Design and Optimization of Energy Systems

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Module No. # 01
Lecture No. # 29
Unimodal Function and Search Methods

We will continue our discussion on search methods. First, we look at the nature of the objective function. There are some important definitions that we will have to flush out. For example, what is the monotonic function? What is a unimodal function? What is a concept of a global minimum? What is the concept of a local minimum? And, all that.

And, these are best understood in relation to single variable problem. Then, once we can understand with relation, single variable problem, we can understand with respect to others, other multi variable problems also. Because, optimality criteria for multi variable problems, we have already established, right; Heisen, and all that. So, we look at some basic properties, nature of the objective function; and, then, we will start of, with search methods.

A very crude search method was already discussed yesterday, namely the exhaustive search method, where, if, when observations, you can reduce it by; how many? Original interval is b minus a, we are able to reduce it to b minus a divided by n by 2, or something like; something like 2 b minus a divided by n, or something like that. Now, we will see algorithm, which are much more powerful than this, which can rapidly, which can rapidly reduce the interval of uncertainty.

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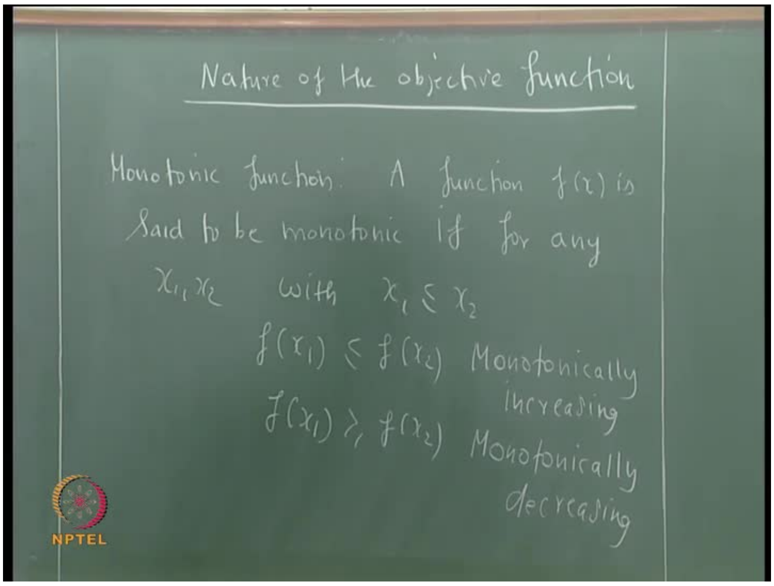

So, the nature of the objective function, in the, whatever discussion we are going to have, is basically with respect to a single variable problem, and, we can extend it to multi variable problem, very easily.

So, a function, f of x, is said to be monotonic; can you complete the definition? A function, f of x, is said to be monotonic; why, you start with some, you start with some x 1, x 2, you start with some x 1, x 2; function f of x is said to be monotonic, if, for the, any x 1, x 2. A function, f of x, is said to be monotonic, if for any x 1, x 2, with x 1 less than equal to x 2; f of x 1, what is this? Monotonically increasing; monotonically decreasing.

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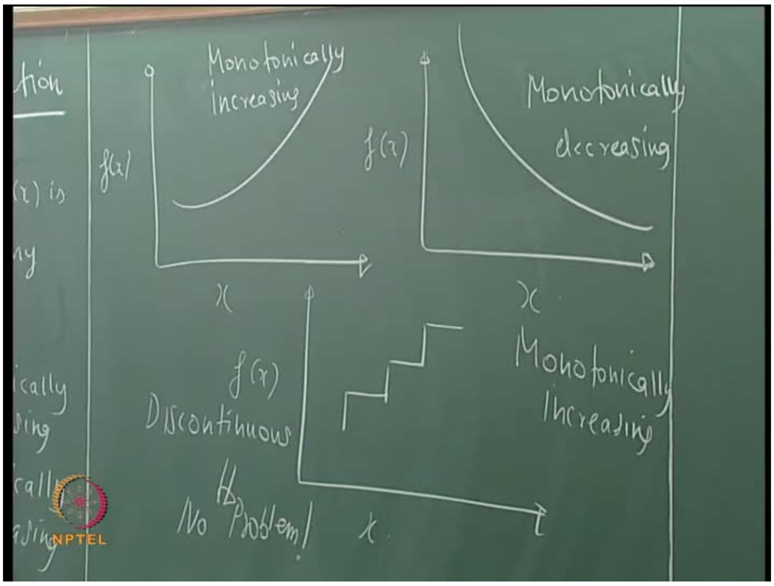

I graphed 3 possible situations, Student: Sir, there should not be equality in the. There should be equality, Student: strictly, otherwise it would be, strictly, there; will be equality, ok; let us complete this. So, this is monotonically; this is also monotonically, monotonically increasing. But, what is the, what is the problem with this? In the function, I have sketched at the bottom, so, discontinuous.

What happened? Student: Voice not clear. What is continuous? Student: Voice not clear. How is it continuous? The same value of x, gives 2 values of y; it depends on what we are talking as continuous. For me, if there is a value of x, they should be unique value of y; any other definition, I will not accept. So, the greatest of mathematics process can tell, if, there should be unique value, isn't it; differential and other things, will vary later. I am coming to that; is it differentiable or, ok. So, this is as, this is with regard to a monotonicity.

What is the next logical step? What is the unimodal function?

Student: Voice not clear. The problem is I have drawn lines, you just replace it by points; if you feel that your problem arises because I have joined it, you can keep it like this. So, that is your problem, ok, remove the line. According to me, if I give 1 x, I cannot get 2 values of y. As an engineer, it will trouble me too much; whether the size of a nut or bold should be 2 milli meter or 20 milli meter, it will really, I cannot sleep. Mathematically so many things are possible, but basically we are engineers, right; so, the way we look at things is different.

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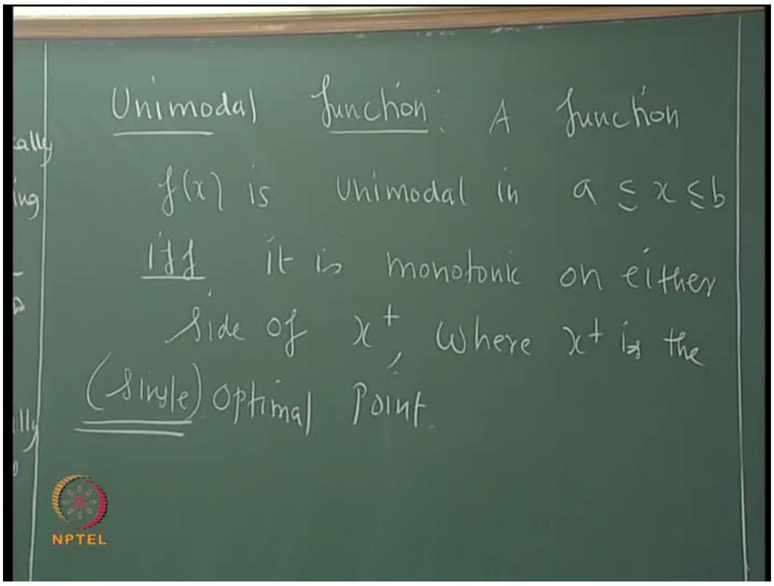

Unimodal function: a monotonic function is said to be unimodal; can you complete the definition? Monotonic function, f of x, is said to be unimodal, ok. A function, a function is unimodal in, a less than equal to x, less than equal to b. A function, f of x, is unimodal in, a less than equal to x, b; if that is, if and only if, it is monotonic on either side of x plus, where x plus is the single optimal point. It can be beautifully depicted graphically.

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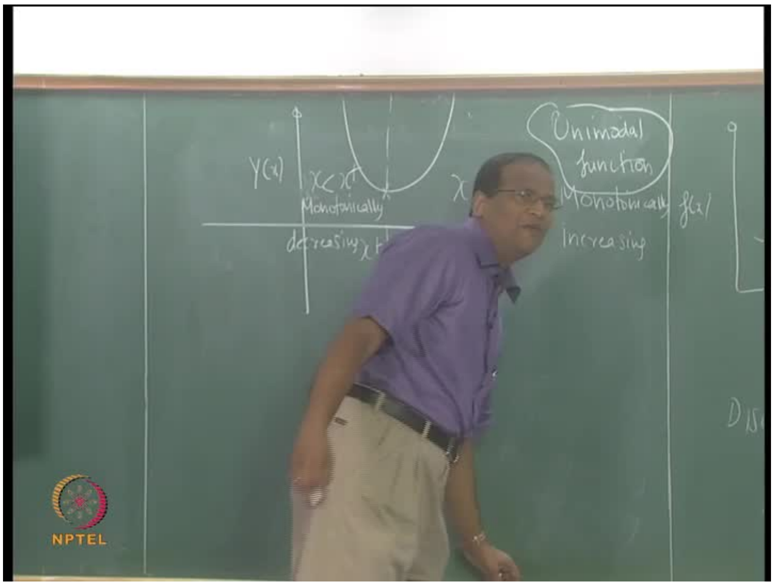

So, y of x; suppose, you draw parabola like this; about the optimal point, it is monotonically increasing to the right side, for all x greater than equal to x plus, is monotonically increasing. It is a; therefore, this is a unimodal function. How it determine the optimal point? Is something different, we have already seen; I mean, we are going to see all. Once you have an optimal point, it is increasing on one side of the optimal point and it is decreasing on the other side of the optimal point.

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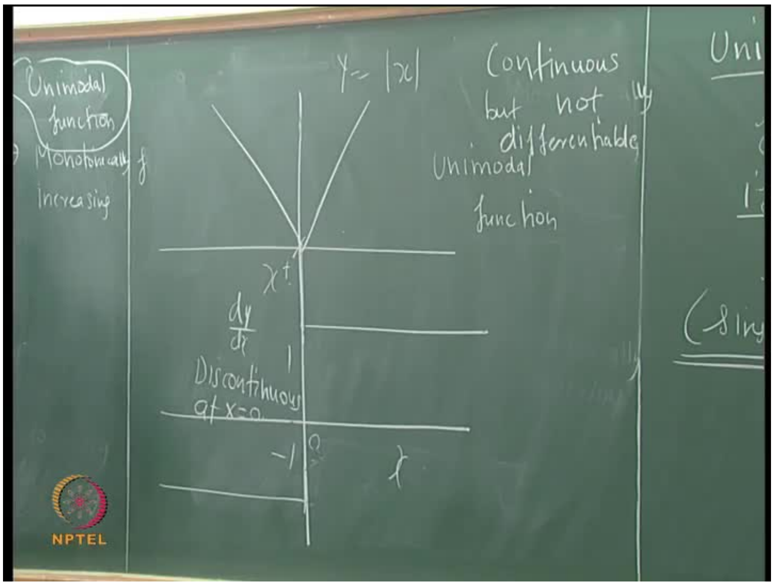

Now, I will come to the function, which all of you worry about, y equal to mod x, ok. Students seem to be obsessed with, y equal to mod x, alright. What is optimum for this? Minimum, if you want to minimize; Student: 0; 0. So, this is the x plus. Is it unimodal function? Sure, it is unimodal. However, so, it is continuous, because there is, whether you approach from the left hand side or you approach from the right hand side, at x equal to 0, y equal to 0. Therefore, it is a continuous function.

There is no discontinuity; that is, when you are coming from one side of x, if it gives one value of y, if you are coming from the other side, other side of x, it gives another value of y, then there is a discontinuity. This function exhibits no such behavior. However, if you try to plot, d y by d x; what is d y by d x? Quickly; correct; what is this value? Student: 1; 1; d y by d x is discontinuous at x equal to 0. Because d y by d x is discontinuous at x equal to 0, y equal to mod x is not differentiable at x equal to 0.

So, so many teachers have told you, from j e you consider is a big thing, it is a trap; you can find out to somebody knows match, what is y, whether discontinuous, difference. So, we will put the led, the led of this, continuous but not differentiable. But, still, you can find an optimum. So, there is no need, some function has to be always differentiable to get the optimum. This is very, very nice example; continuous but not differentiable. So, d y by d x, discontinuous at, ok.

So, we are trying to seek a solution, solution to the optimization problem, for a single variable problem. If, when, whenever or, when, whenever it satisfies its property of being a unimodal function; if it is multimodal, we have to divide it to sub intervals and try to seek the unimodal optimum at that particular point, or the, in the particular interval. Otherwise, we are in trouble; are you getting the point? You have several local maxima or minima; you have to sub divide the interval, and seek the individual local minimum or maximum.

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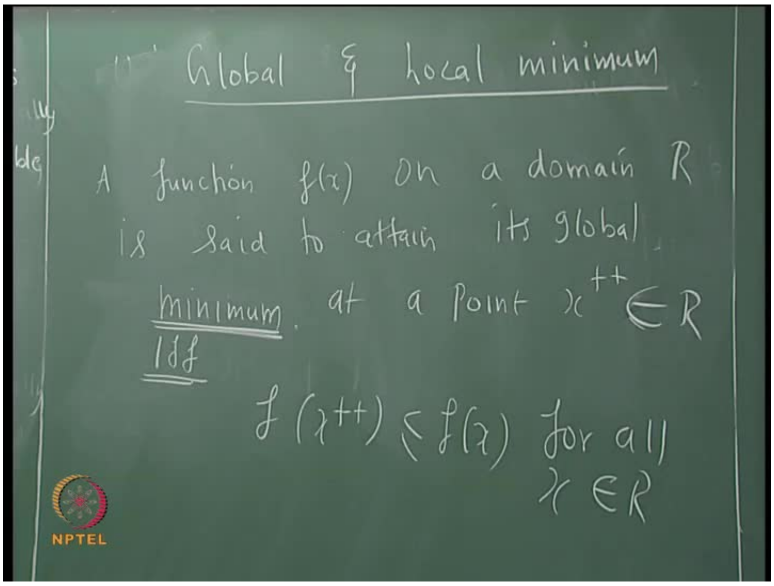

Now, we will come to the definition of global. What is global and local minimum? A function, f of x, is a f of, on a domain; let us say, on a domain R; a function, f of x; a function, f of x, on a domain R is said to attain its global optimum; can you give me the condition? You can see at a point, at a point x plus plus, at a point x plus plus, x plus plus, is contained within the domain R; iff, iff means, if and only if, Student: f of x plus plus is less than or equal to, f of x plus plus, Student: is less than or equal to, very good, f of x plus plus is less than equal to f of x, Student: for all x belong for all x belonging into R. There will be no point x, at which y can take a value which is lower than the value, which is taken by y at the point x double plus. This is the, let us say, it is a global minimum I should say, right.

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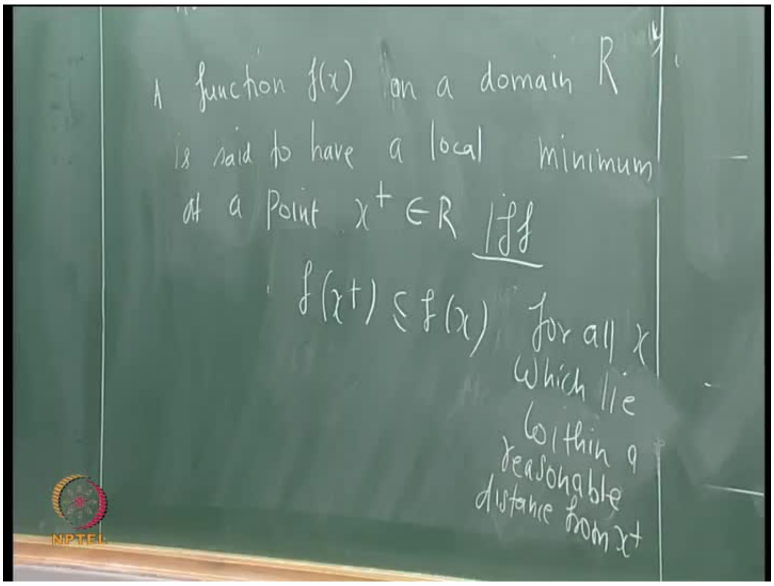

A function, f of x, is said to; a function, f of x, on a domain R is said to attain a local minimum. If x, at a point x plus, at a point x plus is contained within R or x, x plus is lies within the domain R, if and only if, can you complete? Function, f of x, on a domain R is said to have local minimum, I should say; double if is, if and only if, right; yeah, it is tricky. Student: For all x belonging to f of x plus. Very good.

So, if, f of x plus, is less than equal to f of x; for all x; I will make it non-mathematically; for all x, which lie within a reasonable distance from x plus; what is that reasonable? We will decide, ok. Within a, I can make it threateningly formal by calling that reasonable distance, the epsilon, epsilon plus, gamma, whatever. It is not central to our this thing, but conceptually you are understand. So, for all x which lie, all x which lie within.

So, if you located local optimum, if you point out the value of x around that, you will not get a value of f which is Student: better than this, better than this; however, it is not global because if you go far away from this x plus, there could be other positions of x, where the, f of x, could be significantly lower than this. So, this is the difference between global optimum and local optimum, or global minimum, global minimum and local minimum.

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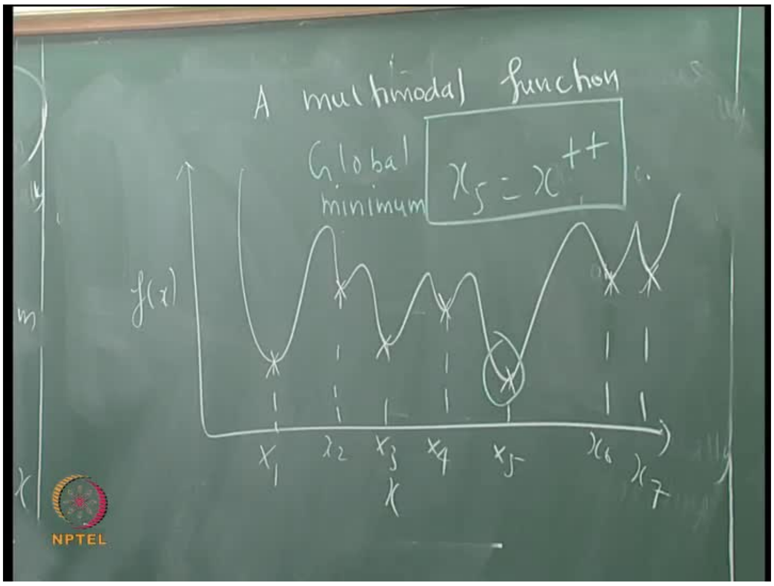

Now, 1, 2, 3, 4, 5, 6, 7, all are all, all are, among these 7, 6 are, Student: local, local. So, x 1; this is f of x verses x; this is a lower, right. Is everybody agreeing to this? These are all conceptual frameworks, which are very important. There are 3 important points which I want you to remember. If required, we will write them on the board.

Point number 1, for unimodal function, for unimodal function, the local minimum and the global minimum coincide. For unimodal function; if you want, you can write it down, in telegraphic language; for unimodal function, the local minimum and the global minimum coincide; there is no doubt, right. For unimodal function, the global minimum and the local minimum coincide.

Point number 2: for multimodal functions, several local minima exist; for multimodal functions, several local minima exist. You saw that on the board. For multimodal functions, several global, several local minima exist. We need to evaluate y at all these local optima. We need to evaluate y at all these local optima, and select the lowest one, for getting that global minima; you write it in your own words. So, we need to evaluate y at all the local optima, and select which of them gives the lowest value of y; that is the global minimum. So, it is a lot more difficult; it is a lot harder to solve a multimodal objective function problem.

So, algorithms having specifically developed, which can handle this, which can handle multimodal objective functions, and which can finally or eventually give you the global minimum. The problem with some Lagrange multiplier, any calculus based technique is, if you start from here, it will say that, this is the answer; then if you go this side, this side whatever takes you do, it will error, but this is the answer; but, it does not know it is in error, because this is the correct answer.

That is why, we are, people have come up with simulated analysis, genetic algorithms and all that. In genetic algorithm, for example, a quick, half a minute course on genetic algorithm; I will come to it towards the end. In a quick half a minute course on genetic algorithm, basically will see, we look at the complete solution space. We will look at this complete solutions space, and at any point of time, we will look at 5 or 6 solutions. And, then, these 5 or 6 solutions will be replaced; just like, parents produce, reproduce and then, it results in new off springs and so on. New generations we will see, we look at this quality of the solution; that means, we are looking at, f of x. By repeatedly searching in the complete domain, we are ensuring that we do not miss out any of these global properties.

There are several ways of; so, the basic problem with calculus based techniques is, they are extremely fast because derivatives, the force the derivatives quickly to 0. They are fast; they can be highly inaccurate; they will be quickly converged to the wrong answer, that means, the local minimum. That is called premature convergence, premature convergence. So, we will talk about it little later.

So, the first point was, a unimodal function, the local minimum is the same as the global minimum; second, for a multimodal objective function, you got evaluate all the local minimum, and then pick and choose the best; third, we can modify this definition for a maximization problem. The definition for global and local minimum can be modified for a maximization problem, alright.

So, now, we know, what is the nature of the objective function; what are properties it has to satisfy; how to get the optimality criteria for using calculus also, we have seen; how to establish, how to look at second order necessary and sufficient conditions, all that we have seen.

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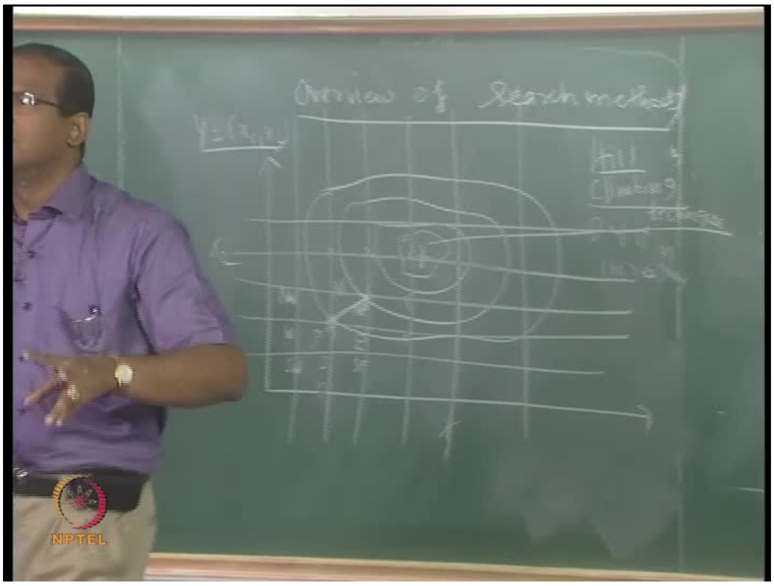

Now, we will go to, formally we will go to search methods. Now, before going into a region elimination method, let us look at a broad overview of search methods.

We looked at one category, where we divided the whole interval into 7 or 8 divisions; maybe, yesterday's 6 divisions and evaluated the function as 7 point; we removed certain portions of the interval, because they do not lead to the minimum; that is called as a region elimination method. You eliminate a portion of the interval, each time you do some functional evaluation; are you getting the point? That is, sometimes you have the interval, you may cotter the interval, have the interval, and make the interval to two thirds whatever; you are cutting a portion of the original interval of uncertainty by using your procedure, by using your logical algorithm or whatever; but, that is not the only, only conceptual way of solving a optimization problem with the search method.

There is another possibility. For example; so, the 2 variable optimization problems, I am looking at a minimum; this fellow is minimum. For example, I am drawing contours. So, y is increasing, iso contour, iso contour lines, iso caused lines, iso caused lines, or whatever. Now, it is possible for me to come out with the rectangular grid like this. I start from somewhere, point p, ok; east, west, north, south, southeast, southwest, northwest, northeast, for a 2-dimensional problem; a 2-dimensional means, y is the function of (x 1, x 2). I will have a rectangular grid; to make matter simple, let us say, we assuming uniform grid for both x and y, that is del del x, del x 1 equal to del x 2.

Now, I look at the 8 neighbors, right; correct, 8 neighbors know. We look at the 8 neighbors, and see, see and evaluate the value of y with respect to this. So, totally, I am doing 9 functional evaluations. Depending on where it is lowest, it will go either in this direction or this direction, anyway; now, I know that it has to go in this direction; this is the direction in which it will go.

Now, this fellow will be in center, I will get 9 points around this. Conceptually, this is an altogether different story, compared to the elimination of the interval by using the exhaustive search. Here, this is systematically climbing towards the top; I mean, if this is, this can be considered as some minute of the hill top, we are systematically proceeding towards the top of the hill. So, this is called a hill climbing technique. Conceptually, it is much different from cutting; I am not cutting the interval, and I am not saying that, this x 1 is not good, this x 2 is not good; I am not doing anything like that. So, this is called a hill climbing technique. I start that one guess, and I am trying to move that guess, till the value of y keeps decreasing. So, this is the hill climbing technique.

So, conceptually, you have two different approaches. You can have an elimination method, or you can have a hill climbing method. So, let us write down the framework for solving optimization problems using search techniques.

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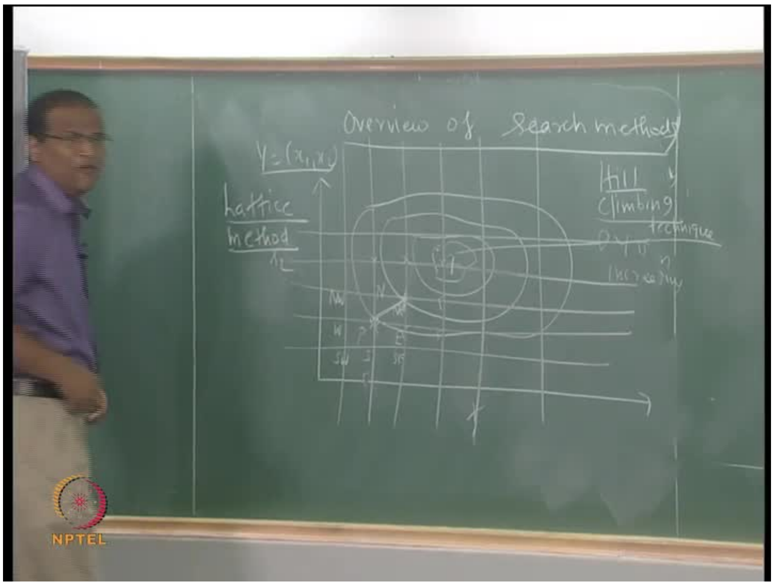

By the way, this is called a lattice method. If you do not have time and money to invest in a sophisticated optimization algorithm, you want a quick solution, instead of exhaustive search, you can go for that, ok; no calculus nothing, just keep on getting y. If getting y is so difficult, you can put, you can use a neural network, and you can get the solution for 10 values of x 1, 10 values of x 2; set of a neural network, which, when supplied with, x 1 and x 2, will automatically give the value of y. That is the neural network based lattice approach, lattice method for solving a n-dimensional optimization problem; there is already a paper.

So, overview of; so, but, if you want write a paper, what you have to do is, first you have, first you have to prove for what is called Himmelblau function or banana function. Banana function or Himmelblau function is something which has 3 or 4 local optima, but there is only one, x 1 square plus x 2, x 1 square plus x 2 minus 7 whole square plus, x 2 square x plus x 1 minus 11 whole square; it is something like that. People are worked on that. You prove that your algorithm works; it leads to some solution, where y is equal to 0. So, I have established my algorithm for a standard Himmelblau function; now, I am proceed into the fluid mechanics, thermal power, energy system, heat transfer problem; it works; this is the approach; send your paper and see how far it files.

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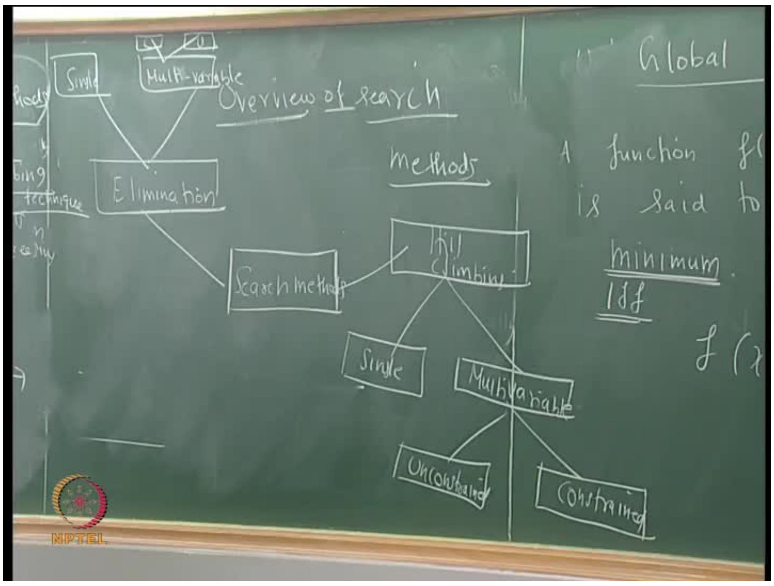

So, overview of search methods. So, the framework; basically, we have elimination, hill climbing; under this, single, multivariable; constrain, unconstrained. I do not want go to constrains and unconstrained here, though it is possible. Already, it is single variable, why you try to constrain it too much. So, this is unconstrained. But, do not ever imagine that, in none of these techniques, we will use calculus; we will use calculus, but we will not use calculus to make the functions stationary; we may use calculus to find out the best direction in which we have to go.

There is the difference; I am still searching. I am starting with the guess point, what is the best movement from that, position 1 to position 2? That, I will take the help of calculus. But, I am not using calculus to make the whole function, functions stationary, like what we used in the Lagrange multiplier method; there is a difference, are you able to get this? Likewise, elimination also you have, single, multivariable; and then, you have 2, constrained, unconstrained.

Let us see, in the coming classes, how many of these, we will be able to discuss. For example, the toughest will be a constrained multivariable optimization problem. And, suppose, or, we able discuss something in the class, a hill climbing technique for a constrained multivariable problem, or an elimination technique for a constrained multivariable problem. How much of this we will be able to cover, depends on the time we have, alright. Now, this is, this gives you the overview of the search method.

Now, I taught you some exhaustive search, right. It comes under which category? We already learned one; Student voice: single variable, single variable, Student: unconstrained, unconstrained; was it unconstrained? No, it was constrained, but you convert it on; for single variable, unconstrained, Student voice: elimination, elimination. So, we already; so, I do not have to teach you any, anything else. No, I will teach you something more than that. But, we have already seen this. Now, we will see, how many of this variable to circle, alright. I need the full board.

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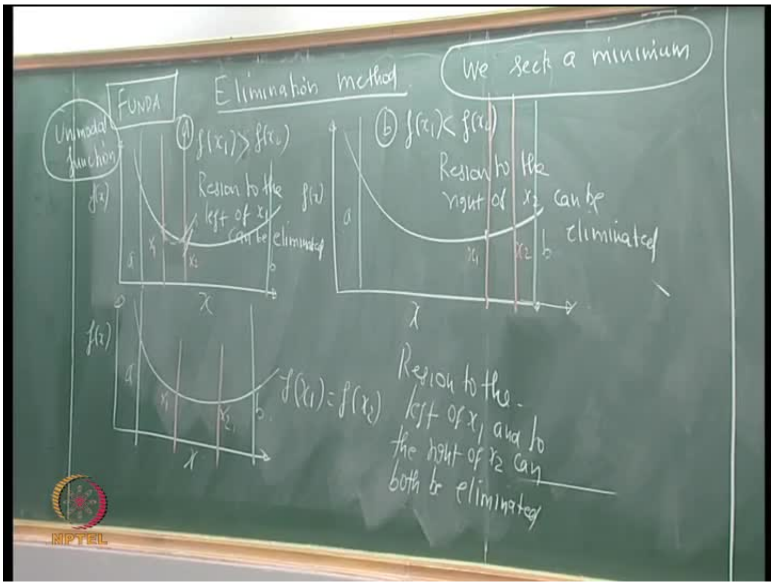

So, elimination method; what is the FUNDA behind this? What is the basic FUNDA? So, we will see that. I am drawing the same function, ok; f of x; a, b; we are seeking a minimum in the interval a, b, ok; alright. This looks like an additional line; we will remove him. Are you able to get this story?

There is the interval a, b. I am trying to seek a minimum in this interval; f of x is unimodal. It is monotonic at; I know that it will become unimodal; that is, once it is unimodal, if I seek an optimum, and one side of the optimum, it will be monotonically increasing; the other side the optimum, it will be monotonically decreasing. The challenge is to get the optimum, without taking request to calculus, by using a search technique, and without final interval of uncertainty.

Now, f of x 1, is less than equal to, f of x 2; case, a, agree? Sorry, greater than; b, f of x 1, less than, f of x 2; this is f of x 1; hopefully, I have drawn it, alright; f of x 1, Student: equal to, correct. I am seeking a minimum. What is the story now? When we are seeking a minimum, if, f of x 1, is greater than, f of x 2, from this, based on a simple, this is called a simple 2 point test; that is, 2 point test means, you take 2 points, evaluate the value of y at these 2 points, y at x 1, y at x 2. And, look at the nature of y? Whether, y of x 1, greater than, less than, or equal to, y of x; very simple. So, what is the conclusion? Logically, we can draw, f of x 1, is greater than, f of x 2.

Student: Voice not clear. No, No, No, No, I am talking about elimination method.

Student: Voice not clear. Region to the left of x 1 can be, Student: eliminated, very good, eliminated. We will write, region to the left of x 1 can be eliminated.

Student: Voice not clear.

Left of; I have written x 1, know.

Student: Voice not clear.

Left of; No, you can go like this, know. How do you know that it is, how do you know you already reached optimum; who told you? I can only say that, since it is decreasing, it cannot be left of x 1; be very careful, because, it is not such a fine division; between x 1 and x 2, so much story can take place.

Student: Voice not clear

It is not decreasing. I am saying with, I do not know whether it is decreasing or not, I am evaluating spot evaluation. I do not, I am not drawing the function, I can evaluate at 2 points; decreasing means, f of x 1 is, f of x 1 is greater than f of x 2; that is the information I have; with that you cannot conclude that, you cannot conclude that everything to the right of this thing can be, everything to the left of x 2 can be eliminated, that is not correct. A very simple logic; I do not know why you are not able to get it.

Student: How can we predict the left hand term left of x 1, and the

I am already telling you it, I am already telling you, it is a unimodal function; wherever you go, the logic will; correct. It is a unimodal function. We are looking at unimodal function. I have taken sufficient care to ensure that; I am looking only at that interval, in which it is unimodal.

Now, everybody will get a doubt; sir, we can take the left of x 2 also we can eliminate, but that is not correct. So, it could have, it could have been like this also; it could have been like this; you still do not know, right. You do not have the luxury of evaluating the value of x; and you know, at every point, you have only, x 1 and x 2, alright. So, region to the left of x 1 can be eliminated. Here region, region to the, region to the right of x 2 can be eliminated. Here? Left of x 1 and right of x 2, can both be eliminated, can both be eliminated.

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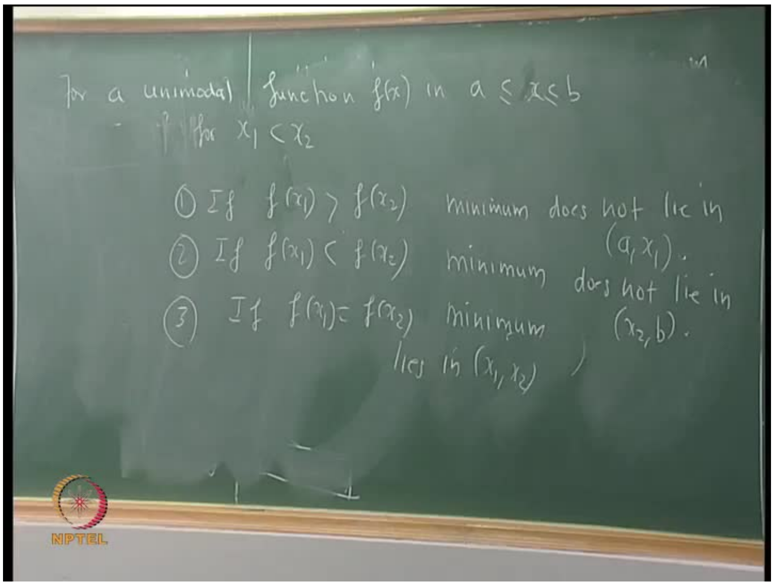

For a unimodal function, for a unimodal function, f of x, in the close interval, a, less than equal to, x, is less than equal to, b; for; minimum does not lie in; Student: (a, x 1); (a, x 1). Good. Minimum does not lie in (a, x 1). Minimum does not lie in, Student: (X 2, b), (X 2, b), very good.

Minimum lies in, if f of x 1 equal to f of x 2, Student: (x 1, x 2). Correct. But, this is very very rare in practice. When you are doing search method, you will do 1.99, x equal to 0.1991, other may be 0.1990, then, f of x, x 1 and x 2, numerically will never be equal, but it is a mathematical possibility, so I discussed. When you are actually doing computation, it will never happen. There will be always a small difference between (x 1, x 2), because round of arise this thing, and you getting the point?

So, we can use this 2 point rule, and come out with various algorithms, ok; we can come out with various algorithms. I will just give you sneak-peek into a very powerful algorithm called a dichotomous search. We will work out a problem in the next class. But, on the basis of this elimination method, we will start off with the first algorithm, namely the dichotomous search. Is this clear? The logic is clear?

Vikram, you have any problem?

Student: Yes sir, it depends on the function right, you defined a monotonic function behavior.

No, No, monotonic is old, after that unimodal.

Student : If it is 2 unimodal, Sir, as in, it should be strictly monotonic, and then we can have the function as an inverted omega, flat line then, and then again a flat line. So, that was,

No, that flat line, we are looking at well behaved function; I think you have something like this, a line.

Student: Yes

I think, sometimes, you try to specializing in finding out the exception, right; where something can go wrong; where the teacher can be caught; that is also ok; that makes us more and more alert, right.

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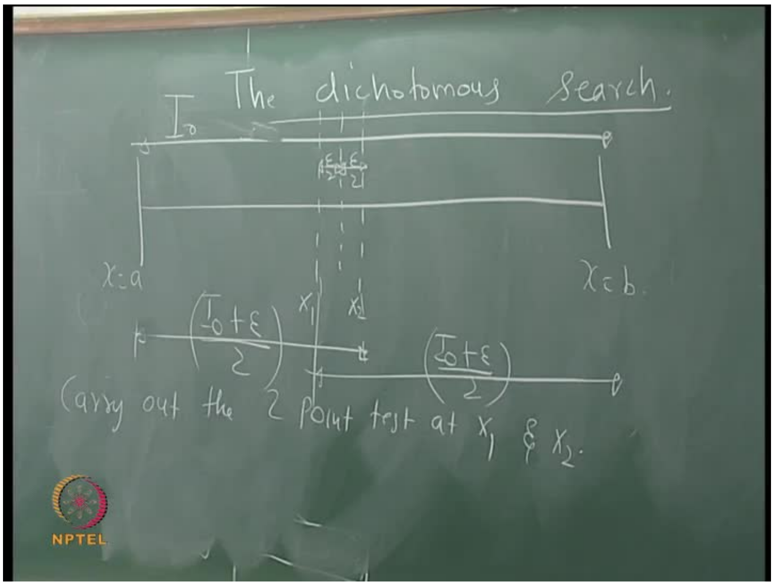

The dichotomous search is very simple. So, if this is, ok, x equal to a, x equal to b, this is I naught. Let us look at the center of this. And, let us take 2 points on either side; these 2 points, r at distance epsilon by 2 from the center, ok. This is x 1, this is x 2. So that, epsilon can be made very small, ok. Now, what is this?

Student: voice not clear.

In terms of I naught; I naught plus epsilon by 2.

Student: Voice not clear.

There, up to this, there is not x 1 up to x 2; be alert; I have done the homework; you want to say, it is I naught minus epsilon by 2, that is a dumb answer.

Student: Voice not clear

I naught plus epsilon by 2, Anand you got it.

Student: I naught is not written on board.

I naught, I have written on the top; shall we go for the eye test. Everything is I, I naught. So, this is I naught; this is also I naught plus epsilon by 2. If I do a 2 point test on, x 1 and x 2, how much of the interval can I eliminate, with just 2 function evaluation, just tell me? So, carry out the 2 point test. Forget about the case one, f of x 1, equal to, f of x 2; either the interval to the left of x 1 or the interval to the right of x 2, which are both equal. So, I naught minus epsilon by 2 is eliminated.

If epsilon is sufficiently small; that is, epsilon is much much smaller compared to I naught, we ditch the epsilon. Therefore, every 2 evaluations will reduce interval by Student: Voice not clear, leave the epsilon. Every 2 evaluations will reduce interval by half, or every function evaluation will reduce interval with 25 percent; 2 evaluations will reduce by 50 percent.

Student: Voice not clear.

Deliberately, I say 25 percent, then you get confused. I am also checking, whether you are awake. If you do 2 evaluations, it reduces by 50 percent; so, it is equivalent to saying that every function evaluation will reduce in 2, will be 25 percent. But, it is the 2 point test, you cannot have an odd number of evaluation; every time, we should have 2 evaluations, so, it is an even number. So, every 2 have evaluations, it reduce by 50 percent; it is fantastic.

In 8 evaluations, this got reduced by 16. 8 evaluations you struggle; what will be the, exhaustive search, how much can it reduce. So, the dichotomous search is indeed a extremely powerful algorithm. So, in the next class, we will revisit the cylindrical polar water storage heater problem and solve it using the dichotomous search. But, you will be dismayed; you will be dismayed to know that, there are algorithms more powerful than this. This is not a be all and end all of single variable search method, right. We will stop here.