Design and Optimization of Energy Systems

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Lecture No. # 31
Fibonacci search method

So, I am continuing with our treatment of single variable searches, we will now look at the Fibonacci search method. It basically works on the Fibonacci series.

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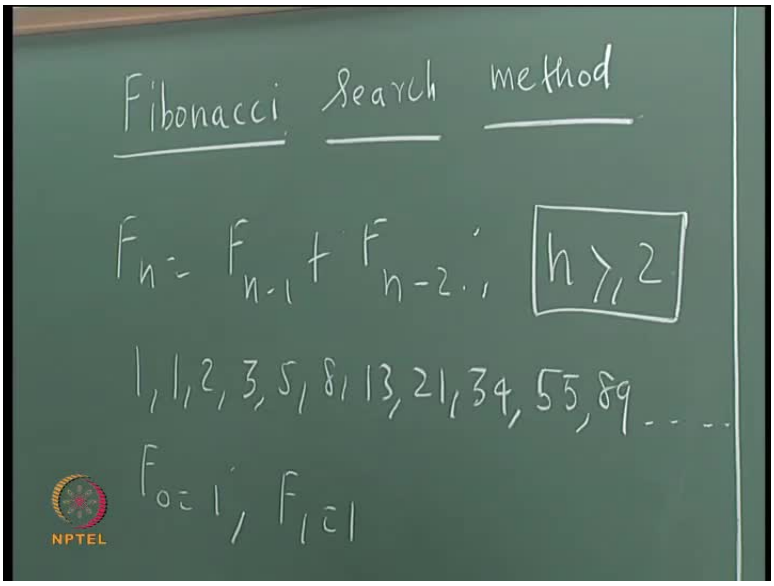

If you recall in the last class, we wrote the algorithm for any number F of n in the Fibonacci series, the first number is 1, right. So, we use the Fibonacci series in order to divide the interval and choose two points. Please remember that always in a single variable search, we use a 2 point test. We find out whether Y of X 1 is less than or greater than Y of X 2 and accordingly a portion of the interval is eliminated. So, Fibonacci search method is yet another elimination method where a portion of the interval is eliminated; it is called an elimination method or a region elimination method.

Please recall that in all these elimination methods we are just looking at whether Y of X 1 is less than Y of X 2 or Y of X 1 is greater than Y of X 2, regardless of how much these values have changed with respect to the previous iteration. Obviously, you can conjure up with visions of coming out with some algorithm which can do better than this; that is if you take into account the nature of variation of the objective function also, how these vary, then it is possible to come out with better single variable search algorithms, but that is not the intent of the treatment at least in today and tomorrow's class. Without having to worry about between two iterations how Y of X changes, we are always looking at the two pointers and seeing whether Y of X 1 is greater than or less than Y of X 2 and remaining certain position of the interval.

So, it comes with certain efficiency and it comes with a certain price, right. The price is the computational cost. Now what is the strategy? And I think in the last class I told you I took an interval where X varies from 0 to 8 and we divided the interval such that X 1 equal to 5 and X 2 equal to 3 or whatever, and then we subdivided and we figured out that one of the 2 points in every new iteration is already an old point in the sense that it was a point in the previous iteration. So, in the execution of the Fibonacci search, every time you have a 2 point evaluation there is only one new point, the second point is already an old point. Therefore, even though the interval is not reduced by 50 percent, because you are evaluating only one at a time asymptotically it has a convergence rate which is far superior compared to the dichotomous search method.

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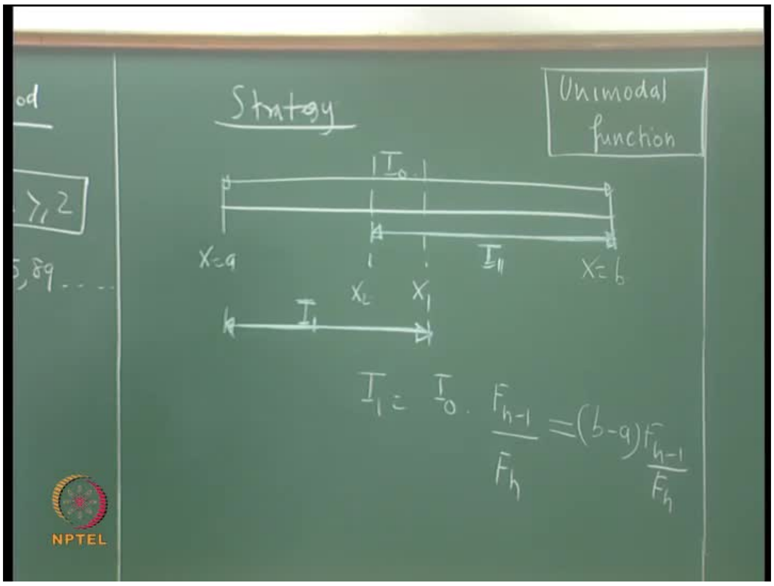

Now what is the strategy? So, needless to say, we are considering a unimodal function. As usual we start with an original interval of uncertainty of I naught, X equal to a, X equal to b which use two points I 1 and this is also I 1. Correct, should I use different chalk or is it fine? I meant different colored chalk differently colored chalk.

This is a simple 2 point test where we are having 2 points which are symmetric about the end points. Now what is this I 1? So, there should be a strategy for getting I 1 from I naught I 1 is equal to I naught by 2 approximately if epsilon tends to 0; that is the dichotomous search. But what is the relationship between I 1 and I naught; that is the algorithm; For the Fibonacci search the algorithm is. So, you choose a particular n. How will you choose a particular n? Finally, we will have to figure out what is the reduction ratio of the algorithm? For the reduction ratio of the algorithm which relates the I naught to I n you know the original I naught.

So, suppose we specify the final accuracy level of the interval of uncertainty you will also know what is I n. So, knowing I naught and I n you know the reduction ratio. The reduction ratio should be a function of n or should be a function of F of n. So, you will be able to get F of n or n. If you are able to get only F of n in the first place from F of n you will get back your n; you start backwards and work like this because in a computer when you are writing a computer program you will not do like. So, purposes of exam or for purposes of class I say 6 evaluations, 8 evaluations, 10 evaluations, we do not do like that in a computer program. You do not say do I equal to 1 to 100 iterations and stop a program that is for starters; to check whether your program is working, who has told you that within 100 iterations the solution would have converged.

You will see whether it works for 50, 100, 200, whether there is math overflow, blah, blah, then you put your convergence criteria. Like that when I say do 6 evaluations, 8 evaluations it is for demonstration purpose in the class. When you actually write a program you will work backwards. For this system I want an accuracy of this much. Then you start backwards and find out and then you will find out how many evaluations are required. I 1 is equal to I naught, so this is equal to. So, it is lighting up, so I naught b minus a.

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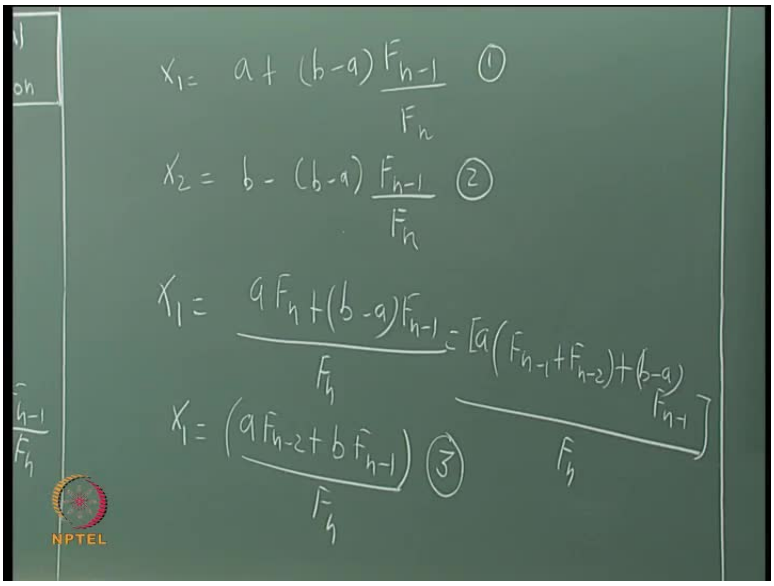

So, can we write X 1 and X 2? X 1 is equal to a plus with dynamical adjustment of to find out whether Renu, Rohith, and all these people are. This is the algorithm devised by Web studio. So, X 1 is a plus b minus a into F of n minus 1 divided by F of n. X 2 equal to b minus, correct. Vikram is it alright? So, early morning you should be up on your feet; immediately you should not wait for me, oh, if F of n is on denominator you have to cross multiply a into F of n.

F of n can be written as F of n minus 1 plus n minus 2, b minus a you can subtract certain things and simplify. Alright, you can do that. So, X 1 equal to a. Can you tell me, can you simplify this? a F n minus 1 goes off, right. So, X 1 equal to a f n minus 2 plus b; you are just exploiting the property of the Fibonacci sequence, let us call it 3. Similarly, is this step clear? I rewrote F n as F n minus 1 plus F n minus 2 that is the property of the Fibonacci sequence of the particular number F n in the sequence.

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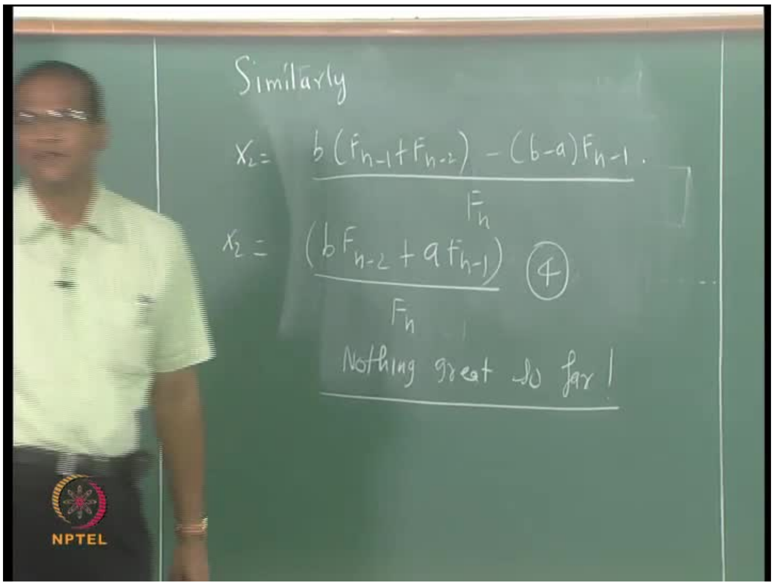

Similarly, minus. So, fortunately X 2 is b F n minus 2 plus a F n minus 1; otherwise, you have to go home. If X 1 is equal to X 2 there is no 2 point test. So, if you get the same expression for X 1 and X 2, go back and check your algebra. This may happen eventually when the 2 points coincide, then you get r g. So, finally when it reaches there is only 1 point; that is ok but not when n is sufficiently large. You should have X 1 should be different from X 2.

So, X 1 and X 2; then you do the 2 point test depending upon whether the function has to be minimized or maximized, you will eliminate the region to the left of X 2 or the region to the right of X 1, right nothing great so far. It is like any other algorithm, right. No great shakes till now, the beauty comes in the next iteration. Now, for purposes of demonstration, we have to take a call on whether region to the left of X 2 or the right of X 1, I always like the left hand side. So, we will eliminate we will assume some Y such that the region to the left of X 2 is eliminated and we will go to the next round; that is we will go to I 2. So, shall we retain this fellow? Shall we go to the other portion of the board?

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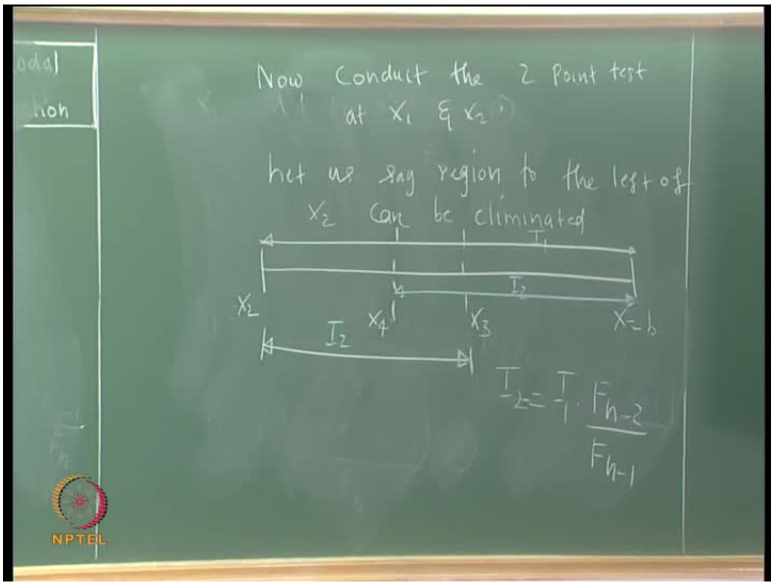

Now conduct the 2 point test. Let us say the region to the left of X 2 can be eliminated. Now we are left with X 2 X equal to b, right. Correct, what is this interval now? I 1, correct, this fellow is I 1. The new interval of uncertainty is I 1. Now we chose two points X 3, X 4 such that they are at a distance of I 2 from the two new ends. What is our friend I 2 himself. I 2 is I 1 into, right. You remember in the last class we took 5 by 8, the next one was 3 by 5, next will be 2 by 3; you have to go backwards. So, now we do not have to leave him like that, we can simplify further.

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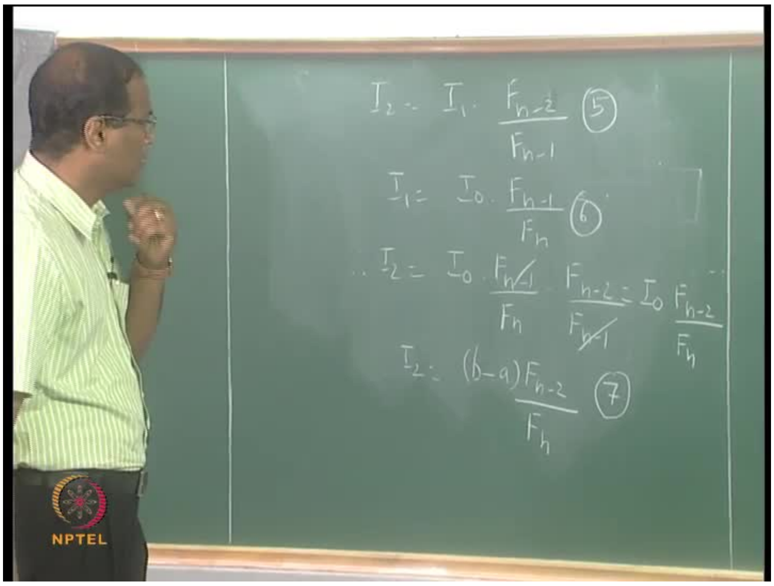

So, I2 is it correct; alright. Now so I2, so what is the equation number we stopped with? Four; so, we will make this 5, 6, 7. Now the time has come to erase this portion of the board I think.

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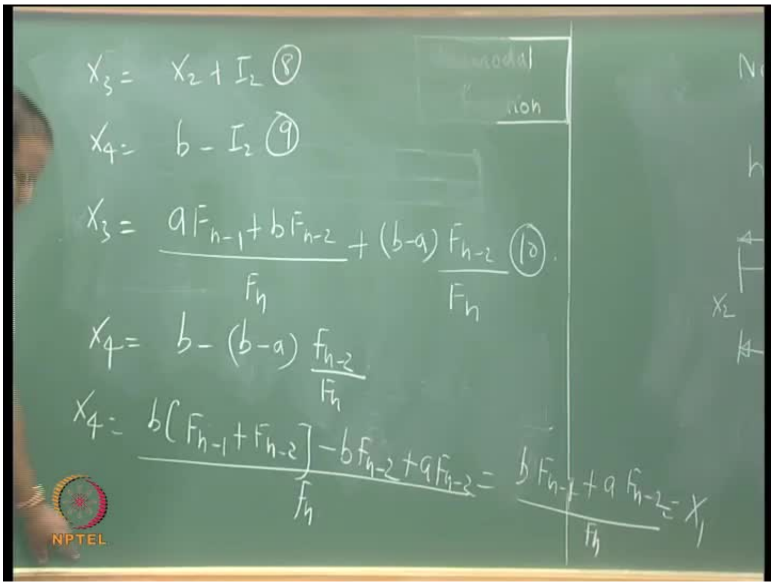

So, X3 equal to X2 plus I2, X4 equal to b minus I2. Please watch carefully. Is it fine; Vivek is it fine? So, X3 is the point from this end is the original point X2 plus I2, X4 is at the same distance I2 but from the other side. It is b minus, okay. Now this is equation number 8 and 9. All this labor is to prove that one of the 2 points is already there. I already demonstrated with numerical example, but I wanted to do that with algebra so that the proof is clean. So, it is good to have this in the notes.

It is very difficult to find it in text books. So, people will say it can be some asymptotic convergence, right. X4 is X3 is equal to X2 plus I2. Now let us substitute for X3. What is X2? What is X2?

Student: a.

No, no we simplified know a F n minus 1

Student: Plus b F n.

Plus b minus a, so something gets cancelled. No hope, this is not the point. I am interested in X4, I do not like him. So, I just leave him. So, I will go to X4. Anyway it is a new point, right. X4 is b minus I2; X4 is fun, so b minus, right. Now I can substitute again for F n. What is happening now? X4 is b F n minus 2 is equal to X1.

Student: b F n minus 2.

Plus a F n minus 2. Yeah b F n minus 2 gets cancelled. So, this is X1, alright.

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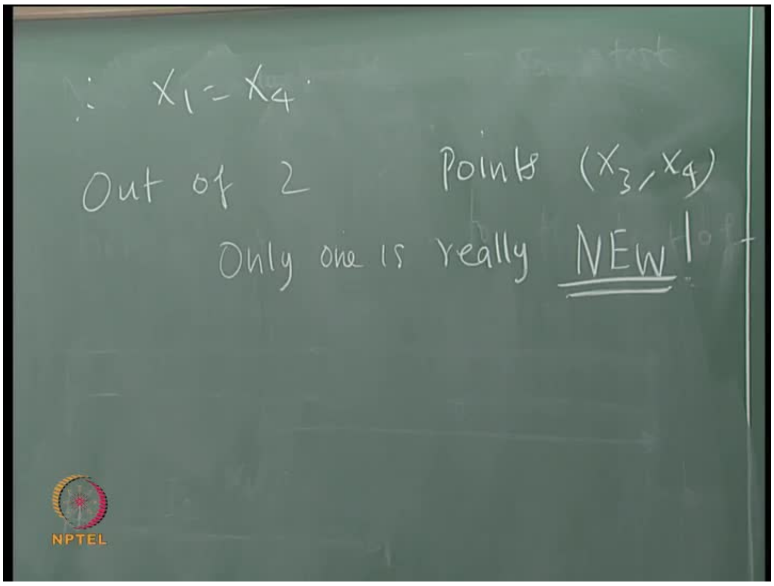

Therefore, out of the 2 points X3, X4 only one is really new, this happen iteration after iteration that contributes to the power of the method. 1953 it was figured out. It was a big rage; it must have been a big rage at that time, now it is so obvious to us. What is obvious to us now may not have been obvious in the past, okay. And what is obvious at 8 am may not be obvious at 1 pm after lunch when you attend the same lecture.

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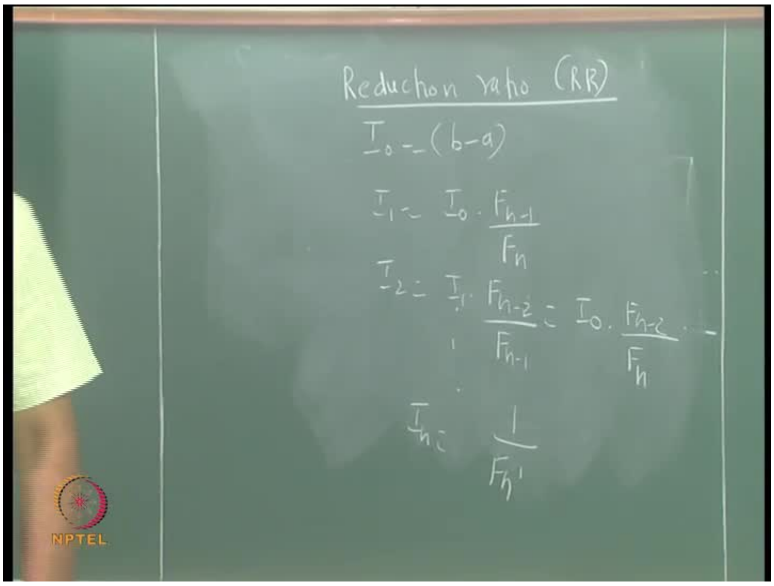

So, what is the reduction ratio of the algorithm? What is the reduction ratio of Fibonacci search method? What is the RR of the algorithm? I naught is b minus a, correct. Will it become one in the end? It should become because the first and the second number of the Fibonacci series is 1.

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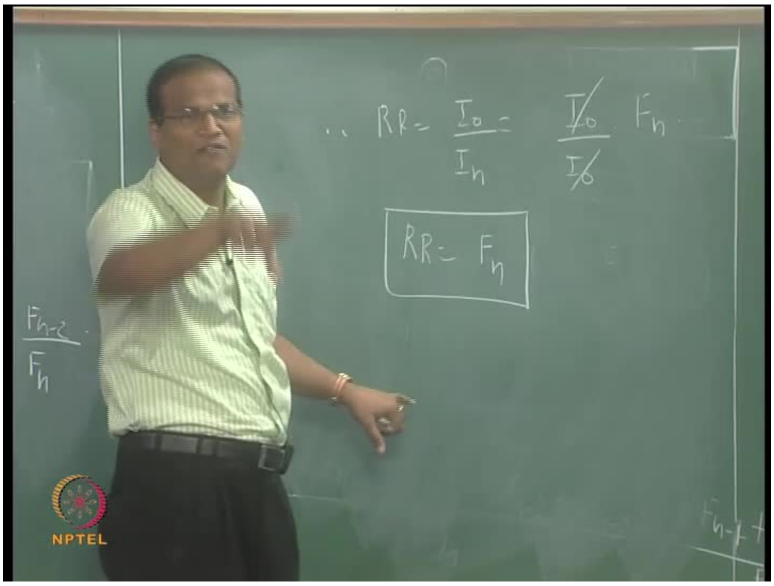

Therefore, RR the Fibonacci number itself is the reduction ratio of the algorithm. So, how do you start with? Suppose you want to find out for the cylindrical water storage problem, solar water heater storage problem if the original interval of uncertainty is 3 meter, you want to have an uncertainty of point 0.15 meter, the reduction ratio is 20. If you have a reduction ratio as 20 you have to look for the nearest possible number for the Fibonacci sequence. What is the nearest number? You should take the next higher number not the lower number, 21, okay.

So, if you start with if it is 21, this n also approximately gives you an idea of the number of functional evaluations. I think you will have to do n plus 1 evaluations. Can you correct me if I am wrong? Correct. So, you have to do n plus 1 not n because the first time you are doing two. So, you will get an idea how many evaluations are required, what is the final accuracy; this is the way algorithm works and you can compare now. I have given you 3 algorithms exhaustive search, dichotomous search and Fibonacci search. For a good number n equal to 14 or 16, you can see that the Fibonacci will just whiz past all the others.

What is the sixteenth number of the Fibonacci, some crazy number? So, this fellow will X1, X2, X3, X4 exhaustive search and you will see all the things and then finally reduce it to one sixteenth or one-eighth; you will reduce to one-eighth of the original interval. So, revisit the solar water heater problem. Now why I am doing that is there is some self consistency check to the same method. We will change the problem after we move out to the 2 variable problems; for single variable search techniques we will stick to the solar water heater problem, right. So, Monday is the quiz, right. So, we will have 3 questions; the paper as usual is very long, I already set is yesterday. So, all the best; it took 7 hours for me to set the paper not to solve the paper. I wanted to ensure that that any of the algorithms you develop to beat the system, do not work. So, what is the problem number? 35. It should be n plus 1 man, Vikram. 34.

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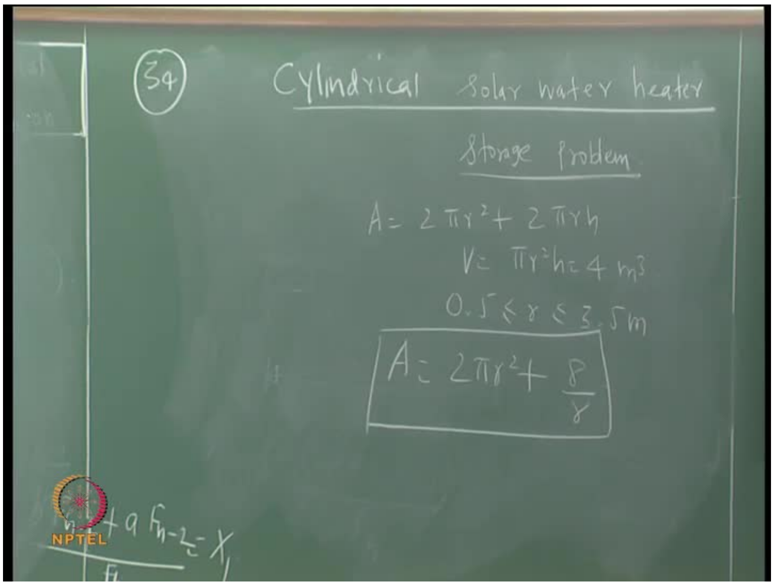

Cylindrical solar, okay. So, for the cylindrical solar water heater storage problem how do we give this n because I want you to do the same thing as what I have done so that we can compare. So, execute the Fibonacci, execute the Fibonacci search, execute the Fibonacci search algorithm for the cylindrical solar water heater storage problem. Execute the cylindrical execute the Fibonacci search algorithm for the cylindrical solar water heater storage problem with the original interval of uncertainty being 0.5 to 3.5 meters. Choose perform 7 functional evaluations. No, this is for class room, right, do not worry. Perform 7 functional evaluations; that means F of n you have to perform 7 functional evaluations. So, I will also say that perform 7 functional evaluations; start with F of 8, understood.

Start with 21 by 34 on both sides because you can compare with, after 7 evaluations you will hit the center. If you do not like you remove that do 7 evaluations, start with F of n equal to 34; that also is alright. Start with F of n is equal to 34. So, take the original interval 3 meter, F of n minus 1 by F of n is 21 by 34. So, 3 meters into 21 by 34 will be your I1. Take I1 from both the ends 2 points. Calculate the value of A at both the points. See which portion of the interval can be eliminated.

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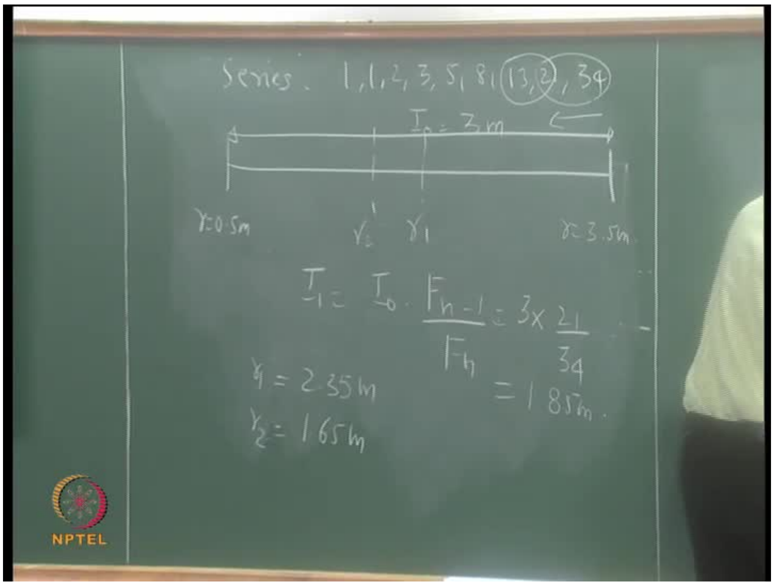

So, first step please write the series so that you do not forget it, right. So, you have to go backwards, right. Please first write the series and then draw the original interval of uncertainty and it is not X, it is r. So, r equal to 0.5 meter, r equal to 3.5 meter, I naught equal to 3 meters, I1 how much is this? 1. 85. So, choose 2 points X1 is 2.35. X2, X2 is how much? Is it 1.15?

Student: 1.65.

1.65, correct; do not worry that X1 is more than X2, you do not worry about this, we have this thing. Our notation is whatever point is chosen from the left interval is X1, whatever is chosen from the right interval X2. Is Use your common sense; do not get worried about whether X2 can be on the left of X1, do not worry about it. If it is confusing, you change it; change X1, X2 accordingly, right, but be consistent. Vaibhav, is it okay, fine. So, now get A of X 1.

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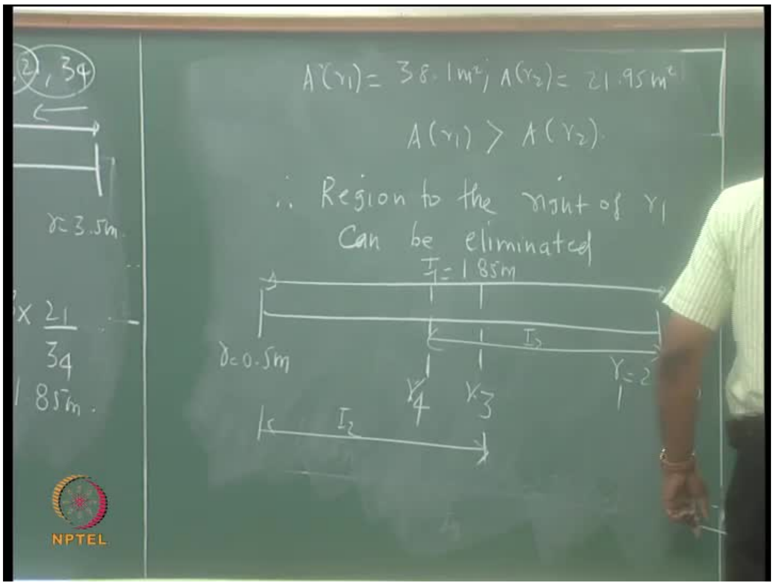

Now we should not say X1, right, what is it? It is not X it is r or we should change the radius itself to X in the original first place, right. So, we will stick to r1; A of r 1. I have the solutions with me, but I want you to solve along with me. So, A of r 1 is 38.1 meter square, correct, A of r 2 21.95 meter square. A of r 1 is greater than. So, region to the right of r 1 can be eliminated, is that right? Therefore, region to the right of r 1 can be eliminated. Abhishek is it fine, okay. What is the new interval now? Yeah, what is the new interval to the right of. So, r equal to 2.35, right. What is I 1; 1.85 meter, right. So, we now take two points X 3, X4. These two are at a distance I2 from the two ends.

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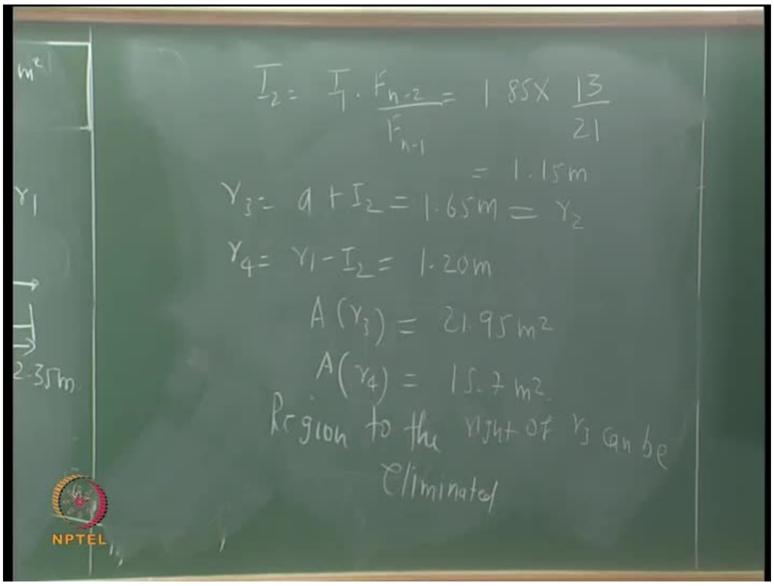

I2 equal to I1 1.85, 13 by 21, yeah. So, first we took 21 by 34, now we took 13 by 21. So, you move from the right to the left till you hit this; now towards the end when you reach the end, the two points X n and X n plus 1 will be at the center, right. That is where you stop the Fibonacci search, clear? Now how much is this? 1.15. We will adjust this such that one of the 2 points is equal to. There may be some small problem with your decimals but do not recalculate, I mean because originally the algorithm tells you that one of the 2 points must be an old point. So, why I am writing X, Why am I doing this? r 1. Please tell me r 3 equal to r 2 plus I 2. r 4, what is this? r 1, right, r 1. 0.5 plus this is 1.65, this is?

Student: 1.2.

1.2, 1.21, 1.20, then we will leave it like that, 1.20. No, sorry this is correct know a plus. This is correct r 1 minus I2, correct. So, now r 3 is equal to r 2, right. Is it correct r 3 is equal to r 2? So, that point we already know. So, A of r 3 equal to 21.95, correct. A of r 4 must be smaller.

Student: 15.7.

15.7, where is r 3 now? The region to the right of r 3 can be eliminated. Babu is it correct? Region to the right of r 3 can be eliminated.

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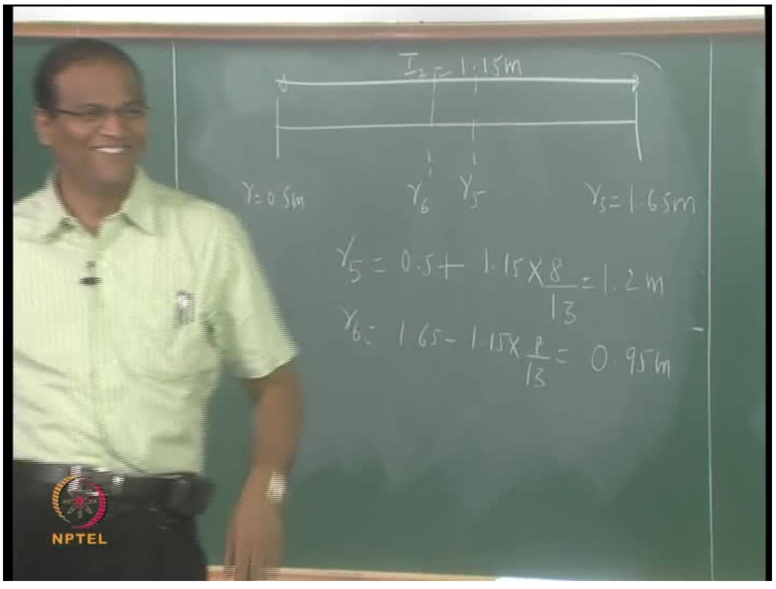

So, r equal to 0.5; I stay with this. r 3 is1.65, I 2 is equal to 1.15. So, I now take two points r 5, r 6. So, r 5 0.5 8 by 13; r 6, what are the 2 values? r 5 is 1.15.

Student: 1.2.

1.2. This is?

Student: 0.94.

0.94. We know the solution is 0.866. So, I know that A of r 6 is less than A of r; I do not have to do but you have to do. We know what the answer is.

Whether you do it as a part of a project or exam it is very easy but you have to be systematic and methodical. You cannot get impatient. You have to do it patiently. It is not such a brain teaser that you cannot solve or something. If you are systematic you will get it. Do not be in a hurry. You do it slowly without making mistakes; you will be home, right.

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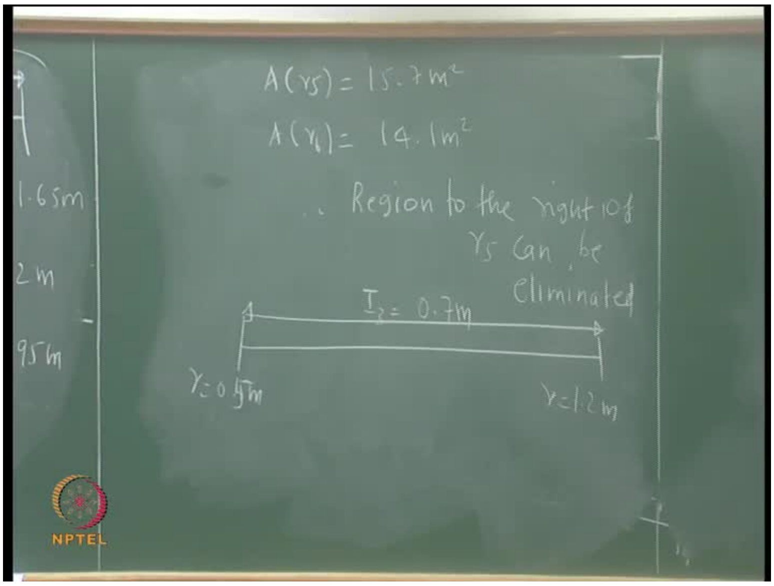

So, A of r 5 he is already there 15.7 but now I have A of r 6 which is lower than him, I hope so, 14.1. Therefore region, see you will keep on eliminating to the right because the answer is close to the left interval. You will keep on chopping off from the right side region to the right of. Now r equal to 0.5, r equal to 1.2; I 3 equal to 0.7, correct. Yes. Now take two points 7 and 8.

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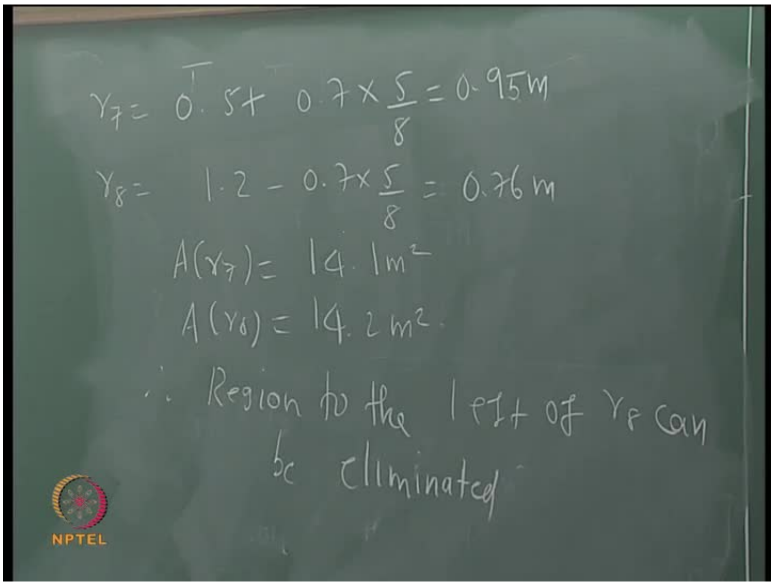

So, r 7 0.5 into 5 by 8, right, 0.94, okay. Please note that in every iteration one of the points is already the previous point. So, though you seem to be working out a lot, your functional evaluation is; in this case it so happens that the functional evaluation is easy and the other step seem to be as difficult or as laborious or as time consuming as a functional evaluation. Sometimes the functional evaluations may require the solution of a Naiver-Stokes equation or you have to solve some using COMSOL multiphysics. In this case it was so simple 2 pi r square plus 8 by r. Then the other steps will be ridiculously less time consuming compared to the functional evaluation, okay.

Now A of r 7 we know that 14.1, I know A of r 8 will be greater than this fellow. He will be some 16, 17, 18, 14.2. Oh, now you have to be careful. Now region to the left, region to the left is eliminated. Therefore, for a change we got rid of the 0.5, right. I expect you to solve a problem like this in 15 minutes if it is 7 or 8 evaluations. 10 to 15 minutes, you can do it if the evaluation is not painful, okay

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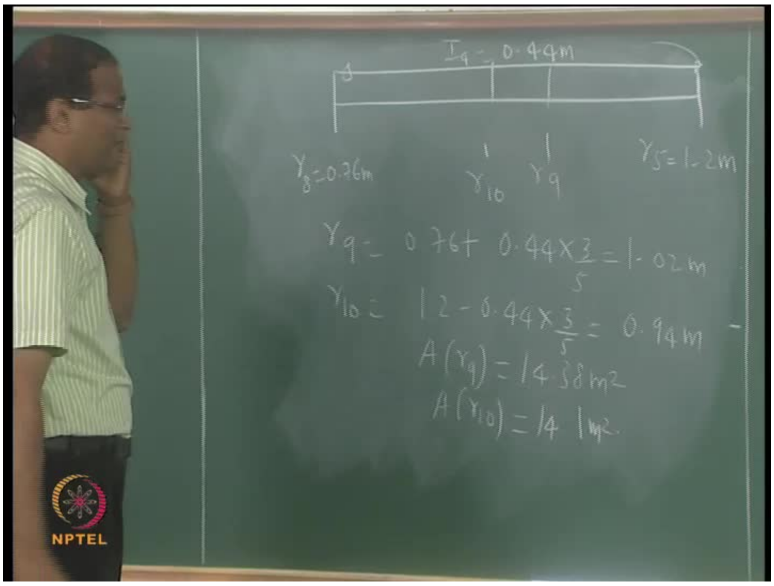

So, now what is this, left side?

Student: 0.76.

0.76, what is the value r? r 6, r 7, what is it? r 8. This fellow is r 5; he is 1.2, I4 0.44, r 9, r 10, 3 by 5. Yeah, tell me 1.02.

Student: 0.95.

0.94. A of r 9?

Student: 14.38.

14.1. Region to the right of r 9 can be eliminated. We are close to the final point. How many have you done so far? 6? 6 evaluations are done, only one more.

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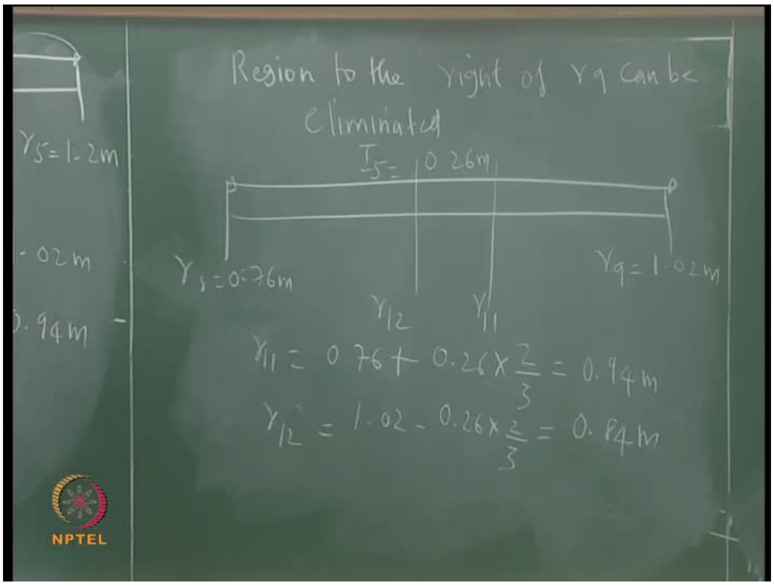

So, region to the right of; I hope all of you are able to follow why am I trimming region to the right of r 9 because 14.3 it is more than 14.1, we are solving a minimization problem. Now what do we have now? r 9 is 1.02, you have already came, is it. What is it? Okay, I cannot ditch that r 8, yeah, r 9, right. How much is this? 1.02. So, I have I 5 0.26. So, I have two points r 11 and r 12. So, r 11 equal to 0.76 plus 0.26 into 2 by 3, r 12 equal to 1.02 minus 0.26 into 2 by 3, 0.94, 0.84. I hope everybody is following this.

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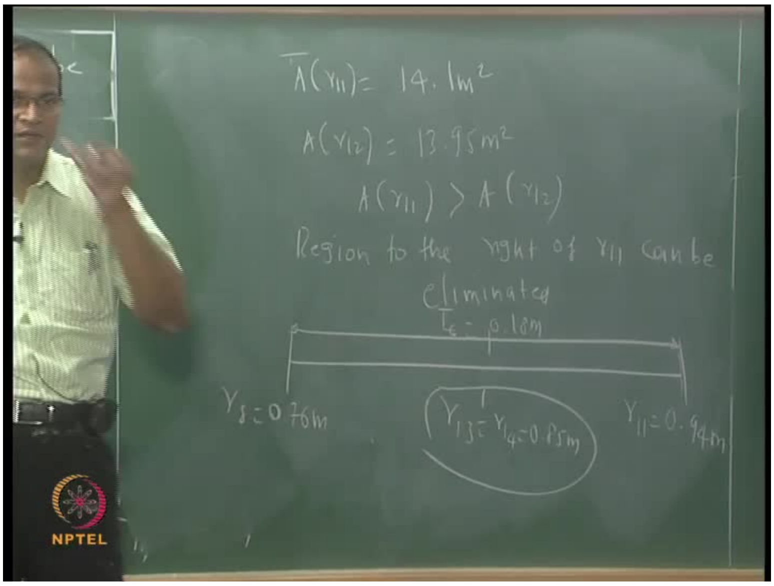

So, what is A of r 11, 14.1, A of r 12 should be more.

Student: 13.9.

Okay, that is closer to the answer, 13. 9. The region to the right of r 11 can be eliminated. So, we have 0.76 to r 11, right. Is that right? What is the r 11? 0.94. 0.18 I 6, the next point will be the center. We cannot do much with that. The buck stops here. So, the next point is 0.85, r 13 equal to r 14 equal to. So, the final interval of uncertainty is 0.18 meter, the solution is now lying between 0.76 meter and 0.94. With just 7 functional evaluations you are able to reduce the final interval of uncertainty to 18 centimeter from 300 centimeter. So, you got a remarkable reduction ratio which is how much? You got an RR now, so final interval of uncertainty.

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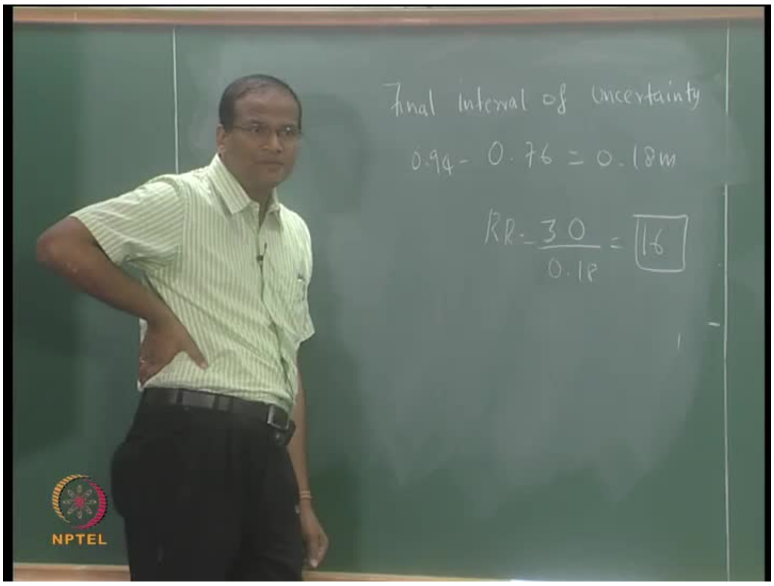

So, RR is this is how much? 16. What is theoretical RR? 34. What is n?

Student: 21.

Theoretical RR is how much?

Student: 24.

F n, so what is the problem now?

In fact you stopped with 6 evaluations, right. We stopped it; we can go for one more. We can go for one more and have a 3 point test or you can go for one more have a 2 point test between these two 2 point tests. It is equivalent to the 3 point test if f of X1 is greater than F of X2 but is less than. So, you can still the bracket the interval and bring it to half which will make it 32. So, you have to go to the last step, right. Are you getting the point? Except that instead of 2 points you will get 1 point, so you have a 3 point test, right. 3 point test also I told you, right; otherwise you can stop here. So, you will get something which is smaller than, what is predicted by the reduction ratio theoretical reduction ratio the algorithm but which is much much superior to what you will get using exhaustive search. If you use exhaustive search, your final interval of uncertainty will be 1 meter, right, after 7 evaluations.

N plus 1 by 2, how much is it? 8 by 2 is 4. So, you will have a final interval of uncertainty of 75 centimeter; it will be superior even to the dichotomous search. But now there are some higher order searches which can do better. For example, after you have reached the center point it is possible for you to get r 3 is equal to r 14 which is already there. r 13 is equal to r 14 was already one of the points. If you have these three points, now if you are pretty confident that you have bracketed the minimum and it is indeed lying between 0.76 and 0.94, it is possible to use the Lagrange interpolation formula and have a local polynomial connecting these 3 points and then make that fellow stationary. d y by d, d a by d r equal to 0 and then he will give you the correct answer. But so much it is overkill for us for a stupid problem like this, but it is possible.

Student: When we are writing the program how do we define that till a date or till?

I have to do that. If I take F n is equal to 34 I have to do that last iteration. So, let us say that if both the points coincide we do the evaluation and take the 3 point test.

Student: Assuming we have some desirable accuracy required in the problem.

Here desirable accuracy required you take n plus 1 go to the next. If you have desirable accuracy required, go to the next nearest Fibonacci number, so that this half problem does not arise. Then you will land; you can work out your problem that way. What is your desired level of accuracy for this problem? Suppose, you declare that the desired level of accuracy is 0.15 meter then you divide; then you divide 3 meters divided by 0.15, 20 reduction ratio; for that calculate the corresponding F n. And since you now know the Fibonacci, the last 2 points will give you trouble, go to the next Fibonacci number; that is all. Because the Fibonacci it seems finally ends up with 1 point where you have to use something other than the 2 point test to reduce the length of the interval.

It is possible still; it is not illegal for us. You can do 1 evaluation. You can do 1 evaluation and decide where it will lie. So, you will decide either it is lying here or here right, fine. So, we will stop with this.