Design and Optimization of Energy Systems

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Module No. # 01
Lecture No. # 30

Dichotomous Search

We will continue our discussion on single variable search methods. We saw the most primitive and most unimaginative one first, namely the exhaustive search, and we briefly touched upon this dichotomous search, right. I do not think, I have given you the algorithm, and we have not worked out any problem on this.

The dichotomous search, needless to say, it comes from the word dichotomy. What is the meaning of the word dichotomy, in English? Student: Voice not clear. Ok. And, it is also used in sentences like; I hope you are able to appreciate dichotomy between the 2 means, when you are able to appreciate the difference between the 2, or whatever.

So, it means, that you are dividing something into 2 equal parts, more or less 2 equal parts. Then dichotomous search, we try to, we try to get those 2 points of the evaluation, very close to the center of the interval; so that, every time you, every time you do 2 evaluations, you can get 50 percent of the interval; or, you do one evaluation, you will get 25 percent of the interval. But, there is no one point test, so you will always have, even number of evaluations, ok.

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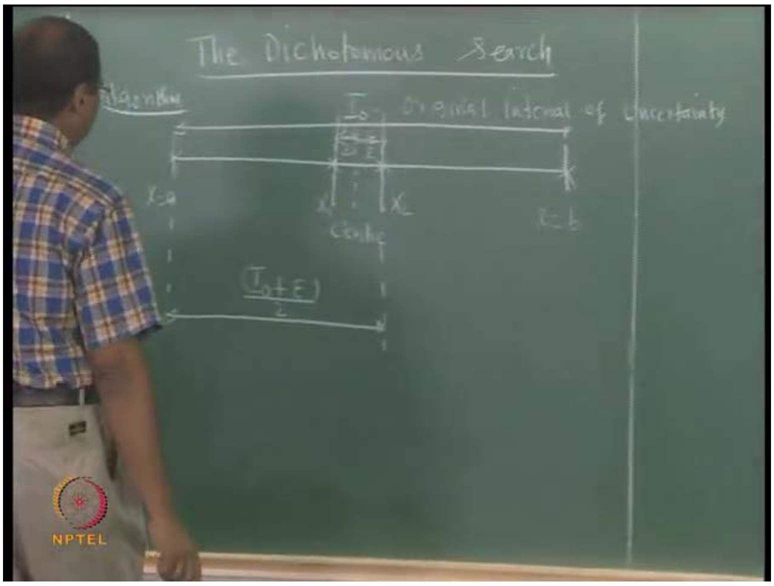

So, the dichotomous search is like this; x equal to a, to, x equal to b; this is I naught, 0 always represents the original interval of uncertainty. Now, the centre. I choose 2 points very close to the centre, so, right. So, if you consider this, the distance between, a and x 2, you know, I naught, ok. So, the distance between x 1 and; I do not have space to indicate there. The distance between, x 1 and b, is there; so, I naught plus epsilon by 2, ok. So, the algorithm is like, the algorithm is like this.

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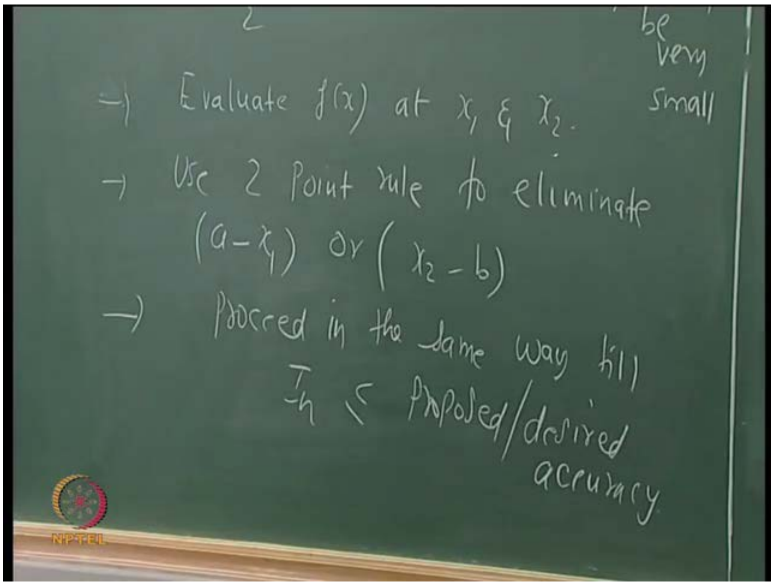

Choose 2 points at epsilon by 2; needless to say, epsilon should be very small. For example, epsilon should not be, I naught by 4, for example; then, there is no point in it. Choose 2 points, choose 2 points; let us say, x 1, x 2; evaluate. Now, use; eliminate, or how do you indicate that, a? You will understand, or, should I say, a to x 1?

Student: We will understand, sir.

You will understand, ok. Of course, that third case of eliminating both, to the left of x 1, to the right of x 2, if, f of x 1, equal to, f of x 2, is very rare. Numerically, it is very rare, because epsilon will be 0.001 something; unless the function is so simple, it is symmetric and all that, it will never happen. If it proposed at desired accuracy; that is the algorithm, right.

Divide I naught into 2 parts; around the, around the centre, choose 2 points x 1, x 2, at epsilon by 2 from the centre; epsilon should be very small; evaluate, f of x, at x 1 and x 2. Use the two-point rule to eliminate, the region to the left of x 1, or the region to the right of x 2. Proceed in a similar way.

Similar way means, that means, whatever interval is remaining, take the centre of that interval; have an epsilon; have 2 points, epsilon by 2 from the centre of the new interval, but each time you will, you will get different points, right. If this is eliminated, here 2 points; from here 2 points, and 2 points and; the points will never get repeated. You will have to compulsorily evaluate, x 1 and x 2, every time you employ this procedure, till a final desired accuracy is obtained.

Now, it should be possible for us to evaluate the reduction ratio of this algorithm. Obviously, it is very powerful, because every 2 evaluations, 50 percent of interval is cut. Shall we start writing now? How do you establish the reduction ratio of this algorithm?

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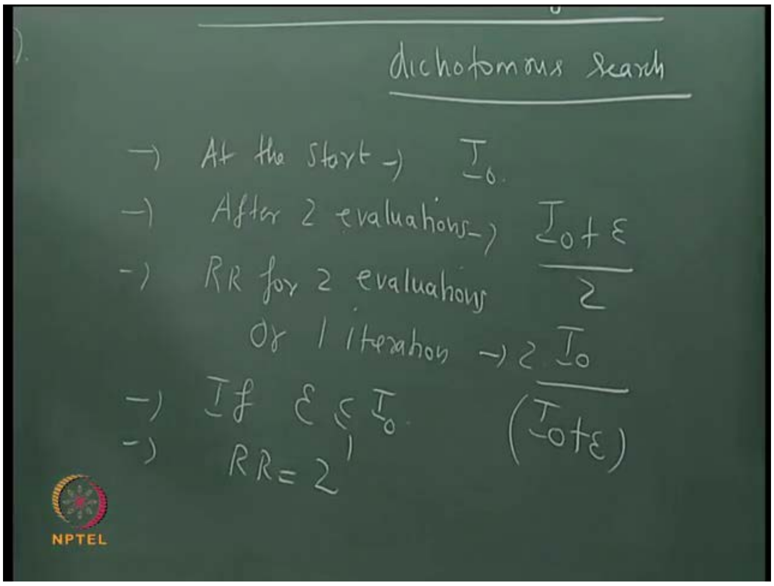

So, after 2 evaluations, what is the new interval?

After 2 evaluations, what is the new interval of uncertainty instead of I naught?

Student: I naught by 4.

Not I naught 4

Student: I naught plus epsilon by 2.

I naught plus epsilon by 2, correct. The remaining interval will be this. What you are removing each time is, I naught minus epsilon by 2, is it clear? So, you have to be alert. So, after 2 evaluations, what remains is, ok. If epsilon is small, it is just I naught by 2, but after sometime we will remove the epsilon, now we will keep the epsilon. Now, what is the reduction ratio for 2 evaluations, or one iteration, whichever we want to say? I naught divided by, 2 I naught divided by I naught or.

Student: 2 I naught by I naught plus epsilon.

Is it correct?

Now, if epsilon is much much smaller compared to, I naught, ok; if this is 1 meter, I take it as 5 millimeter, which one? I naught is 1 meter, epsilon is 5 millimeter or 2 millimeter, it is ok. Then, R R will be equal to, R R is equal to 2 to the power of 1; is that clear?

Now, I want to retain this, so, shall I erase this portion of the board? Is everybody through with this?

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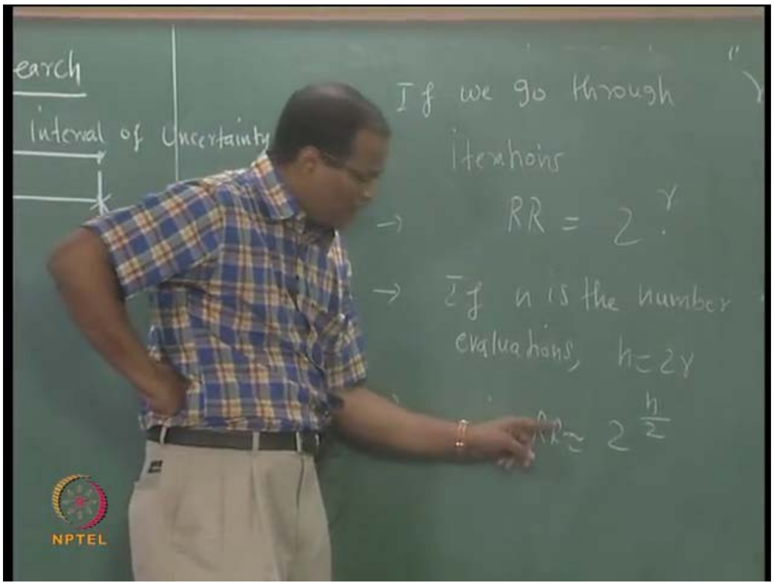

We go through r iterations; we go through one step, it is 2 to the power of 1; we go through r steps, the reduction ratio is 2 to the power of,

Student: r.

r, not gamma, r. What is the relationship between the number of iterations and the number of evaluations? If n is the number of evaluations, and r is the step number of the iteration, then the relationship between n and r is;

Students: Voice not clear.

If n is the number of evaluations, n equal to; Student: 2r. So, I should say, r will be; whether this, how much this approximately equal, will become, will be replaced by the equality sign, depends on how small an epsilon you choose. But, epsilon is finite and hence nonzero, therefore, you will get a reduction ratio, which is always smaller than this. If you take big epsilon, then the reduction ratio will fall down substantially, ok.

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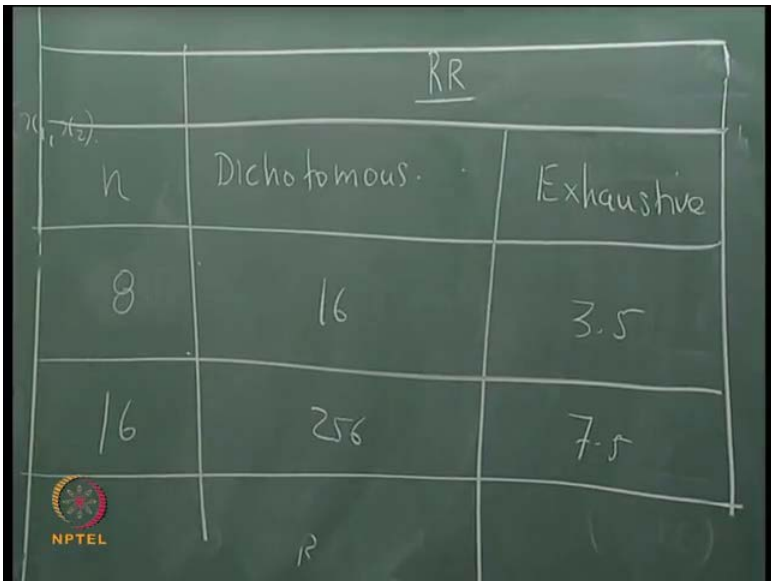

For example if n equal to 8, n equal to 8 dichotomous, tell me the R R? Exhaustive? 16; exhaustive, how much is it? 3.5. 16? 256. What is this?

Student: 7.5.

So, if n is very small, you do not see much, you do not see much difference between the R R of a exhaustive and dichotomous. But, as n increases, as the number of function evaluations increases, either because you want, either because you want an increased accuracy or the cup function is so more complicated, that is not very easy for you to figure out the optimum. It can be seen, that the reduction ratio of dichotomous search is far superior, compared to the simple exhaustive search.

So, there is a need for anybody who is same to study optimization; if somebody says no, I will you divide the interval into 100 equal parts and get the, and evaluate the function, as stupid as it gets. For very simple problems, if you do not know any, what is called the naive, what is called the naive procedure? It is ok, one half, and you will learn to optimization for the rest of your life; otherwise, it stands to reason, that we study optimization technique; is it ok? So, this is the power of the dichotomous search.

Now, we will revisit the cylindrical water storage problem, we will just do 8 evaluations and you can see the power of this method. Unless we work out an example, certain things will not register. So, for example, different people may take, may take different epsilons, and then finally, they will come and argument that, they are not getting the reduction ratio which is claimed on the blackboard and all that. So, I will have to tell them, what is the value of epsilon you have to choose. Is this fine?

If I want, if I want to solve an optimization problem with desired accuracy; how will you work, using the dichotomous search?

Student: Voice not clear.

Because in life nobody will say, I will do say 16 evaluations, I will do 32 evaluations, I will do 64 evaluations, that is not going to work; finally, I say, I want this with an accuracy of 0.01, 0.001, something. So, you will start with the last step; you will start with the reduction ratio first. So, the reduction ratio is very very important. Reduction ratio is, I naught, by the new interval of uncertainty or the final interval of uncertainty. The final interval of uncertainty is the desired level of accuracy, that will be in the denominator. You know the original interval of uncertainty, I naught, from that you will calculate the R R.

R R, if it is some complicated number, you will put it in a form, such that, you find the nearest equivalent, where some, 2 to the power of n by 2, will be equal to, R R. From there, you will find the number of ns. And, then, anyway you will start with dividing by 2 and dividing. So, you will not, so you will not expect a, you would not, you would not expect a convergence, unless you have reached n, which is very close to what this formula has told you, right. So, you would not get terribly impatient. What is this I am keeping on evaluating? I am not getting the accuracy; why? Because, it requires 2 to the power of n by 2 evaluations, and it requires n evaluations, ok, 2 n evaluations, whichever you look it; fine.

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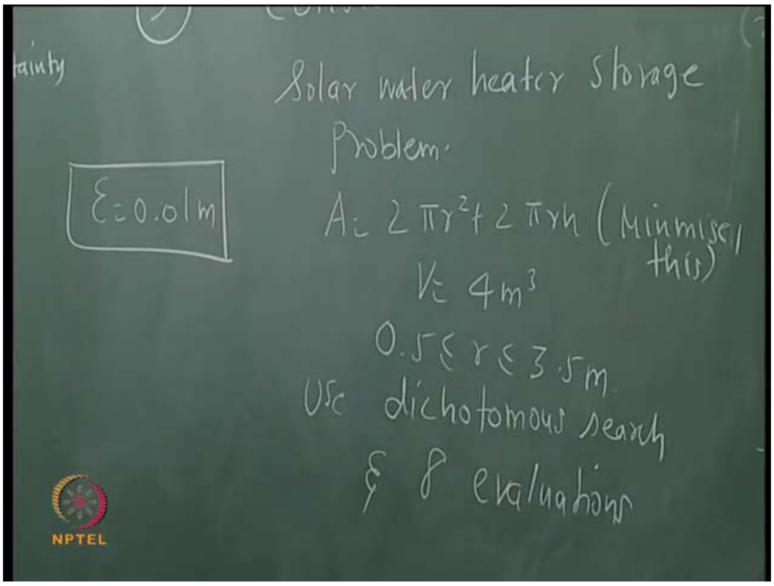

Now, problem number

Student: 33

33, ok.

Consider the cylindrical, consider the cylindrical solar water heater storage problem. A is, 2 pi r square plus 2 pi r h, that is the surface area, which you want to minimize, subject to the volume. So, volume condition, volumetric constraints, the volume has got to be 4 meter cube; and, the original interval of uncertainty is 3 meters; so, we start with 0.5 and end with 3.5 meters; use dichotomous search and 8 evaluations, that means, 8 functional evaluations.

All of you may take an epsilon of 0.01 meter. You may take an epsilon of 0.01 meter. If somebody does not like it, you can take 0.0001 also; it becomes messy, if you want work it in with you calculator; if you want you can take 0.0001, there is no problem. Do not take something higher than this. 0.1 is no good, it will bring down your reduction ratio, ok.

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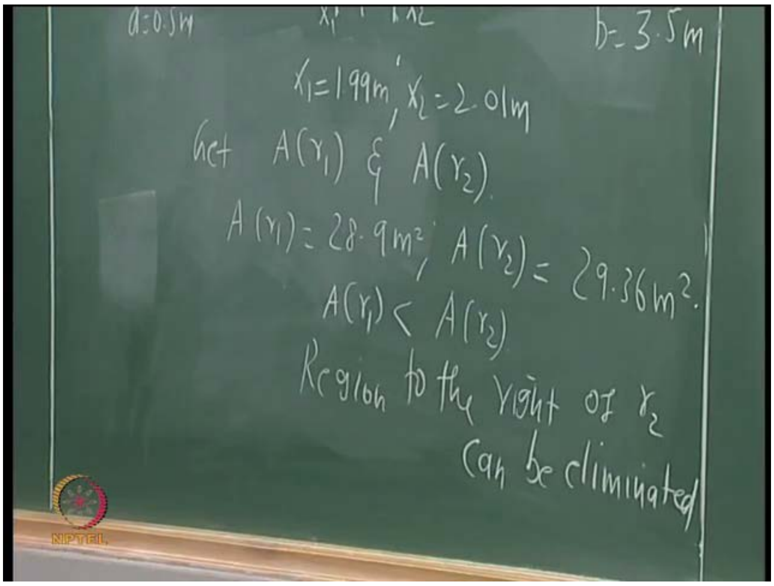

So, this, we have this here (Refer Slide Time: 19:40). So, a, right; x 1 equal to

Student: 1.99 meters.

Good. 1.99 meters; x 2 is 2.01. Get, A of r 1 and A of r 2, and decide based on the two-point rule, right. This is the first step. Vikram, what is the point?

Student: Fine sir.

Not able to get something?

Student: I am trying to remember from that.

What is A of r 1?

Student: 28.9.

A of r 1 is 28.9. A of r 2?

Student: 29.36.

A of r 1 is less than A of r 2. So, what can you say? It is a minimization problem; region to the right of r 2 can be eliminated. So, that is the, that is the logical conclusion you arrive at, after you do the two-point test; region; because, we are still talking about a unimodel function.

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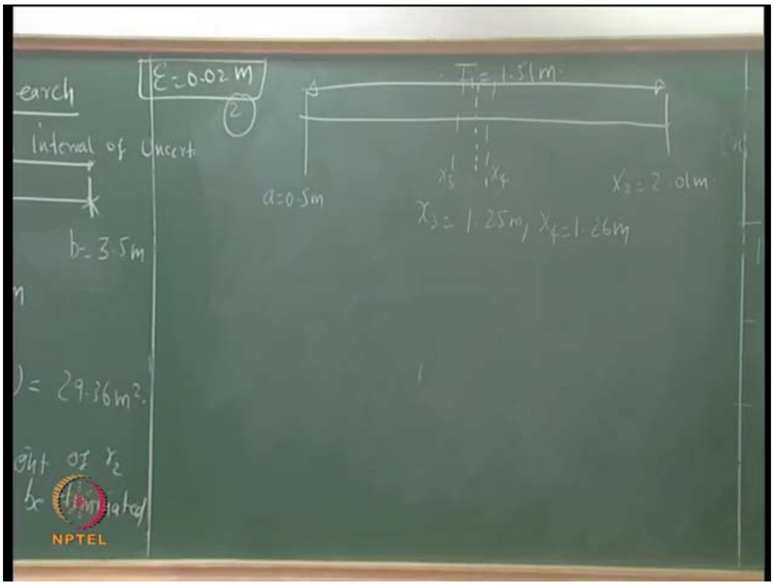

Now, what is I 1, value of I 1?

Student: 1.5.

Not 1.5

Student: 1.51.

1.51; correct. So, r 2 is

Student: 2.01.

So, this remains. Now, you bisect this. 2.51, 1.255, it gets messy.

Student: 1.245

So, you make it 1.25; that is a centre, is it?

Student: Voice not clear.

No, what is a centre?

Student: 1.255

1.255. 1.2255. So, you can make 1.25 and 1.26, you are reducing the epsilon; you can take, x 1 as 1.25 and x 2 as 1.26. Now, what is it, Vinay? 2.25 or 1.245.

Student: Center is 1.25, x 1 is 1.25.

Centre is 1.255. Centre is 1.255, so x 1 can be 1.25, x 2 can be 1.26; do not get it in the third decimal place. Or, if you want, you can, I have no objection, you proceed; you will have 3 decimal place, 4 decimal places, it gets messier. I am just suggesting one way out, but you free to.

Student: Sir, within the first iteration we should have used 0.001 because we are using epsilon by 2, but we know; Now, it is r epsilon by 2, we added r epsilon by 2, x 1 will be 1.995, Which one, x 1 is 1.995.

Come again, what did I say?

Student: Sir, basically the interval is epsilon sir, but we used 2 epsilon.

Then you change that 2. What correction should we make, ensure that whatever you are done is correct.

Student: Epsilon equal to 0.02

Ok take it, I do not want a rework, take epsilon is equal to 0.02; let us not repeat that. Now, do not call it, x 1 and x 2, we will get confused; well already there is x 1, x 2, proceed with x 3, x 4, x 5, x 6; x 3 equal to 1.25 meter, x 4 is 1.26, is that ok? I will write it a little bigger. If there are any objections, you can raise now. The center is between these two, you are sure, because Samarjeet is saying, vehemently he is saying no.

Student: Sir center is 1.255

That is ok. Epsilon you have taken 0.02 now.

Student: Sir, you take 1.24 and 1.26.

No, I change the epsilon now, dynamically I am changing. Why do we get saddled with same epsilon? See, when it 1.255, it seems to tell us that, we better take 1.25 and 1.26, right; something close to that, right. We create rules, so that it helps us to solve the problem, right.

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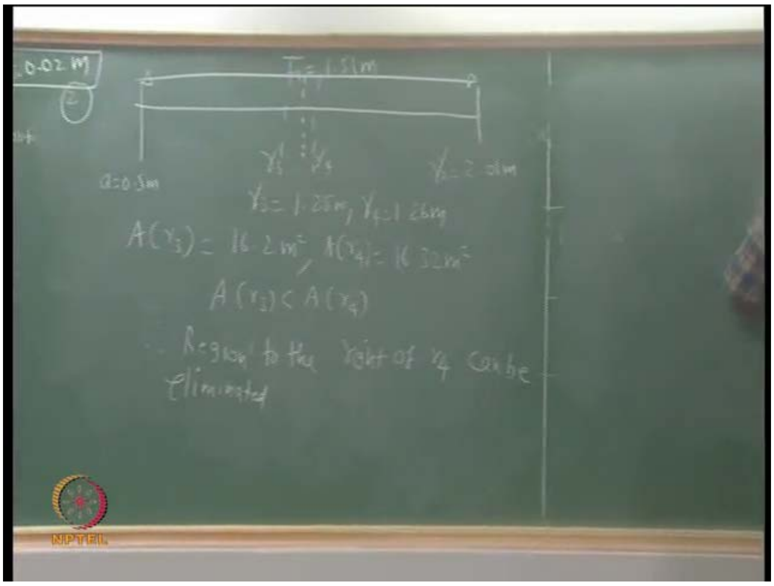

Now, A of r 3, A of; it is not x 3, I think we have to be, we have to be clinically correct; it is r 1, it is r 2, r 3; here, the variable is radius. What is A of r 3?

Student: 16.2

A of r 4?

Student: 16.32

So, A of, region to the right of,

Student: Voice not clear.

Therefore, region, can be eliminated.

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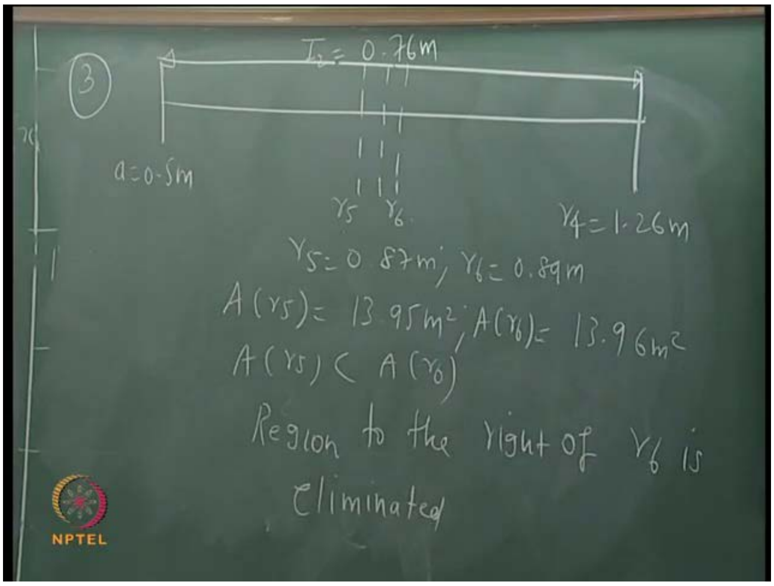

Now third step, how many evaluations are over?

Student: 4.

We need 4 more, right; we have to do it patiently. Yeah, tell me, what is I 2 now, I 2? That is, the interval at the end of 2 iterations, or 4 evaluations.

Student: Voice not clear.

How it can be become?

Student: Voice not clear.

What is this?

Student: 0.76

It should come down, right, with iteration; otherwise, we are not going in the right direction.

So, shall we retain A; A is there, right. So, r 4 is what? 1.26. Now, you take the centre r 5, r 6, what are the, what is the centre now?

Student: 1.88.

1.88. So, what you want to do?

Student: 0.789

0.789; so, it is a dynamically changing epsilon; that is ok. So, what you want to do is now, r 5, 0.87 meter, that is ok. So, the epsilon is small; it is fine. We will ensure that this is not violated. We can take something smaller than this, but we do not want to take something which is larger than this. A of r 5?

Student: 13.95.

A of r 6?

Student: 13.96.

Please remember, this is the answer; we are close to that, right. 866 is the answer, but you are by passing that. But, you not working out using calculus base method, it is just searching. That is why, if you take 2 points, r 5 and r 6, the function is not changing so much. But, itself gives indication that, nearing optimum. Of course, our interval, it takes a long time to come out the dichotomous search. But, it gives us some ideas about the nature of the function.

So, what do we say? Again, A r 5 is less than A r 6, right. Always, region to the right is eliminated, is it? That is very bad. So region. I expect you to write all these in the exam; do not just put r 1, r 2, r 3, r 4, in the exam.

Student: Sir, can we get it done as a table.

Table or whatever, I want you to go through. Do not say, do not say, some people I have seen previous years, A r 1, r 2, r 3, r 5, r 6, it will be, it will be just like, you know, it will be like a table; and, I would not know, whether they have really understood, whether they have applied the algorithm properly or not. We will just take a few more minutes, you write it like this, please be patient and write this out, ok. Now, we just have 2 more iterations, and then we are home.

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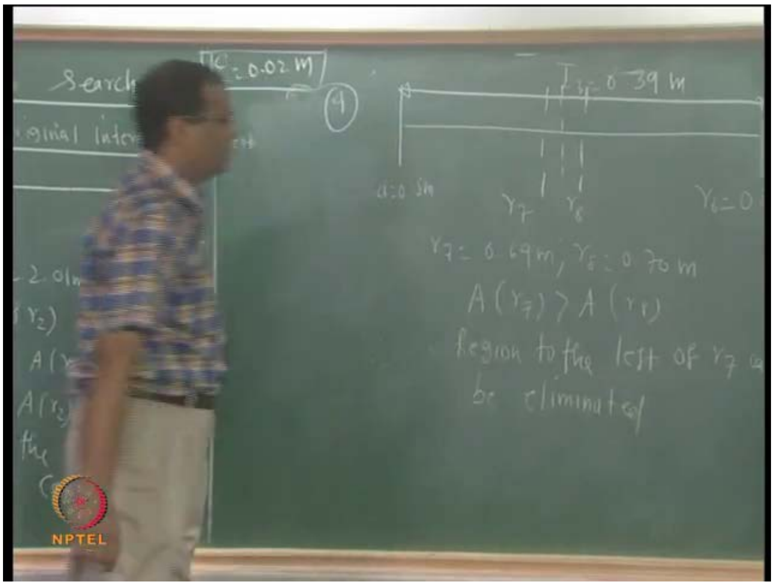

So, we are now at end of 6 functional evaluations. A equal to 0.5, and r 6; what is I 3 now?

Student: 0.39.

How much, see by how much the interval has reduced by using 6 functional evaluations? Using 6 functional evaluation, 3 meters has come down to 39 centimeters; if we use exhaustive search in 6 iterations, how much would you have got? Half a meter, 0.6. As you keep proceeding, the difference will grow. Now, what about this? This is centre; last 2 r, r 7, r 8. What is r 7? Please tell me.

Student: 0.69

r 8?

Student: 0.70.

A r 7 is greater than A r 8, that is the intuition, please keep it; A r 7 greater than A r 8; therefore, region to the left of r 7 can be eliminated, is that correct?

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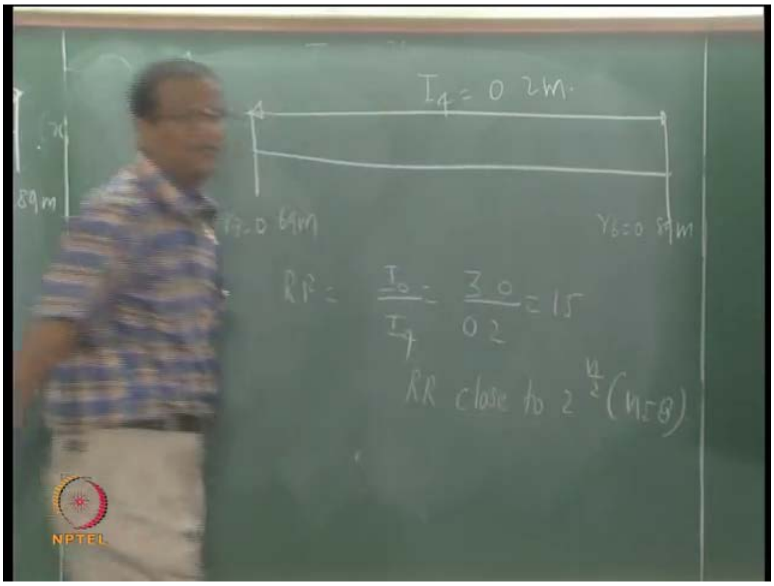

You not going to one more iteration, but you will leave it like this. Now, A equal to, A is gone; r 7 is 0.69, r 6?

Student: 0.8.

I 4?

Student: 0.2.

0.2. What is the R R? What is the R R?

Student: 15.

15. What is expected?

Student: 16.

Why is it not 16?

Student: Epsilon is not 0.

Epsilon is not 0 epsilon will change by 0.00501, but if we take more pains and make epsilon very small 0.0001, you can get, you will get something like 15.5, or in the limit epsilon tending 0, your R R will approach, 2 to the power of n by 2. It is a powerful method.

Even if you have to solve a multiple variable problems, some people what they do? Why will so much of fuss, repeatedly I am saying, why so much of fuss about single about having a sophisticated optimization technique for a single variable problem is; several problems are, most optimization problem or multi variable problems; each of this multi variable problem can be broken into single variable problem.

So, what you can do is, you can keep all variables, but one, at some values fixed in a particular level of iteration and then you optimize with respect to one variable, using the most powerful single variable search algorithm. Then change the variable, and each, when use a particular variable, choose the best optimization technique available to you. Then, after you are finished with one, one round of iterations, for all the variables, then you go to the next round; that is one way of doing it. That is why in literature, in literature lot of papers are written on, coming out with a very efficient single variables search algorithms. So, the R R; I naught; where n is the number of functional evaluations, fine.

Can we increase the reduction ratio beyond this? Is it possible to increase the reduction ratio beyond this? Intuitively, the answer is no. Because, the best you do is, you can cut the interval 50 percent; anything, anything less than that, is sub optimal, correct?

So, if at all there is a technique which claims, if at all there is a technique which claims that it can have a reduction ratio which is superior to the dichotomous search, then what should support that claim? What is the logic which is possible, I would not use a two point test; computational economy suggests that we have to use; do not say I will use three-point each time and I want use only each of my functional evaluation comes with a cost of computed time, I may have to run these things on a supercomputer.

So, somebody claims, that he has come up with an algorithm, which is better that the dichotomous search. Does it sound like violating the Kelvin-Planck statement of the second law of thermodynamics? It is impossible to construct a device, which when operating in a cycle, will have it is whole effect, has its whole effect? The production of work by having interaction with only single reservoir; there are algorithms which are superior to this.

So, what is a FUNDA? Please wait, I will go through this history. So, the, one of the most powerful single variable search is basically the Fibonacci search.

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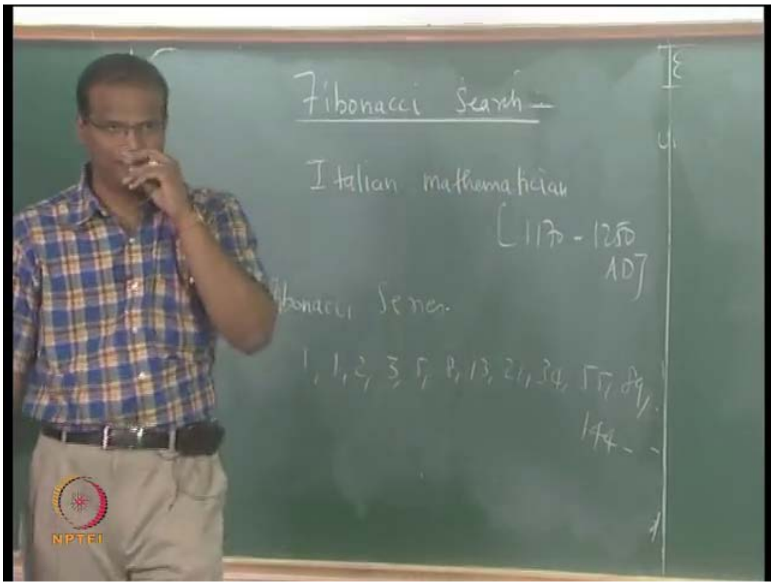

It is quite obvious, it is quite apparent, it is quite apparent that it should be based on the Fibonacci sequence; the Fibonacci sequence should be employed. So, Fibonacci, of course, as name suggests that, he was a Italian mathematician. I guess, maybe he was supposed to be a celebrated Italian mathematician in the middle ages. So, what is this Fibonacci series? Yeah, can you tell me the Fibonacci series?

Student: 1

1, 0 is there? Leave the 0.

Student: Voice not clear.

So, this is basically a Fibonacci series.

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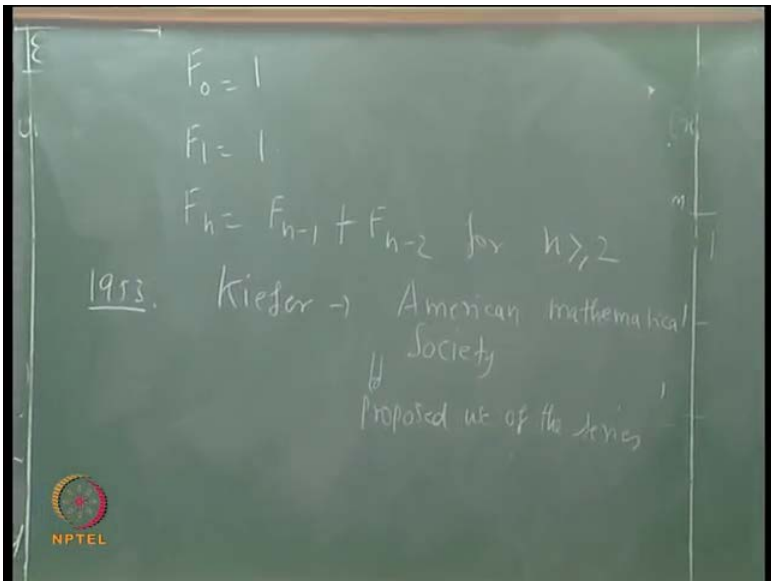

Any particular number in that series is sum of the 2 previous numbers, for n greater than equal to is, alright. How does Fibonacci enter the story here? So, this fellow was there from 1170 to 1250, has developed all these, but optimization is relatively new, right. In 1953, a scientist called Kiefer, a mathematician called Kiefer proposed in the journal "American Mathematical Society", proposed use of the series.

What is the FUNDA about this now? Any clues, now I told you. The FUNDA behind, the FUNDA behind the Fibonacci search method is this; with level playing field, nobody can beat the dichotomous search, so we have to do something shady. In the level playing field, you cannot beat the dichotomous, because fellow just has 50 percent, I mean how can we get better than 50 percent; all these fellows are saying I do not, I do not need 50 percent, but I still achieve. The FUNDA is, the Fibonacci series is so beautifully suitable for this, because the sum of a particular, the entry at a particular point, the number in the Fibonacci series in the sequence, is equal to sum of the previous 2 numbers.

So, if you are using a two-point test, when you are always doing it in, doing it in pairs, if you chose, if you so choose the 2 numbers, that one of the 2 numbers is already there, which is evaluated; then, each time you do the functional evaluation of a pair, 2 points at a time 1 point is already there. So, it does not matter, whether you, whether you have 50 percent or not; you have only 30 percent, does not matter, 35 percent, because one of that, you do not do 2 evaluations at a time, it is only 1 evaluation at a time because other is already there; very quickly, very rapidly, it converges; is it, ok. So, simple demonstration is like this, we look at the algorithm in the next class.

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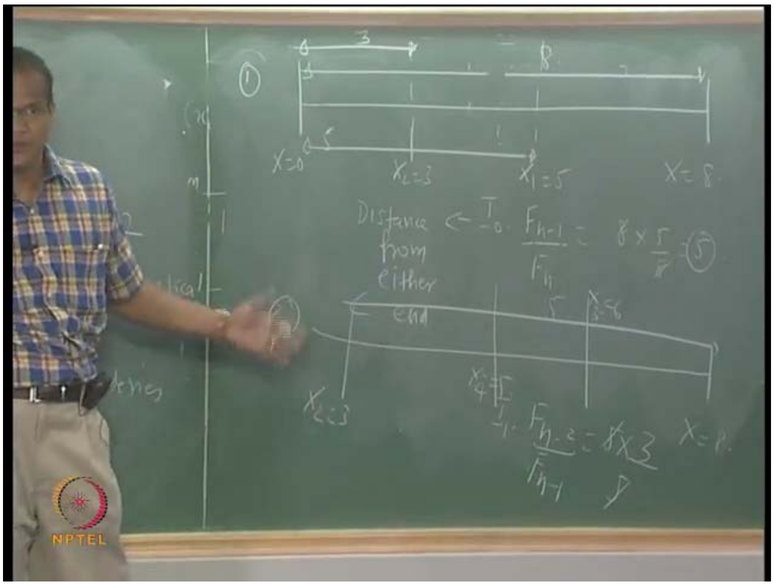

For example, I naught, I naught equal to 8, 8 meter, millimeter, kilometer, whatever you want, some appropriate units; x equal to 0, x equal to 8; I want to employ the Fibonacci series. So, where is this 8? 8 is there know; that is why, I chose this; 8 is there, here, ok.

The first 2 points are chosen in such a way, the first 2 points chosen in such a way that they are at the distance of I naught into F n minus 1 divided by F n. So, what is this? 8 into 5 by 8, distance from either end; that means, it is 5 from, it cannot be so close know, that is dichotomous, right. So, this is 5 units from this, so here it is 3 and here it is 5. So, x 1 is 5; the first point is 5, x 2 equal to 3. Do not worry about, whether, how x 2 can be smaller than x 1, it does not matter; choose the 2 points.

Certain portion of interval will get the removed. How much of the interval will remain?

Student: 5.

5. Is it ok? See, either f of x, if you are looking at a minimization problem, f of x 2 is less than f of, f of x 2, or f of x 2 is greater than f of x 1. In either case, in either case, you will remove either 0 to 3 or 5 to 8. You will have, the interval which remaining is 5, this is the first step.

Next step, suppose I say, the function such that, 0 to 3 is removed. So, what is remaining is, x 2 equal to 3, this x 2 equal to 8. Now, what is remaining is 5; upto this there is nothing great in the algorithm, nothing great; something you took some ratio, some arbitrary ratio, and then got the 2 points, you did the functional evaluation. It is far inferior, compared to the dichotomous search till this point, because the FUNDA is not yet in, is not yet in place.

Now, we will apply the FUNDA. Now, this one, distance is I 1 is equal to; this is equal to I 1 into F of n minus 2 by F of n minus 1. What is this? 5 into, what is n minus 2? 8, 5 what was the previous number? 3; divided by 5. So, what is the distance now? 3. So, if you take the distance 3, you will have what, this is x 4 or x 3? No, this is only 3, so this will be x 4, ok; x 4 is 5, the other one is x 3 is equal to 4. So, this will be x 3 equal to 6, x 4 equal to 5, agreed? But, 5 was already evaluated.

Why is this fellow behaving like this? Because, F of n minus 1 plus, F of n minus 2, equal to F of n. So, you start dividing like this; first iteration 55 by 89, then 34 by 55, 21 by 34, 13 by 21 and 5 by 8, then you will come to half. That means, you will just have one point in centre of the interval. But, each time you divide, it so happens that, 1 of the 2 points is already evaluated. Now, with 3 points you get substantial reduction in interval. How much will the interval reduce now?

So, finally, you will be left with, finally you will be left with, you will be left with 3. In 3 evaluations you reduced, from 8 to 3 you reduced. So, we will formally state this in the next class. I will call this as a, b, F of n minus 1, F of n minus 2, we will prove this, and then I will also tell you the algorithm. Then, we will revisit the cylindrical water storage heater problem. And, using the, using the Fibonacci search method, we will see how this works, alright.

But, what is cumbersome about this method is, that there was some elegance in the dichotomous search, that any interval half, half, half. But, here it is somewhat ugly, right; each time the ratio was different, the ratio with which you divide the interval, but people try to fix that also, we will see that in the next class.