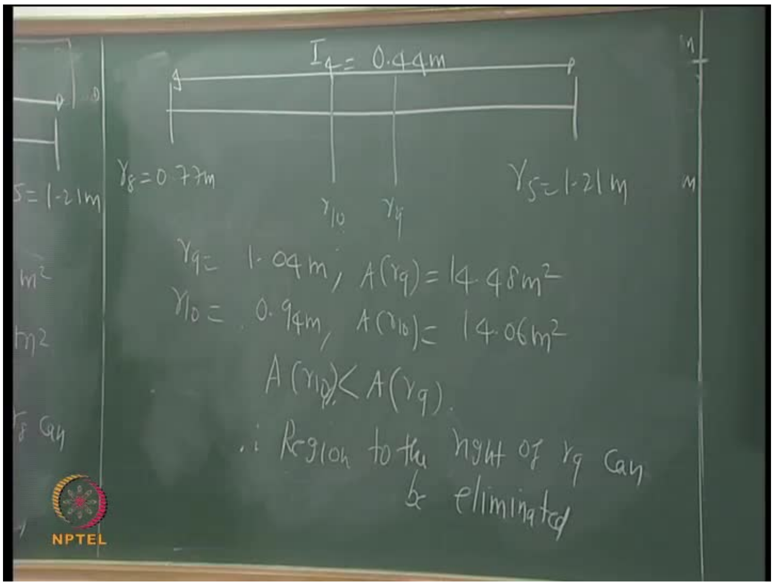

r 8 is what, 0.77; this is r 5, 0.44. Yeah, this is where we stopped in the Fibonacci method. Now you can keep on going in the golden section search. No, r 10 r 11; yeah, now tell me r 10.

Student: r 9, r 10.

9 and 10 sorry, yeah, r 9 1.04; r 10 is the same 0.94.

Student: 14.48.

Is everybody through with this? So, A r 10 already there. So, what is the story now? A r 10. Have we completed 7 now?

Student: No sir till 6.

So, we did not complete 7; therefore, region to the left of A r 10, A r 10 is less than, right, correct, therefore, the region to the right of r 9.

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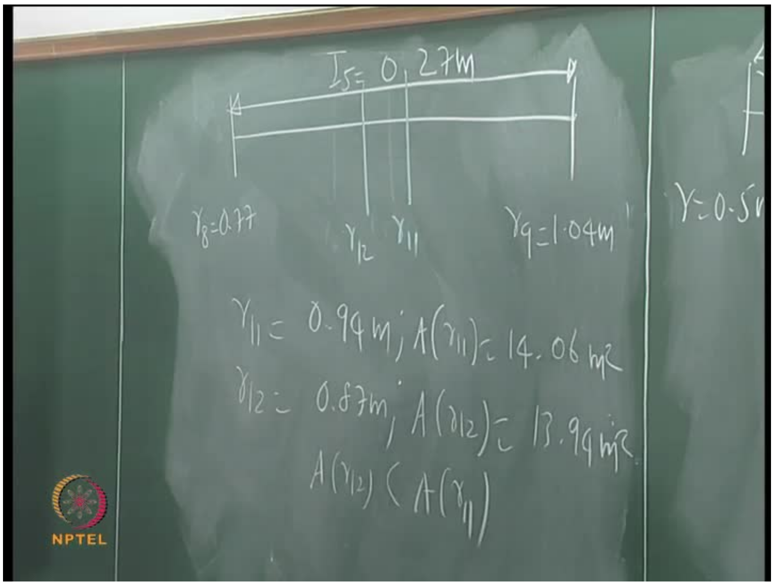

r 9 is what? Region to the right of r 9 know. r 9 is, what is it? r 8 0.77 is it. It is pretty slow. No, I should not say that.

So, I 5?

Student: 0.27. Yeah, I think we have come to the end now.

Student: 0.94.

Student: 14.06, 13.94.

So, A r 12 is less than; therefore, region to the right of r 11 can be eliminated.

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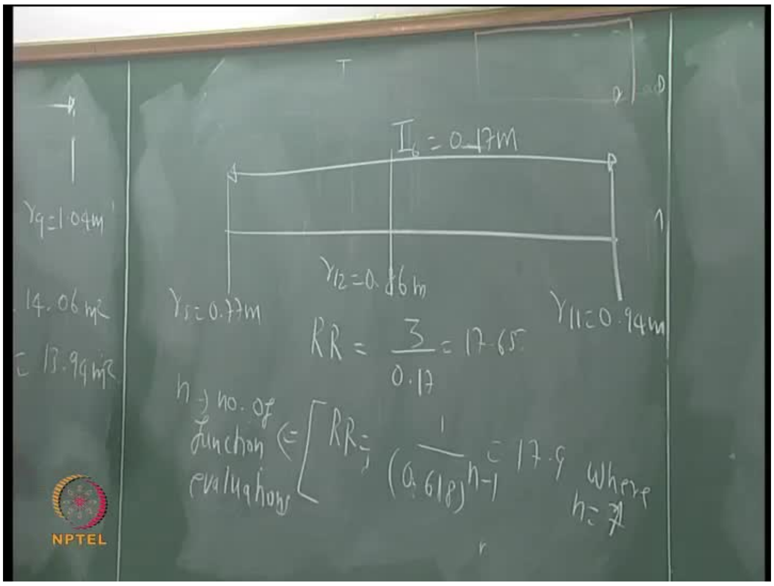

And the other one is 0.94, is it. It is the same thing what we get in Fibonacci, right. It should be the same; the reduction ratio will be the same. You did the same number of evaluations; only thing the advantage is you do not have to remember the Fibonacci number and 0.618 and, fine. So, this is the story now. This is the final interval of uncertainty. What is RR? RR is where n is the number of evaluation; just check it. It would not be n because the first time we do 2, right. The first time we do 2. Therefore, it has to be n-1. What is the value for this?

Student: 17.9.

17.9. Yeah is equal to 17.9, n is the number of evaluations, okay.

So, we have learnt quite a few powerful techniques for performing single variable searches. These are all only first order accurate in the sense that we do not exploit any information on the derivatives or the higher order derivatives or we do not even look at the nature of the variation of the Y between two points. We are just seeing whether numerically it is greater than or less than; that is all. Obviously, optimization especially some mathematicians would not have left it like this. Therefore, other powerful methods it will exploit all this. Because of lack of time, I cannot discuss on many of this but at least one I would like to give you a sneak peak of how powerful we can get by employing some additional from slightly superior or more advanced techniques to this problem. Let us stay with this r 8 and r 11. Let us see whether we can do something to improve this further.

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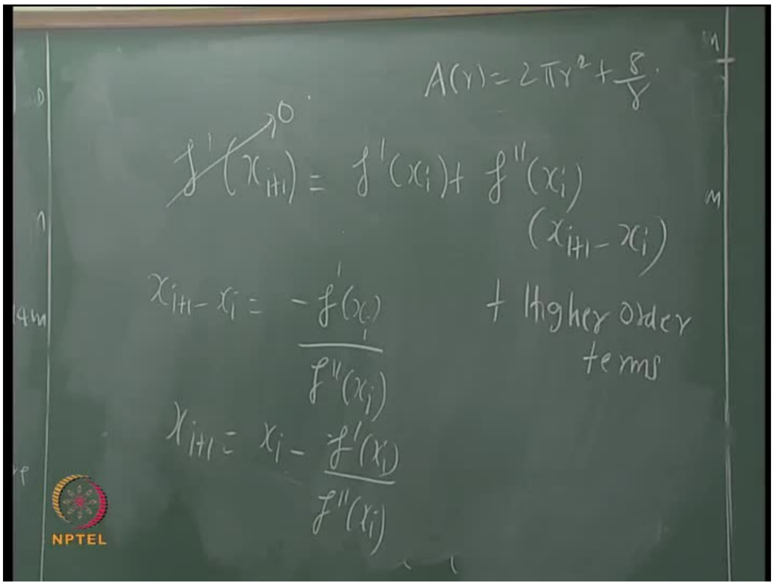

So, A of r. Suppose, I want to expand f dash of X around X of i plus 1, I expand it as. This is perfectly legal to do it using the Taylor series expansion. I am not expanding f of X, I am expanding f dash of X. f dash of X around X of I, in the next iteration f dash of X plus i is equal to f dash of X i plus f double dash into delta X plus higher order terms and now the goal is to make f dash of X zero because we are looking at an optimization problem, right. Therefore, if f dash of X is made zero, now you can see that X of i plus 1 minus X of i is equal to minus f dash of X i divided by f double dash of X of i

Therefore X of i plus 1 is X i minus. This is the Newton Raphson method for an optimization problem. This is quite different from the Newton Raphson method for the systems simulation where we have X of i plus 1 is equal to X of i minus f of X by f dash of X. Here we are having f dash of X divided by f double dash of X because we are trying to make f dash of X stationary and not f of X stationary; is it clear, okay. So, this is the Newton-Raphson method for optimization.

So, it is exploiting information in the first derivative, second derivative. So, it is demonstrably superior, we will demonstrate it. It is far superior compared to the other search techniques but what is the pain? First it should be possible for you to evaluate f dash of X and f double dash of X. Suppose you are solving a CFD problem or you are solving the vector Helmholtz equation or you are trying to determine stress and all this, getting the value of Y is itself difficult, outside of that you want to get f dash f double dash.

So, calculus is not the only route to solving optimization problem; this may look very attractive but it is simply not possible to get the higher order derivatives when you are working with complicated optimization problems where each function evaluation itself is so difficult. You may argue I can get all these derivatives numerically. All that is fine but when golden section search and other things are available or I will go for a neural network. If time permits I will talk about neural network, right. So, let us take this problem we still have 5 minutes. So, let us take r 12. r 12 should be what?

Student: 0.86.

0.86.

Let us take r 12 as 0.86. Let us apply the Newton-Raphson method.

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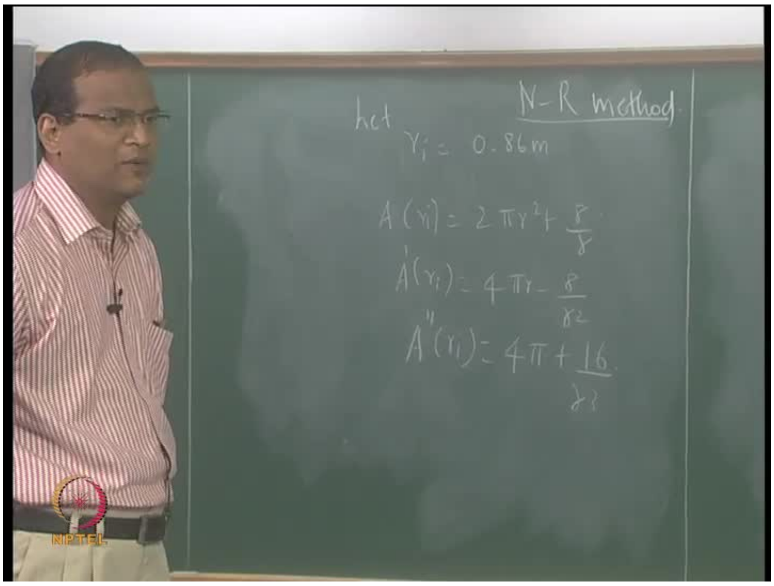

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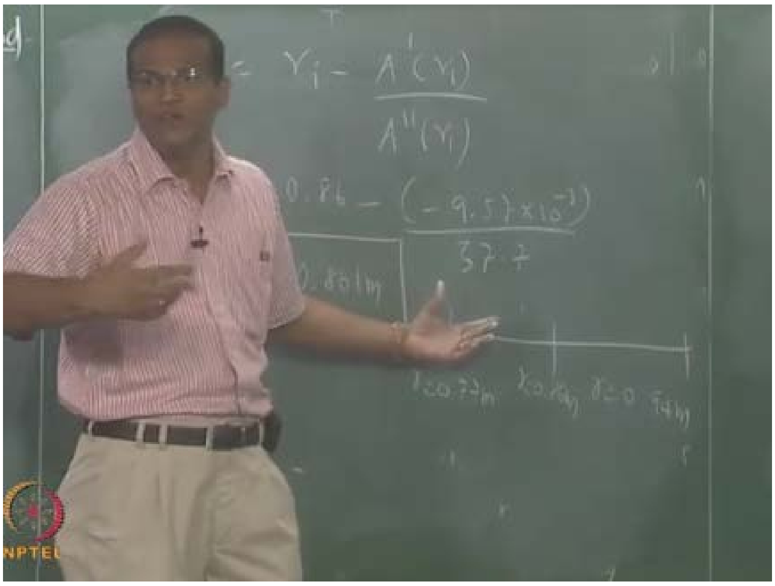

Correct? Can you give me the two values? I have already given the derivatives. At 0.86, it will be 0; I am already there, I am home but I just want to demonstrate.

Student: Which is 0.861?

Which is 0.8? No, no, you tell me the two values Sampath, numerator and denominator. Abishek, you have the values? You gave up? Yeah, give me the two values one of you at least.

Student: Numerator is minus 9.6 into 10 power minus 3.

Denominator is positive, I know that. Cupped? That is a change.

Student: 4.4 into 10 to the power of minus 4.

4.4 into 10 to the power of minus 4 which one, the ratio. No, no, no, tell me separately.

Student: Minus 9.57.

That is correct?

Student: Yes sir.

Minus 9.57.

Student: 10 power minus 3.

Divided by?

Student: 37.7.

Already you know that A dash of r has taken a very small value; therefore, you hit A dash of r is already becoming 0. Therefore, that is the optimum. So, r i plus 1 is how much? 0.861 or what is it, okay? So, I have taught you what is called hybrid optimization technique. You start with the search technique and find Newton Raphson method is also a search technique. But towards the end you are using a search technique which exploits the power of the first derivative and the second derivative in order to narrow down the solution.

There are other possible routes to this problem. You have 0.77 and 0.94; you have one more point 0.86. Go ahead and fit a Lagrange interpolation formula between the three or go ahead and have a Newton’s divided difference formula. Y is equal to A naught plus A1 into X minus X naught plus A 2 in to X minus X naught and X minus X1. Get all A naught, A 1, A 2, A 3 from these three values and then get the derivative of that function with respect to zero and make it stationary and find out where it becomes zero.

So, so many possibilities are there, okay. This is only if asked. If or you want to get a very sophisticated, you want to get a very narrow interval, then you can do this. The assumption behind this is when these points are sufficiently close if you approximate it by second order or second degree or third degree polynomial not much error is incurred. But this cannot be done from 0.5 to 3.5 meters or if you are searching from 1 meter to 200 meter, you cannot take three points and put a parabola; it would not work.

But when you sufficiently narrow down the interval and you are pretty sure that the interval is already very tight, then you can do all these tricks. So, these are all higher order methods. Because you are exploiting you are looking at the first derivative, and in fact you are looking at even the second derivative. So, this puts an end to a study of search techniques in single variable. In the next class we will look at multivariable search techniques first unconstrained and then constrained.