Design and Optimization of Energy Systems

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Lecture No. # 28
Kunh-Tucker conditions contd…

So, we were looking at the Kuhn-Tucker conditions in the last class, but some students approached me and said that certain concepts are not clear. So, we will fix all the doubts today; first 10 to 15 minutes we will go through the Kuhn Tucker condition, we will try to understand it from the same example which we considered in the earlier class.

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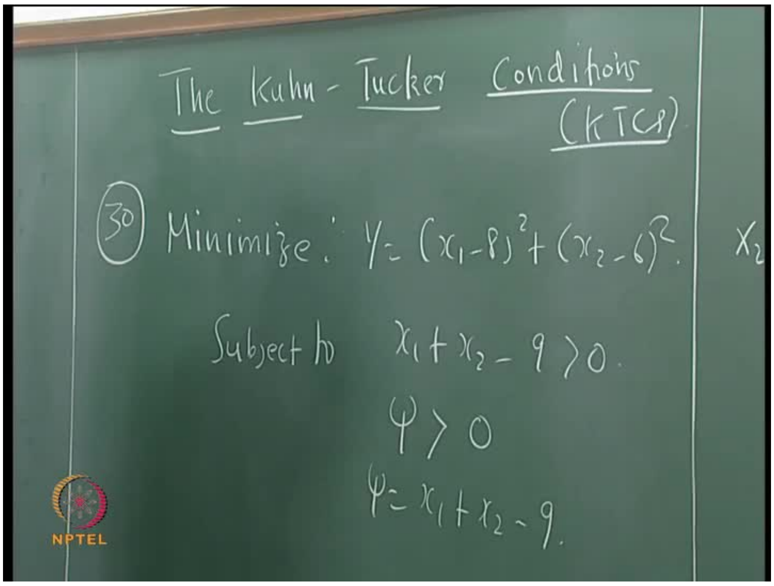

So, this is a minimization problem, this was what 31 problem number, 30 okay. Minimize Y is equal to x 1 minus 8 whole square plus x 2 minus 6 whole square subject to x 1 plus x 2 minus 9 greater than 0, I call this is greater than or greater than equal to.

Student: Greater than.

Okay, it does not matter, greater than.

So, now we can say psi equal to x 1 plus x 2 minus 9, so psi is greater than 0.

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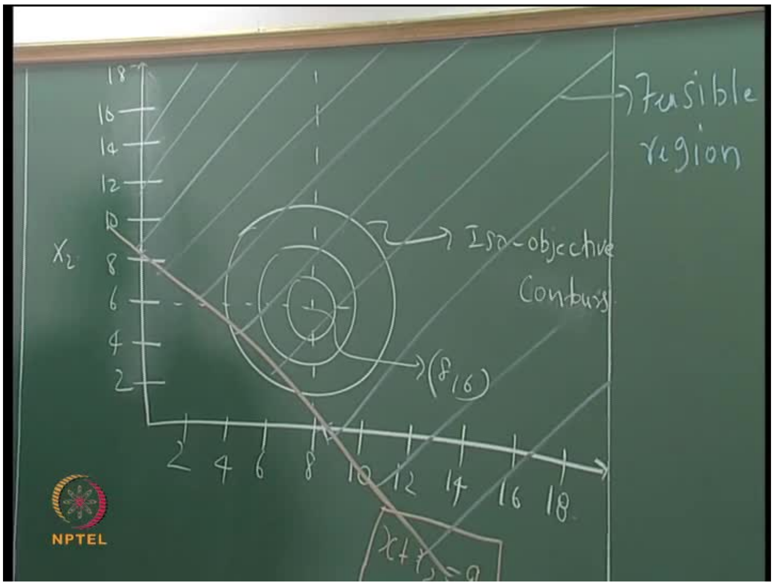

Now I plot the ISO-objective lines, ISO objective it is not ISO objective contours on the x 1, x 2, so this leads to different values of y. So, this center point corresponding to 8, 6 correspond to y equal to 0; that is the solution to the problem in the absence of any constraint. Now introduce a constraint x 1 plus x 2 minus 9 is greater than 0 or x 1 plus x 2 is greater than 9. So, we are evaluating the Kuhn-Tucker conditions and when we use regress Kuhn-Tucker condition procedure, we founded that u is negative. Since the negative value of u is unsustainable, this is not actually a binding constraint for this problem. In a subsequent recalculation this constraint can be omitted; that was the story, right.

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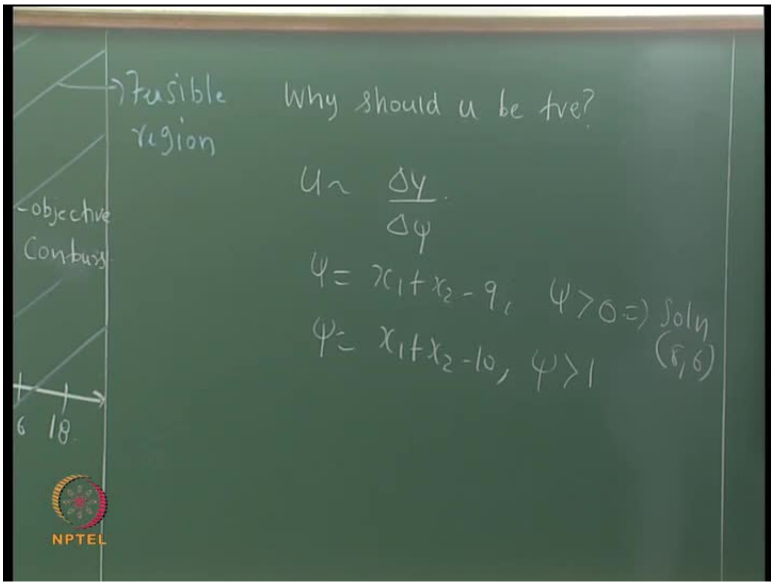

So, now we will look at it in conjunction with why should u be positive? Let us look at this example, right. Now I have said that u is del Y by del psi, psi is x 1 plus x 2 minus 9. So, psi greater than 0, what are the solutions? Solution was 8, 6, okay. Now I say psi greater than 1, so what is the psi greater than 1, where is this new line?

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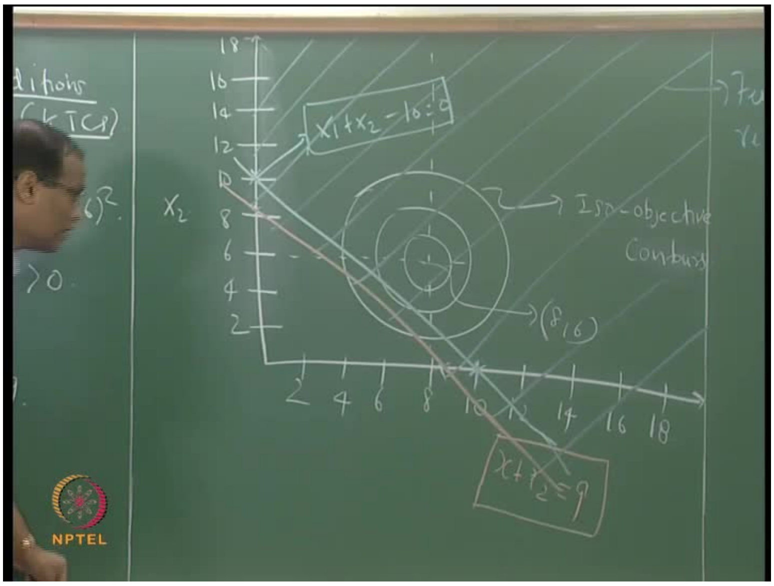

That is x 1 plus x 2 is greater than 10, where will that line be? Here correct, this line is.

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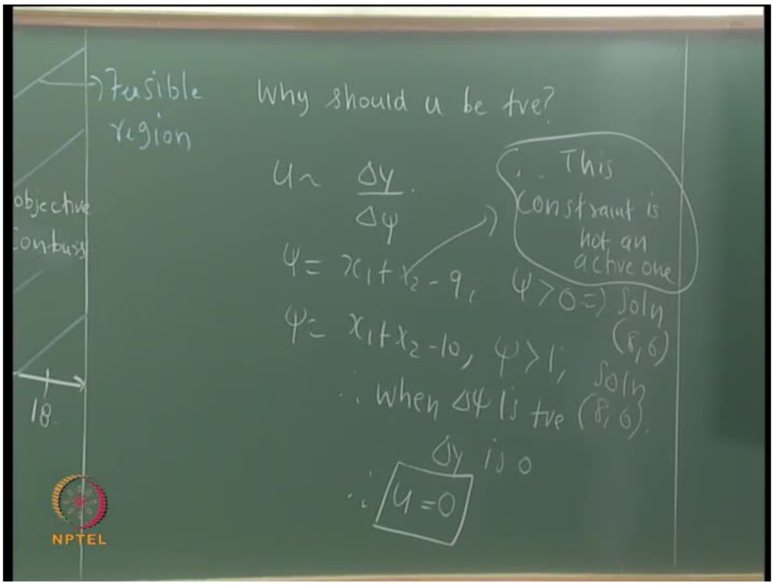

Now, what is the solution for this? What is the optimum solution for this? 8, 6, when del psi is positive that is from 0 psi was increased to 1; that is del psi was plus 1 Y did not change. Del Y was, what was del Y? 0. So, therefore u is equal to 0. If u is equal to 0 that particular constraint is inactive, you should not immediately say x 1 plus x 2 minus 18, del psi is small change in psi, you should not have x 1 minus x 2 minus 200 is greater than 0 or something, okay.

For incremental changes in psi there is no change in the solution; therefore, this is not an active or a binding constraint, therefore, u is 0. So, there is no confusion about this, okay. Let us now go to problem 31, is this clear? So therefore, this constraint is not binding; it is not an active one. Now I only partly answered your question, if the constraint is not active psi equal to 0, but I have not answered the question if the constraint is active, why u should be positive, u is 0 not psi.

Student: Feasible region is reduced sir. Can’t we reduce in such a way that the new Y optimum can be less in the old Y optimum.

I am proving it know, you come out the situation where it will violate; it is a greater than it is an inequality that is why it will not work, for an equality it will work, right, we are putting an inequality condition.

Student: If you take it as x 1 minus x 2.

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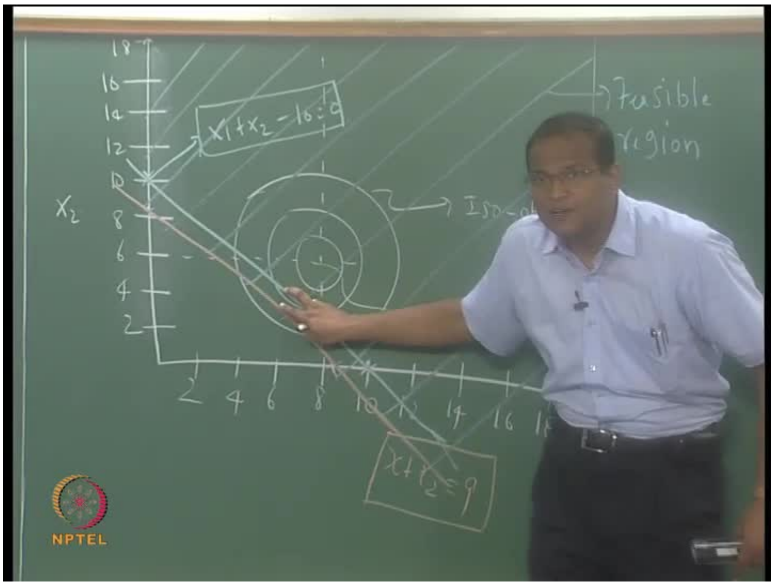

You try whatever combination it will not work, you can try, people have tried; for so many decades you can try. Now we will go to some when the fusible region is reducing the new optimum can at best be equal to the old optimum but cannot be lesser than that for a minimization problem, okay. So, what is actually happening here is compared to psi greater than 0 psi greater than 1 you lose certain points, you lose certain portion of the feasible region, okay.

Therefore, any solution which satisfies psi greater than 0 will anyway satisfy psi greater than 1; therefore, we cannot hope to find a new optimum which is having a value of y which is less than what would have been found by psi greater than equal to 0. In this case it is so happen that even psi greater than 1 you have still having the same optimum, but now if I change to x 1 plus x 2 minus 18 is greater than 0 the situation is different. So, since the feasible region is reducing it is but logical that you can have the y outside the new feasible region, but you cannot ever knew Y which is lower than what could not be captured by less restrictive feasible region.

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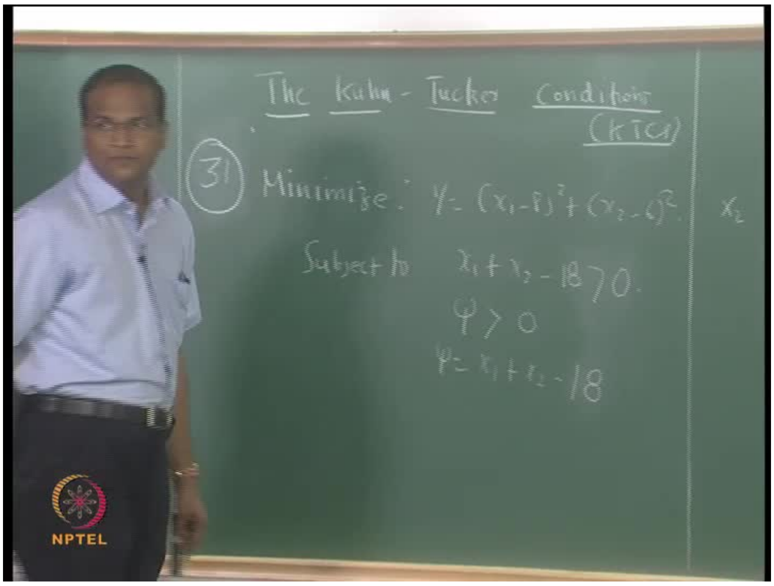

Now let us do 31 subject to shall we redraw the whole thing? Ok, fine.

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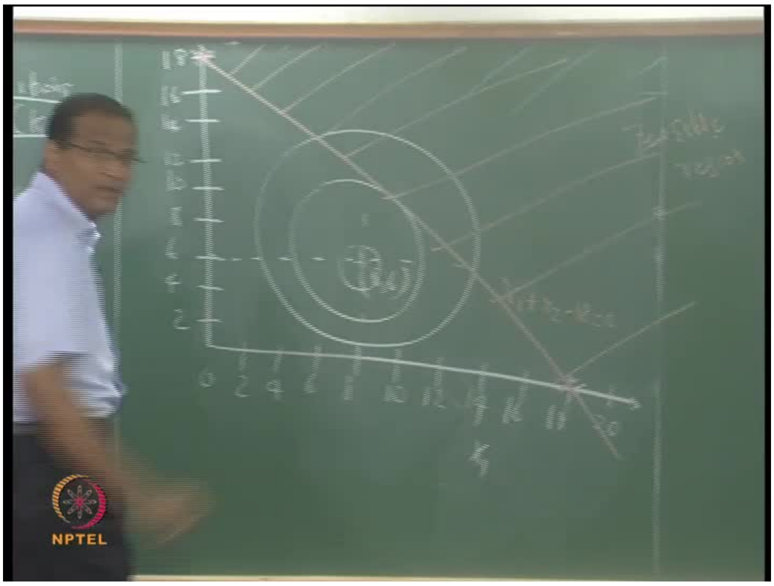

Now let us work this problem out. 2, 4 x 1, I am going to have trouble, right? So, I have to joint this 18 and 18, okay. So, now if you take this 6 and there are some ISO-Y contours; what color we used? Now I have to joint this right. So, this is 0, so this is a feasible region now. The original solution 8, 6 is now below this feasible region, so it has cut; it has cut this ISO-objective curve in such way that the original 8, 6 is lost. It is no longer a valid solution to the problem because it violates the inequality constraint x 1 plus x 2 minus 18 is greater than 0, right fine.

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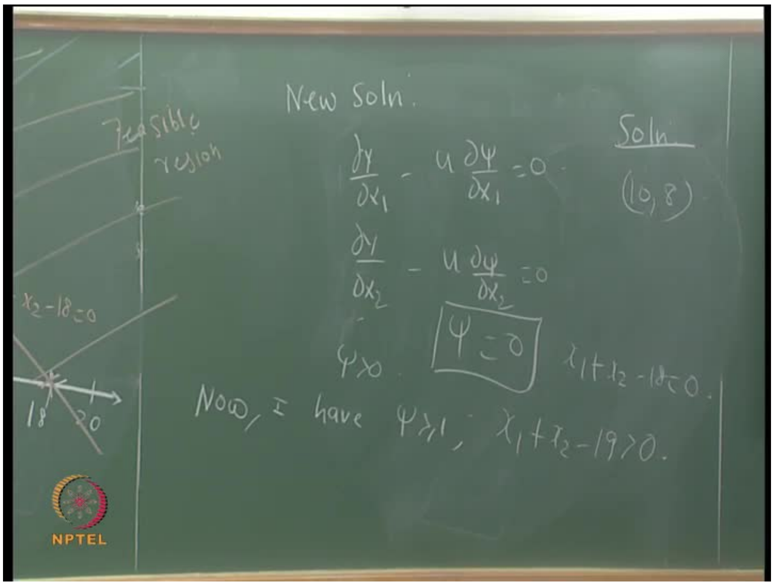

Now what is the new solution? New solution we worked out, we can do this, right, dou psi equal to 0 where psi is x 1 plus x 2. What are the solutions? So, solution is 10, 8. Now let us say that instead of psi greater than 0, now I have psi greater than equal to 1.That means x 1 plus x 2 minus 19 is greater than 0; that means I am cutting the feasible region further.

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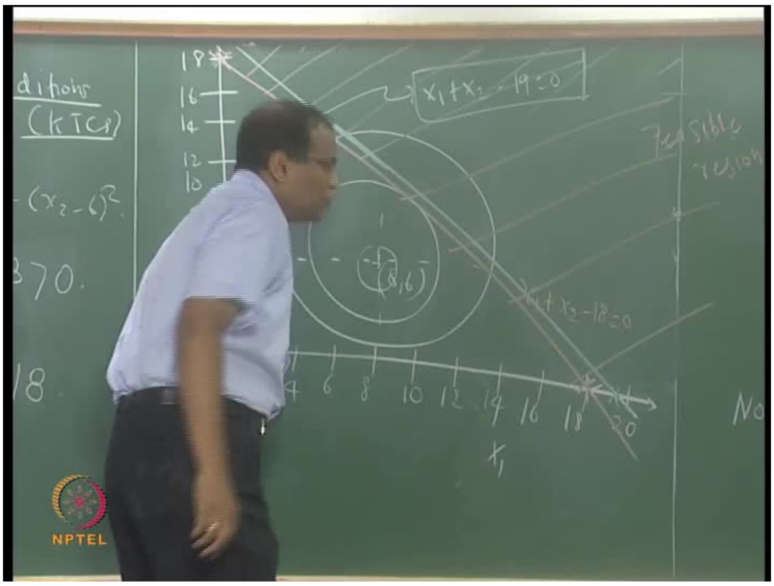

So, I am not able to show 19 here. So, this is, so I can rework, we can rework.

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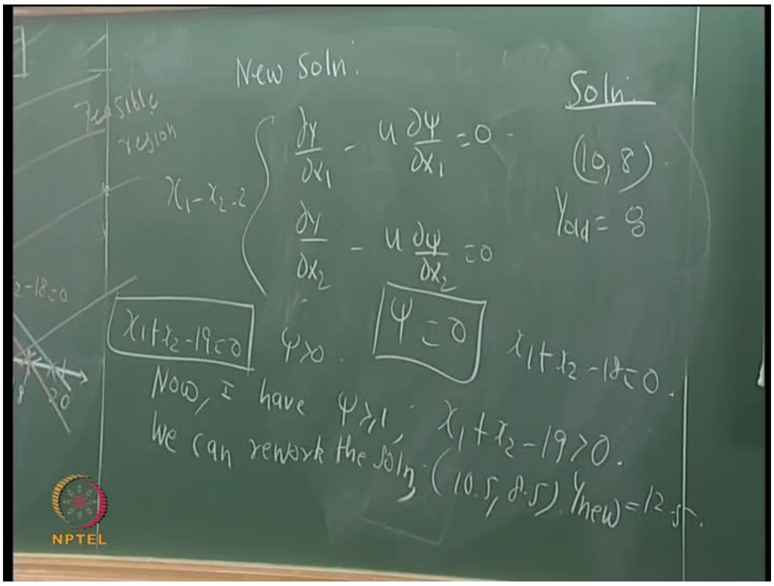

You have the basic from framework; can you tell me the solution? 10.5, 8.5, correct, yeah, what is this? This 2 x 1 minus 8 minus u equal to 0, 2 x 2 minus 6 minus u is equal to 0. If you combine these two x 1 minus x 2 is equal to 2, right. This one says x 1 plus x 2 minus 19 equal to 0; therefore, this fellow has to be 10. 5, this has to be 8.5; it is so simple, right. So, the new solution is 10.5, 8.5. What was the Y old for this x 1 minus 8 whole square plus x 2 minus 6 whole square; yes, what is the problem?

Student: X 1 plus x2 minus 19 greater than 1, x 1 plus x 2 minus 19.

X 1 plus x 2 minus 18 is greater than 1; therefore, x 1 plus x 2 minus 19 is greater than 0; you have to be alert. So, now for this the Y old is 8, what is the Y new? 12. 5.

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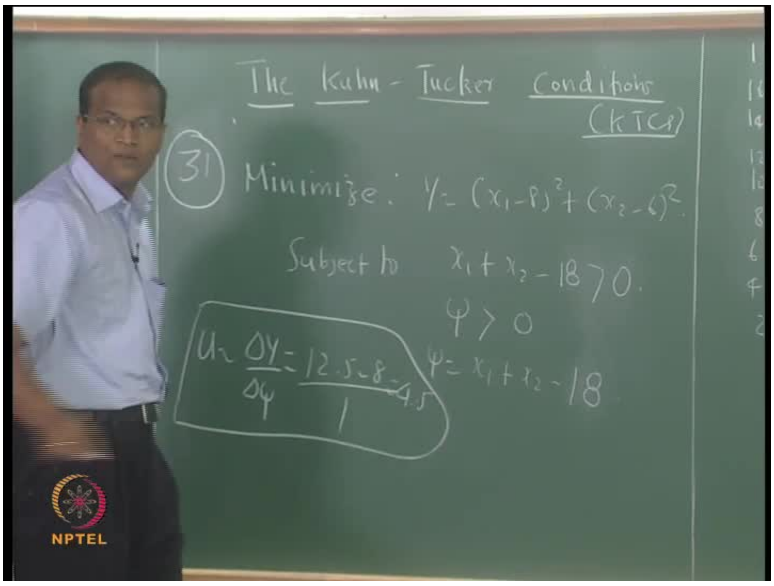

Now tell me u is equal to del Y by del psi is equal to 12.5 minus 8 divided by 1 that is 4.5. So, when it was an inactive or a non-binding constraint, the value of u was 0. When it is an active constraint u is positive, there is no other possibility for u. Therefore, u has to be greater than equal to 0 which is the last condition, the condition which follows the complementary slack condition complementary slackness condition, okay. So, I come again while lambda the Lagrange multiplier is unrestricted in sign, u has got to be greater than equal to 0. Now it is clear why u is positive? So, we will close our discussion on Lagrange multiplier with this; we will start a new chapter now. Yeah, the story is therefore u is positive, right.

Now we start off new chapter search methods, okay. What is the key point? The key point in search method regardless of whether it is a calculus base method or search method, there is always an objective function, you have a set of constraints, you also have bounds for the variables and all that. You want to solve the optimization problem; you want to find out the maximum or minimum of a particular function. But for some reason it is very difficult or you do not want use the Lagrange multiplier because there are too many variables, it is not differentiable, blah, blah, whatever.

You are seeking an alternative route to solving the optimization problem, because you have access to big computers; for example, you are good at programming and this thing or the number of variables is too many, the Lagrange multiplier method is unwieldy.

Now what will be the key point if you start using a search method? What will you first do if you want to use search method, what will be the cardinal principle of the key point? Not reduce, I am just asking for starters, for starters how you will start?

Choose a point or choose some points, some point means if it is x 1, if it is a two variable problem x 1, x 2 pairs you will choose, that may be 1, 2, 3, 4, 5, whatever. You will calculate the value of y at each of this point? Among these 5 points you will decide in which way y is increasing or decreasing and then choose the best point among this, then you can build some other points around this, okay. So, on what basis you decide which point is better and which point is not better; it is completely based on the value of the objective function. There is no place for d y by d x 1 minus lambda d psi by d x 2 or derivatives do not come in to picture at all.

In fact, the Lagrange multiplied method we solve the problem even in quizzes and exams I have seen people solve the problem and forget that ultimately you have get the value of y, they forget. They just say x 1 is equal to 4, x 2 is equal to 6; the problem does not end there. What is the y corresponding to that value of the combination or variable? So, in fact the calculation of the objective function is relegated to the end generally in the calculus method. You are obsessed; you are more or less worried about where the function becomes stationary, where the function becomes stationary. But in real life sometimes the function will not become stationary at all, because the function itself may not be differentiable. Allocation of job, you have got a milling machine CNC layered this thing, NC machine all this thing, and you want the minimum cost or you want to have the best allocation for a two specialist and all that.

We cannot have a curve and differentiate and all; that is all nice mathematically but it is not possible to implement. That is why it is engineers who have developed genetic algorithms, simulated and attending all these things where it is possible for me to repeatedly get the values of y for several combinations of x 1 to x n. However, it is impossible for me to fit a function which could be differentiated. Why should calculus be always a route to solving problem? Why should every function be differentiable? Why? That is one way of looking at things; that is not the only way of looking at things, okay.

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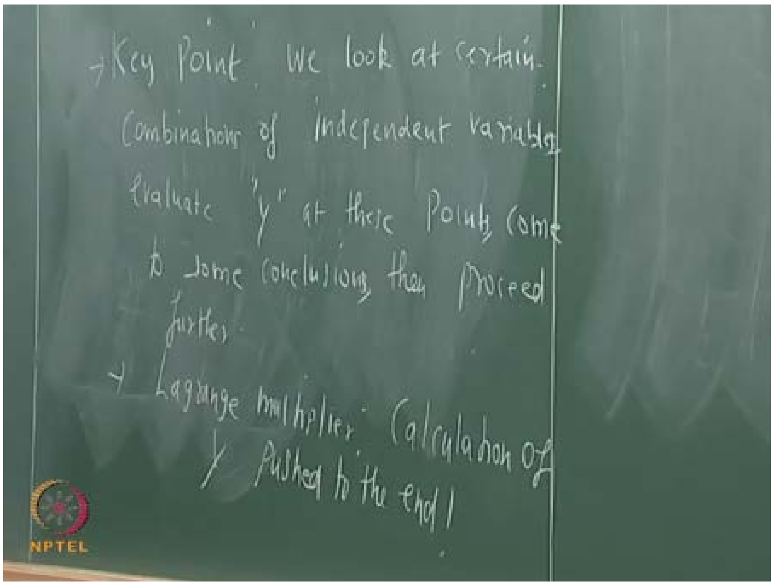

So, the key point is we look at certain combinations of variables, evaluate y at these points, come to some conclusions, then proceed further. Lagrange multiplier calculation of y, push to the end; this is more physical. Each and every iteration we are seeing how Y improves or not that does not improve and all that. Of course there are so many issues if it is multiple local minima and all that how do you handle, we will systematically go through all that, okay. So, this looks like an engineer approach some problem; an engineering approach to problems where it is not possible or we do not want to take that derivative, right. Now what is the difficulty with search method?

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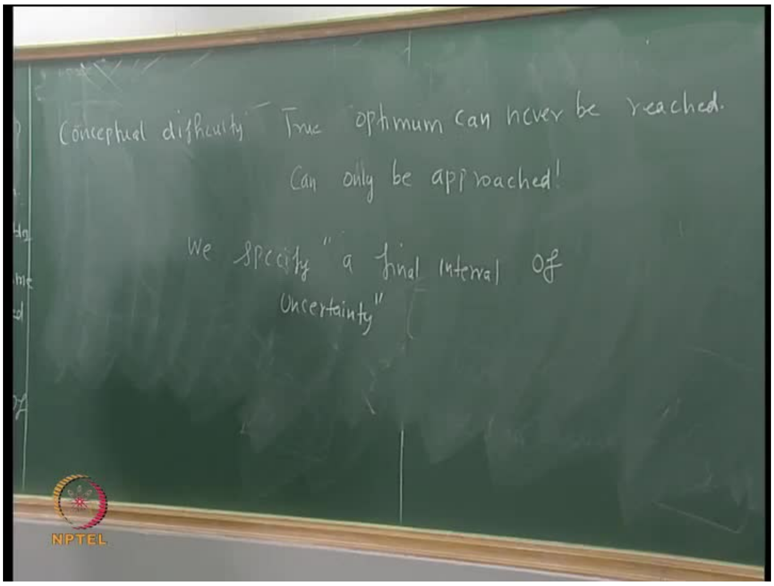

Conceptual difficulty, true optimum can never be reached. You will say that the optimum could lie between 1.4 and 1.5 it is an accuracy of; for example, the optimum radius of the cylindrical solar water heater storage problem varies between it lies between 0.8 and 0.9; that is if you decide an accuracy of 0.1 meter. If you say no, no, no, I want an accuracy of 0.01, we will say between 0.85 and 0.86, but you are always giving bounce because you are not trying to solve d y by d x equal to 0. True optimum can never be reached; true optimum can only be approached. But that is ok we are ready to live with it, okay.

So, we specify a final interval of uncertainty; that is we say that why optimum lies between these two values of x or between these two combinations of x 1 to x n. So, this is a key concept the interval of uncertainty. So, the search method essentially now you can comprehend that the search method will essentially be in iterative technique where you originally start which some interval of uncertainty which may come from your background knowledge or from engineering this thing or from pervious knowledge, right. You have to start, you cannot say that cylindrical storage water problem I will start with 1 micron to 300 meters. It is after all a storage tank it has to be kept outside; you cannot talk like a mathematician.

So, you will say that at least it should be 10 or 20 centimeters or 30 centimeters; it cannot be more than 4 or 5 meters. You bracket it and then start using the search method. So, your original interval of uncertainty is specified before starting the problem. Then you use the search method and hopefully with every iterations the interval of uncertainty will go down. So, as the number of iteration increases the interval of uncertainty will go down. So, we will now try to understand this concept of interval of uncertainty with an example.

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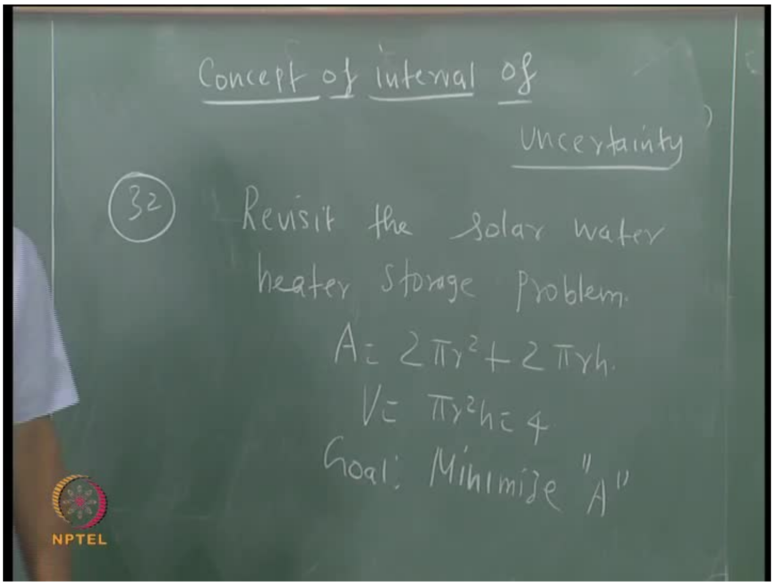

Let us start off with the problem, problem number 32. Revisit the cylindrical revisit the solar water heater storage problem. So, A equal to 2 pi r square plus 2 pi r h, V is equal to pi r square h is equal to 4. Goal: minimize A because minimizing A minimizes the total heat transfer; that is it minimizes the heat losses, so it can be used during the night time. So, we made some approximations to the heat transfer coefficient as constant, the t unit is constant and all that.

This can be treated as single variable problem and you can solve it by directly by taking d A by d r is equal to 0 as a single variable problem or as unconstraint or you can treat it as unconstraint optimization problem and use Lagrange multiplied method. You can treat it as a constraint optimizing problem using Lagrange multiplier method and get an additional parameter called lambda which is the sensitivity of A with respect to v; all these concepts you have already studied. Now we will see how to use search method.

There is no need to use search method for this problem, but we are always trying to benchmark, we are always trying to compare with the Lagrange multiplied in this case, because we know what the true solution is; that is the standard way of we have a frame of reference. Now we want to treat it as a single variable problem in r, then we will take reasonable values of the original interval of uncertainty for r, then we will work out and let us see how the solution proceeds, okay.

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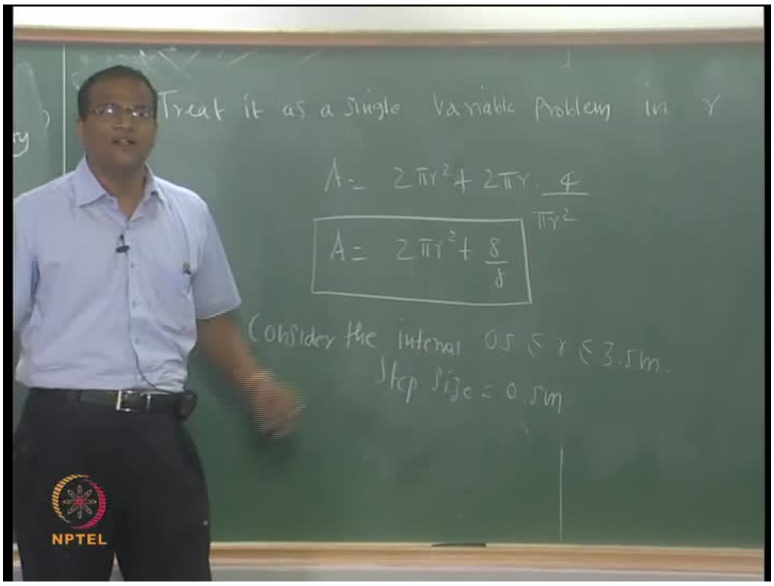

So treat it, so A will be equal to, is it right? Now, not 0 is too bad, 0 it will be singularity. So, consider a reasonable interval you except the solution to lie between 0.5 meter at least it should be 50 centimeter because it is a 4000 litter tank; otherwise, the height will be too much 3.5. This is basically to start from somewhere and let us say step size is 0.5 meter. The problem is like this use the single variable search given this interval and step size and get a new interval of uncertainty. The problem is like this, for the cylindrical, for the solar water heater storage problem, convert it to single variable unconstraint optimization problem in radius r and start with an original interval of uncertainty 0.5 to 3.5 with step size of 0.5 using the search method; this is called the exhaustive search method, using the exhaustive search method arrive at the final interval of uncertainty.

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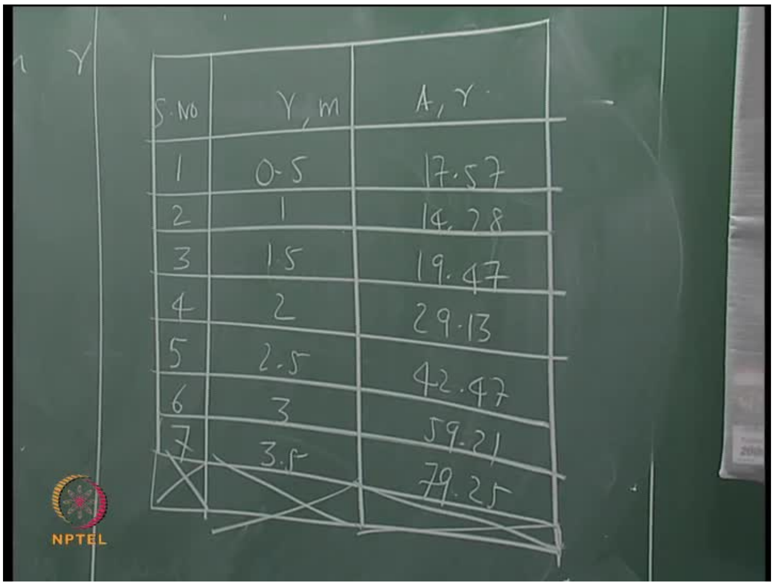

To those people who do not know how to start, you have to just draw a tabular column and then use commonsense. How many divisions are there? 1, 1.5, 2, 2.5, 3, five divisions, how many function evaluations are there? Seven functional evaluations, five intermediate points and two end point, so I need 8 columns here. So, serial number, r, A of r; I got 8 is it, okay, the last one is not required. Yeah, please get me the values of A. You should just take 3 to 4 minutes that is all, then we will also plot it.

Should I take attendance now, so that you fill up the values, yes start?

So, please tell the values 0.5, 17.57.

Student: 14.28, 19.47, 29.13, 42.47, 59.21, and 79.25.

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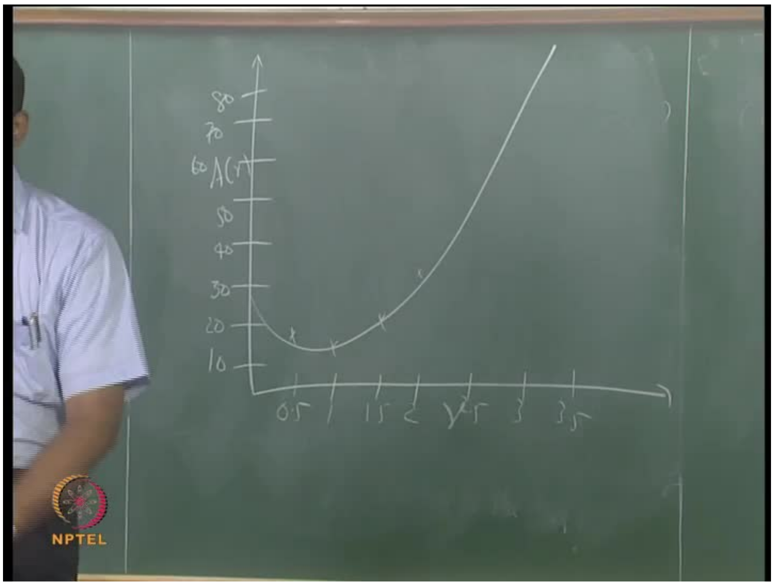

So, we will do it approximately; 0.5 it is about 17, 14, then it goes up 1.5 to, okay. Now what is the funda? What is the funda?

Solution lie 0.5 to 1, no, what is it?

Student: 0.5 to 1.5.

Solution lies between 0.5 to 1.5. We do not know whether the minimum is reached between 0.5 and 1 or 1 to 1.5, but between 0.5 and 1 it takes a turn; from 17 it became 14 it again became 19. Therefore, the culprit is between 0.5 and 1.5, I cannot say it is in 0.5 and 1.

So, what did we achieve? We did seven functional evaluations and reduce the interval of uncertainty from 3 meters to 1 meter; it is not great but it is not very bad also, okay. So, what we have done now is called the exhaustive search method, okay. So, equal interval exhaustive search method; you divide the whole interval into so many numbers of points, you had intermediated points and then you had two boundaries, you evaluated the function and you could bracket, you could say that the optimum now lies between this values of r and this values, okay. So, we will say this.

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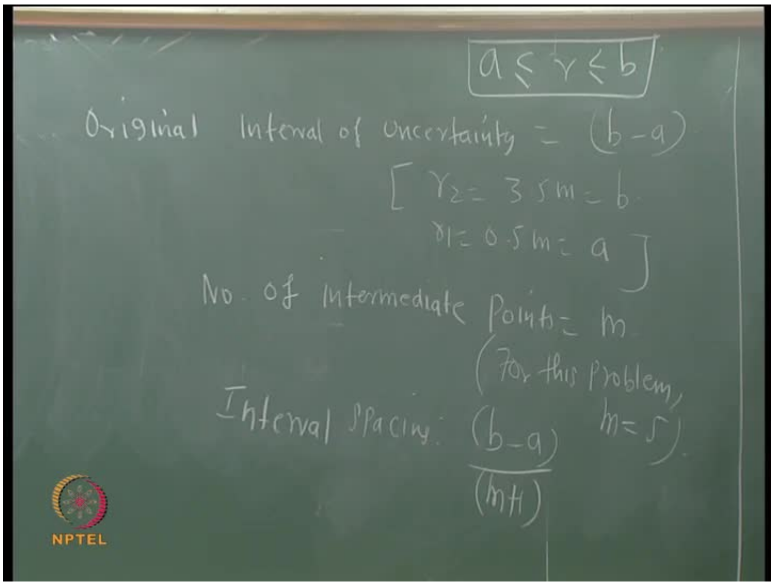

r 2, b a and b are the end points. So, we are evaluating a less than equal to r less than equal to b. Number of intermediate points equal to m, for this problem m equal to 5. Vikram what is the problem, is it ok, fine. So, interval spacing 3.5 meter minus 0.5 meter, 3 meter divided by 6; that is equal to m plus b minus a divided by m plus 1, correct. What is the new interval of uncertainty? b minus a divided by m plus 1 into 2 by half, two times b minus a by m plus 1, what is b minus a plus m plus 1?

Student: 0.5.

0.5, but what is actual final interval of uncertainty? 1 meter because it is between 0.5 and 1.5 meter; therefore, it is two time space, therefore, final interval of uncertainty.

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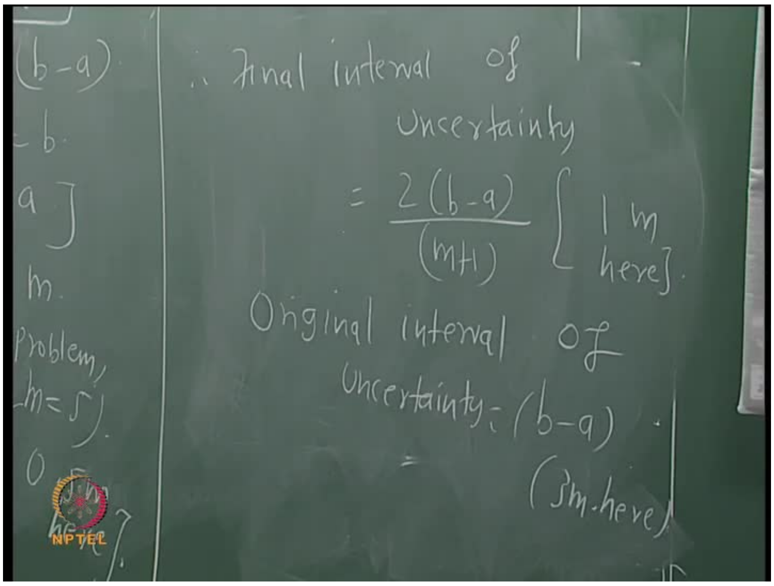

So, this is 0.5 meter here; here means in this problem, final interval 2 times m plus 1, right? All equal space, what was the original interval of uncertainty?

Student: b minus a.

b minus a, why cannot you say in the language of mathematics? Original interval of uncertainty was I am trying to derive the general relationship; original interval of uncertainty is b minus a.

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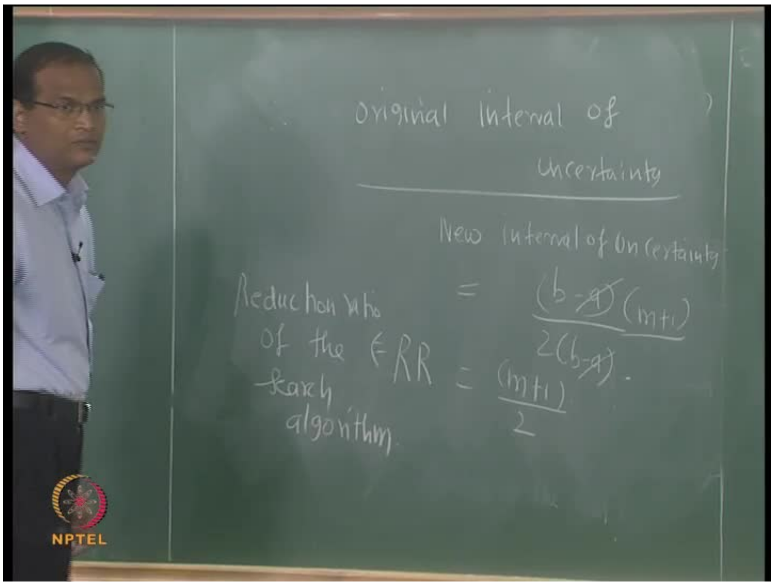

Therefore, original by is equal to, correct. This is called as RR; RR is the reduction ratio of the algorithm. How much the interval reduces? RR is the reduction ratio, it is a performance metric just like mileage for your car, CGPA for the academic performance, RR gives the power of the algorithm in reducing the original interval of uncertainty. RR has to be read in conjunction with the number of observations. If you have m observations the reduction ratio is m plus 1 by 2 if you use exhaustive search method. Therefore, scientist would not have kept quiet; they would of developed more and more advanced methods where for a given m observation you will have a reduction ratio RR which is far superior to what could be achieved by the exhaustive search method because the exhaustive search method is almost stupid. It is least unimaginative to say the least, but in the absence of any other technique for starters it is ok.

You can start with it. If you do not know anything about the nature of the function and all that, you can use the exhaustive search method to bracket between 0.5 and 1. 5, you have done that. After you get 0.5 and 1.5 you can use very sophisticated methods and quickly get to 0.866 meter which is the original solution to the problem. Therefore, this RR is the reduction ratio of the algorithm of the search algorithm, okay. What is the relationship between m and the number of observations? Number of intermediate point is something which I do not like; I would much rather work with the number of observations; that is number of observation is a number of functional evaluations.

Why we are saying observations? It could also be experimental. In this case we just got it as 2 pi r square plus 2 pi r h. Sometimes each of this a could be the output of a finite element program from Comsol or for Fluent, from Fluent or some experiments or could be the data from a cardiologist, radiologist, oncologist, whatever, you trying to seek maximum minimum something; therefore, we are interested in the number of observation.

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Therefore, if n is the number of observations n equal to how much here? In this problem n equal to I mean 7, n is equal to m plus 2. Therefore, RR is n minus; see some books will give n plus 1 by 2, some books give n by 2 and all that. When n is 100 or 200 or 300, it does not hardly matter whether it is, okay; just like when you work out the standard error of the estimate divided by the root of some n minus 2 and all that. I told you know because n minus 2 is the number of degrees of freedom for a linear fit because two points; you do not have any freedom if just have two point. Like that, when n is 100 or 200 it hardly matters whether it is n by 2 or n minus 1 by 2. Can you solve this same problem using the same exhaustive search method in a little smarter fashion? Same funda, same logic, exhaustive search, little more smartly, so that sometimes there is hope that with less than n minus one observation, you can.

I do not want any fundas exhaustive search.

Student: It is to optimize the number of m.

You should not write one more program to optimize m.

Student: Here m equal to 1, just take a midpoint.

If you take the midpoint already it is some other algorithm, interval psi everything is same number of points everything is same. So, what we can do is instead of taking seven functional evaluations instead of dividing you divide it into five points intermediate five points and have seven evaluations. You can do the same thing and start form the left hand side and take only three points at a time, right. You have 0.5, 1, 1.5, 2, 2. 5, 3, 3.5, you can start looking at 0.5, 1 and 1.5 f of x 1 is greater than f of x 2, f of x 2 is less than f of x 3, your home but it is fortuities. It is fortuities that at one side of the interval, you got it; you will get orgied if you start from 3.5 using my method.

You start to 3.5, 3, 2.5, Anand you did not quite get in? If you start with 0.5, what I am saying is take three points at a time; logic is very simple take three point at the time 0.5, 1 and 1.5. If f of x 1 is greater than f of x 2 but f of x 2 is less than f of x 3, then the optimum lies between 1 and 3. Terminate the program else x 1 is equal to x 2, x 2 is equal to x 3, x 3 is equal to x 2 plus delta x; that is you will take points 2, 3, 4, find out, go to points 3, 4, 5, till you go to the end point. When you finally reach the end point, still you are not able to get an optimum; either there is no optimum to the problem or the optimum lies in one of the two boundaries.

This is a very another smart way of writing the same algorithm. On an average, if the solution is as Venugopal says likely to be around the middle, it may take only half the number of observations which I have put here; this is what is called as the worst case scenario for the exhaustive search. But if you adopt the method which I have discussed in the class, you are already dividing it into intermediate points and you a priori your upfront you are evaluating all, but beyond 1.5 it was not required. But before starting the problem you do not know whether it is 1.5 or 2.5 or 3. So, an alternative approach is start three at a time. Yes, Vinay what is your problem?

Student: Two places if it turns one being less than all of them.

What, what, what?

Student: Suppose basically if there is a local optimum.

Then there is a local optimum and all that. So, I am coming to it. Now I have to start discussing what is a unimodal function and what is a monotonic function and all that? Shall we write the other algorithm now and then we will close? Yeah, somebody has a doubt? Yeah go ahead, is this clear now? So, I will just write the algorithm.

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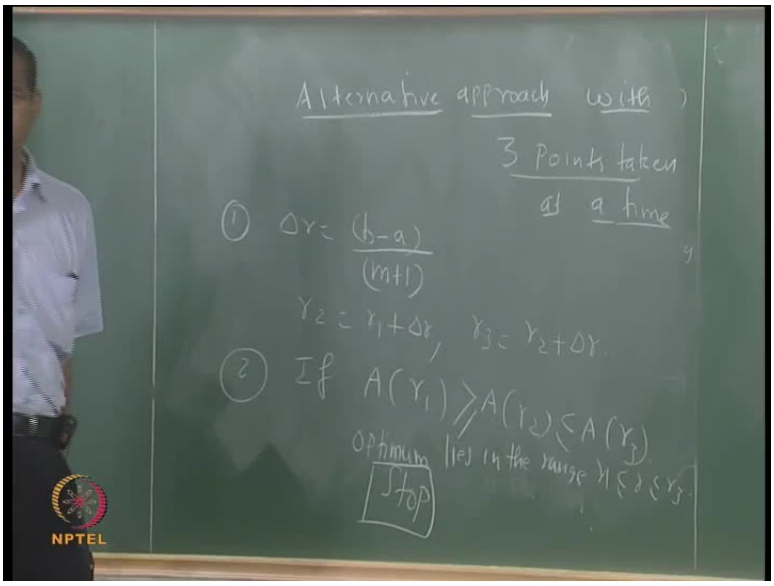

Alternative approach, so delta r, okay. So, please correct me if make a mistake m plus 1, m minus 1 or n plus 1, am I doing fine, okay.

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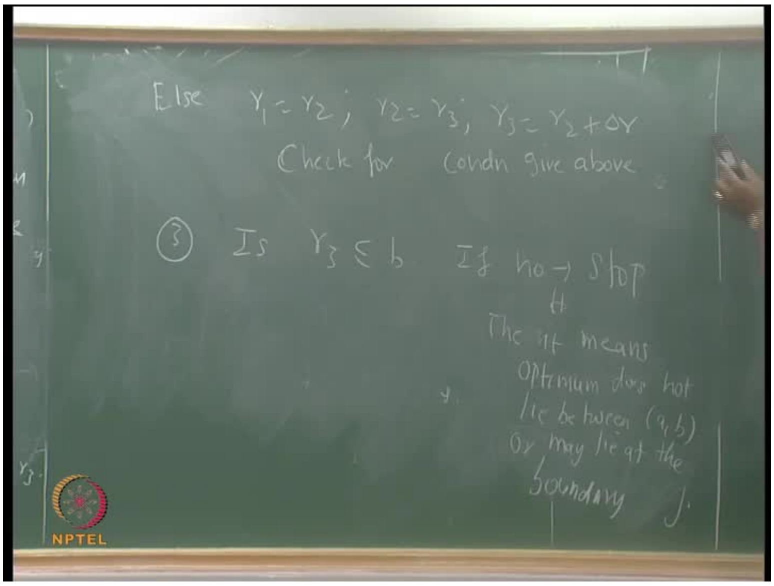

Now stop else r 1 is r 2, right? From programming point of view whatever I am saying is right; r 2 equal to r 3, r 3 is equal to r 2 plus delta r, but this is new r 2 man, okay. Already I changed it; it is in the language of programming. Check, then what happens? If it is yes, we will go back. So, if no stop, then it means r may lie at the boundary.

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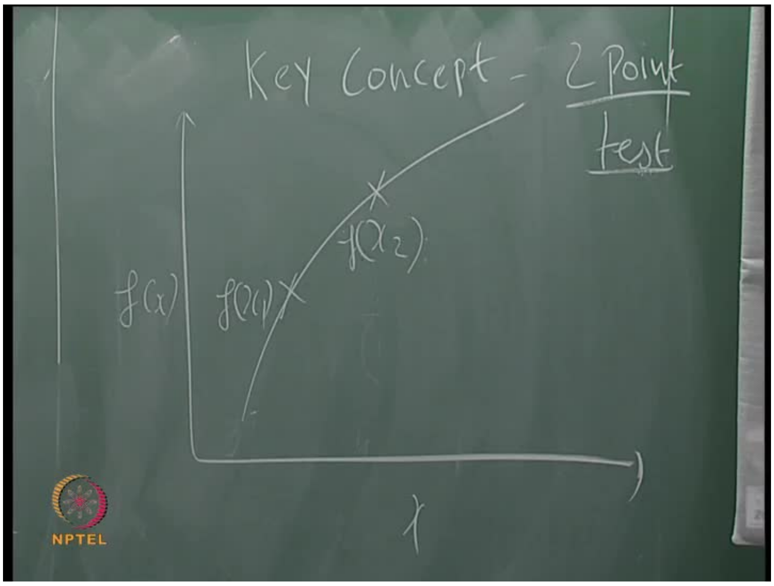

So, essentially what we are doing is this is f of x 1 and this is f of x 2. If you are looking for a maximization problem if f of x 2 is greater than f of x 1, so basically key concept 2-point test; suppose you take two points which will become clear when we look at other search techniques. If you are using a 2-point test what it means is you take two values of x, calculate the value of f of x at these two points. If you are actually looking for an optimization problem, if f of x 2 is greater than f of x 1, what does it mean?

The region to the left of x 1 can be eliminated because it may also go like this. But I am sure that since it is a monotonic function, I will tell you that what are the conditions for monotonocity and all that. If it is a monotonic function what I can do this when I do a interval search, when I am applying a search algorithm, I can confidently say that the interval all values of x to the left of x 1 can be eliminated but whatever is to the right of x 1 has to be retained, okay.

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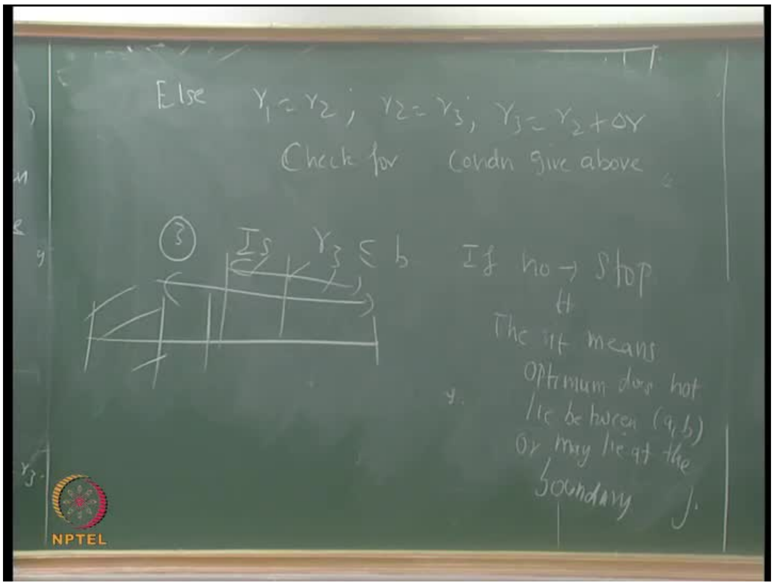

So, if I have an interval like this if I have two points, then I can eliminate one portion, this will get an eliminated, then I will have two points one will get eliminated, I will keep on cutting it. So, obviously you can see that when I propose the 2-point test, I am talking about some search algorithms which will be much superior compared to the exhaustive search because if I choose the two points very close to the center, I can cut 50 percent of the interval right way with just two functional evaluations; you can never hope to do that using the exhaustive searching, okay.

Why so much fuss about a single variable optimization? When you know in life it is only multivariable problems which you encounter. The point there is most multivariable optimization problems are reduced to single variable optimization in the particular variable. If it is x 1, x 2, x 3, you will optimize with respect to x 1, x 2, x 3. Therefore, the key to the whole thing is the development of an efficient single optimization technique. Therefore, all papers, most papers, most research has focused on single optimization techniques for a single variable problem. So, we will meet tomorrow.