Design and Optimization of Energy Systems

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Lecture No. # 27

Handling in-equality constraints

Last part of this topic on Lagrange multiplier will be the Kuhn-Tucker conditions; the KTCs which help us extend the concept of Lagrange the method of Lagrange multipliers even to problems involving in-equality constraints.

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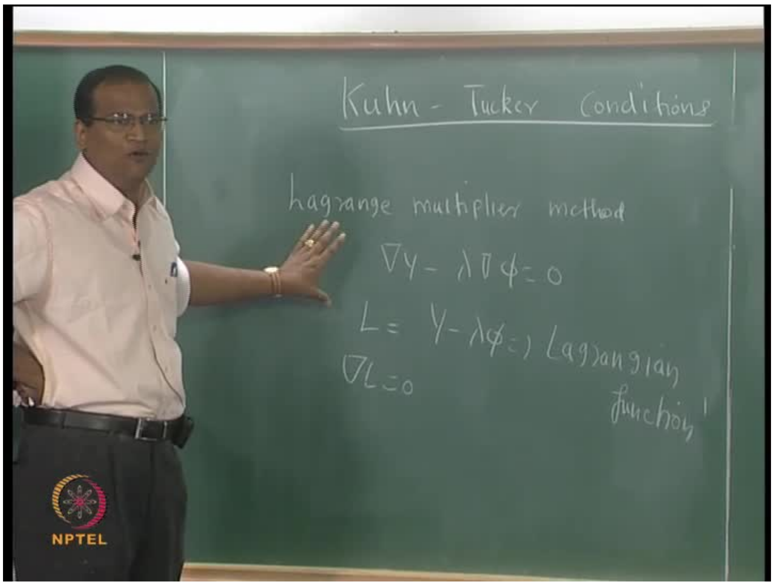

Basically this Lagrange multiplier method, so we are actually doing. So, actually it is analogous to defining a function L which is Y minus lambda phi and we are trying to make it stationary, okay. Then you will get, you could solve for del L equal to 0 and then you will get all the answers in terms of lambda and technically it is possible for you to adjust the value of to find tune the value of lambda such that you get realistic solutions. But because we do not want to do those exercises, apart from making del of L is equal to 0 you also add phi equal to 0 and solve it solve the n plus m resultant equations to get m Lagrange multipliers and n values of x 1 to x n, right.

So, this L is called the Lagrange function. Some people like Ravindran and (( )) they will say that this is analogous to converting the constrained optimizing problem into an unconstrained one; though I do not fully agree with this because the constraint is still hanging around there. If they are saying that once you put this lambda del phi, you have made this you have incorporated the constraints, but still this alone if you solve you will get all the answers in term of lambda. You have to actually use an additional equation phi equal to 0 for one constraint problem to get the value of lambda also apart from the other things, right.

So, this is in so far also L one other concept yet another concept is we are actually trying to solve for the Lagrange function where the Lagrange function is basically Y minus lambda of phi; this is basically one way of looking at it. Operations Research people will look at it this way. If you look at any OR book when they talk about Lagrange multiplier, they will introduce a Lagrange function or Lagrange function first.

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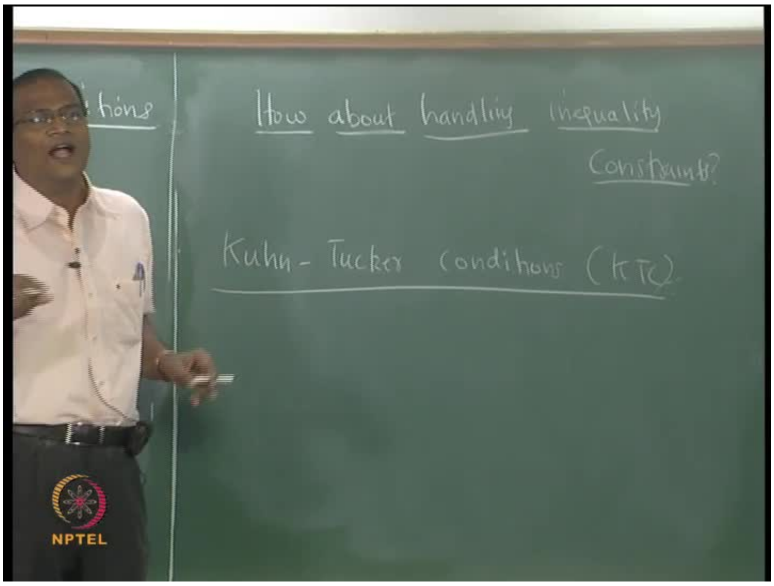

Now how about handling in-equality constraints? The simplest approach would be to ignore the in-equality constraints and solve the problem. Ignore the in-equality constraints and solve the problem with equality constraints and the resulting solution you find out whether it violates in-equality constraints or not. If it does not violate then it is no longer an active or a binding constraint. If it violates then convert the in-equality constraint into an equality constraint and put it as an additional constraints in your

Lagrange multiplier formulation and solve it, and then any way that constraint will be obeyed by the solution which we eventually obtain. So, had we known this information in advance, if we had known which of these constraints which are active or binding, there is no need for us to go through this exercise.

In advance we would have made some in-equality directly into equality and then completed our solution. But there is no way appropriate upfront, you know whether a particular constraint is active or binding. So, this problem was thought about by Kuhn and Tucker and then they finally came out with their process of mathematics at Princeton University. So, they came out with these Kuhn-Tucker conditions, so KTC or KTCs. Before going into what these Kuhn-Tucker conditions are, something from point of view of history, Professor Tucker is no longer alive. He was the PHD adviser of John Nash; John Nash who won the PHD for economics in 1994 who was the subject of this movie 'A Beautiful Mind'. So, he was the adviser of John Nash, then Kuhn was a contemporary of John Nash; Kuhn is a contemporary of John, he is still alive and Kuhn continues to work in Princeton University.

He was a mathematics consultant for that movie, right. He was largely responsible for them making this movie and also largely responsible for nominating Nash for the Nobel Prize. These two guys have done a lot of work in Operations Research and Tucker has worked on basically game theory. In game theory, there is something called prisoner's dilemma, right. Some of you are I cannot go deep and there is a prisoner's dilemma, two fellows are caught; they are kept in separate cells, they have to now vouch. If both of them keep quite they will get minimum sentence, but if they orgy they will get more sentence and then one fellow, so what should they do? I mean what they should do that people have come out with lots of these thing and lots of papers have been written that is a prisoner's dilemma. Tucker has worked a lot on this prisoner's dilemma and so on. That is a historical perspective, but what are these Kuhn Tucker conditions. So, we have to now formulate the optimization problem with in-equality constraints also.

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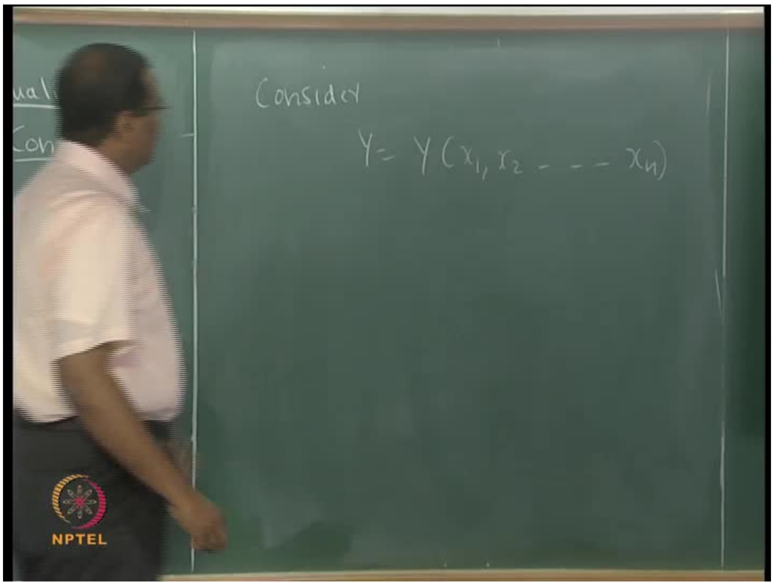

Consider they also developed the Hungarian method for the assignment problem. There are various problems, transportation problem, travelling salesman problem; that is what is called as the beaten to death problem; just like my cylindrical water storage which I will beat to death, by the time we are done with the course. So, assignment problem, transportation problem, north-west corner route, you start from the north-west. Why cannot you start from south-east; that will also give the same answer. What is so great about north-west, we can start from any corner. Anyway, they call it is as a north-west corner rule and all that. So, he has worked on Hungarian method assignment problem.

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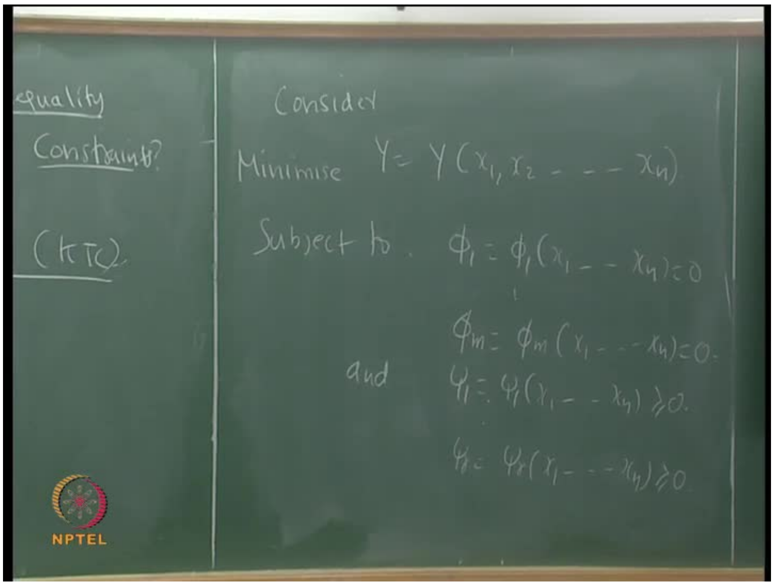

So, consider Y equal to x 1 to x n subject to and greater than equal to 0. All these fellows are now starting up to psi r where psi’s are the in-equality constraints. See I am writing this for a minimization problem minimize. We can have an analogous formulation for the maximization problem also, why minimization greater than equal to constraint is there? For the furniture company making tables and chairs or something, if you want to minimize the cost you do not make anything at all; I mean but that is a very trivial solution. So, these greater than equal to constraints ensure that entropy is greater than 0 and some activity is generated. So, if entropy has to be 0 everything has come to a standstill.

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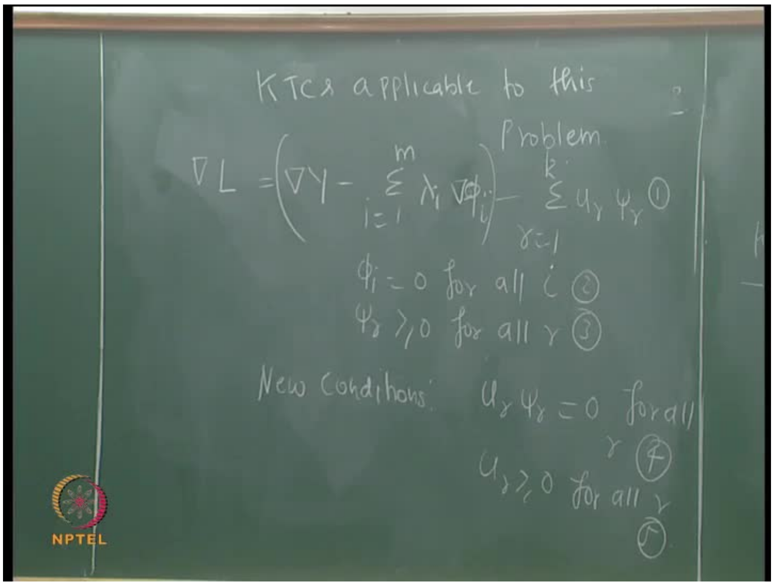

Subject to now we have to write the KTCs applicable to this problem, okay. Now it involves some tricky less than equal to greater than equal to. So, we will go through this very carefully. So, it is analogous, so this is the Lagrange function which you want to minimize, okay. Please note that up to this portion it is same as the Lagrange multiplier method. I have handled the objective function Y, I have handled all the constraints equality constraints for which phi equal to 0 and it is applicable for i equal to 1 to m. Now I put one more minus r equal to 1 to k, what did we do? Already there is I will make it k here.

I thought about all these things j, m, k, something we have to do, alright. So, I have put r, r is equal 1 to k u r psi r where psi r basically represents the in-equality constraints and u is something which is similar to the Lagrange multiplier; it is a sensitivity coefficient whose nature is not yet known for us, right.

Now we have to write, we can call this as equation 1, then phi is equal to 0 for all i that is i equal to 1 to m, then psi r is greater than equal to 0 for all r. So, I will put is as 2 I will put is as 3.

Student: It is that lambda a del phi or lambda a.

Yeah, what is that? Yeah, yeah lambda a, now comes the new conditions. Up to this there is no problem right, except that you got something extra in terms of psi which is an in-equality constraint, but so far there is nothing which is new to you. But now we will introduce some new things, basically u r psi r is equal to 0 for all r and u r is greater than equal to 0 for all r. Do not worry; I will explain all this, okay.

For a maximization problem a minus of Y has to be minimized. If you want a maximum of Y, you can do minus of Y that has to be minimized. This is a very important condition introduced by Kuhn and Tucker, okay. This four means either psi equal to 0 or u equal to 0 or both are 0; I come again. So, it is a very important condition either psi is 0 or u is 0 or both are 0. If u is equal to 0 there is no problem at all, u is the sensitivity of the objective function with respect to that constraint. If u is equal to 0, we are home. It is no longer an active or binding constraint, because it no longer affects the solution if u is equal to 0. We are worried about those cases where u is not 0. If psi is equal to 0, then it is a problem because it is in-equality constraint it is a strict condition, equal to is a strict condition.

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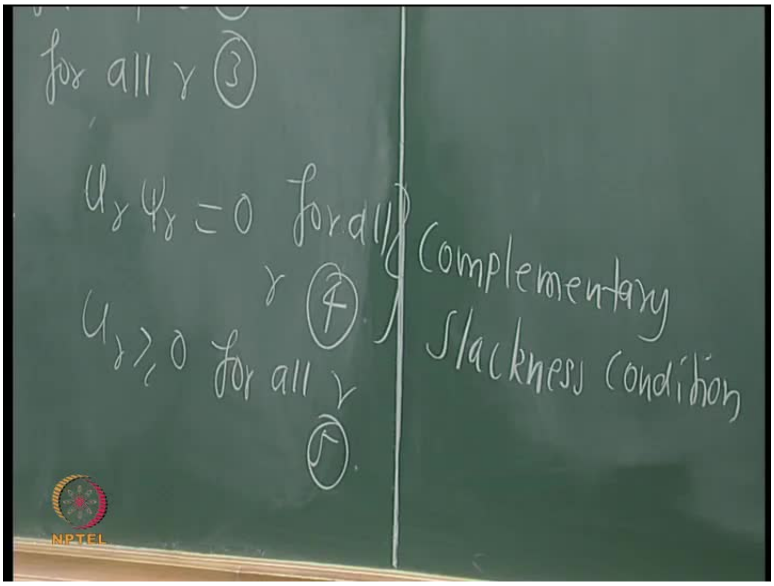

So, there is a complementarity involved in this, either this or this. So, this is called the complementary slackness condition.

Student: Objective is equal to zero.

In priority you do not know, you do not know whether psi has to be considered or not. So, Kuhn and Tucker evolved some conditions where after going through one iteration it will let you know whether it is binding or not. After you go through the example you will become very clear, okay. u’s are very similar to lambda’s; watch very carefully u’s are very similar to lambda’s except that there are not unrestricted in sign. Lambdas are unrestricted in sign; lambda’s can be 0, lambda can be negative or lambda can be positive. It is possible for you to have an increase in the objective function with an increase in the constraint or a decrease in the constraint or no change in the objective function. However, the u’s are restricted in sign; the u can adverse by 0 or it has to be more than 0 because it accompanies an in-equality constraint, it does not accompany an equality constraint, okay.

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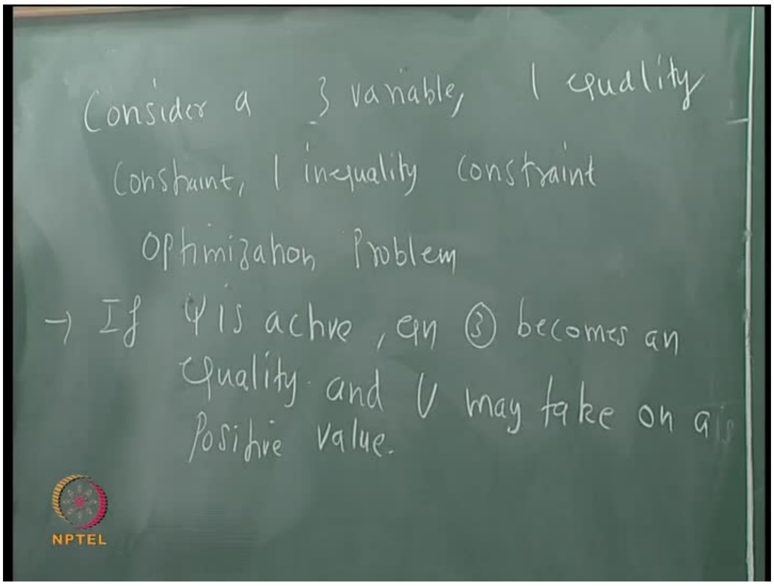

So, now let us consider a three variable, one in-equality constraint optimization problem. I will give you an example and it will become clear. That means there is one psi and one phi I have made.

Student: In-equality constraints should also have certain. No, he is saying that u r into del psi r.

Here, I thought you are talking about the other thing.

Now if you consider three variable two constraint problem in which one is an equality constraint, one is an in-equality constraint, if psi is active equation 3 becomes an equality and u may take on a positive value. This is basically the KTC, right.

So, I come back to the same point again; if you want to handle Lagrange multiplier, if you want to use Lagrange multiplier method for a problem with m constraints and which are non-linear but which are differentiable, it is possible for you to completely ignore the in-equality constraint and proceed with the solution and see whether it is violated or not. If it is violated then you have to reinsert the in-equality as equality and proceed; next time it will be obeyed. Alternatively you can use the KTC condition, assume that a particular in-equality constraint is active, solve the resulting set of equations, apart from lambda you will also get u. Look at the nature of the u; if u is negative, your original assumption that psi is active is wrong.

Therefore, that can be omitted from the calculation in the next iteration. How many of you did not follow this? The funda behind the KTC is like this; when you start with the solution, if you want to assume that a particular psi is active you just go head and assume that the particular psi is active and treat it; if it is active means you treat the in-equality as an equality, then come out with the complete solution. There may be other equality constraints in the problem, but this in-equality is also treated as equality.

So, for this psi you have an appropriate u which will give u 1, u 2, u 3, whatever. If it is only one psi, let us assume that there is only one psi you get a value of u at the end of the solution. If u is positive then it means it is binding; u is positive it is binding. Because then when psi is increasing the Y is also increasing you are seeking a minimum to the problem. Therefore, it is binding and what you have done is correct. If you have assumed psi is active, you have proceeded with the solution u is negative, then your assumption that this is an active constraint is incorrect. So, it is telling you from the next iteration onwards you can drop this constraint and proceed. This is the KTC.

In any case in either case your r g d in either case, it is an iterative process; you do not know, you have to do iterations and find out whether the particular constraint is active or inactive, okay. So, we will demystify the Kuhn-Tucker conditions with examples.

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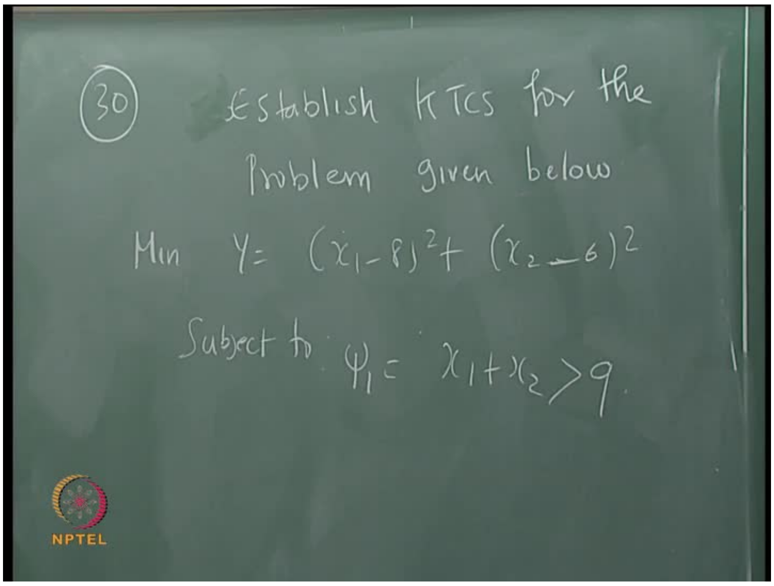

Problem number 30, okay; consider establish KTCs for the problem given below. Min, it is the same circle problem where Y are ISO-objective lines and basically square of the radii. Now I have give an additional constraint x 1 plus x 2 must be greater than 9. So, this is an in-equality constraint. So, we want to use the KTC. The first step in technologies assumes that it is active constraint and see whether u is positive or negative. Can you do this? It is pretty straight forward. Then I will draw x 1 x 2, I will draw ISO-objective lines and graphically we will see whether it is binding or not. It is a straight forward problem, right. What is your guess, what is the solution? 8 and 6, x 1 plus x 2 is greater than 9 is below the solution or above the solution? I mean it is below on to the left of the solution, it should not affect we will just see.

In this case it is obvious, so you will feel the KTC is not required. In many case it is not so obvious; therefore, the KTC is very helpful. Let me draw the solution on the graph, depict the solution graphically.

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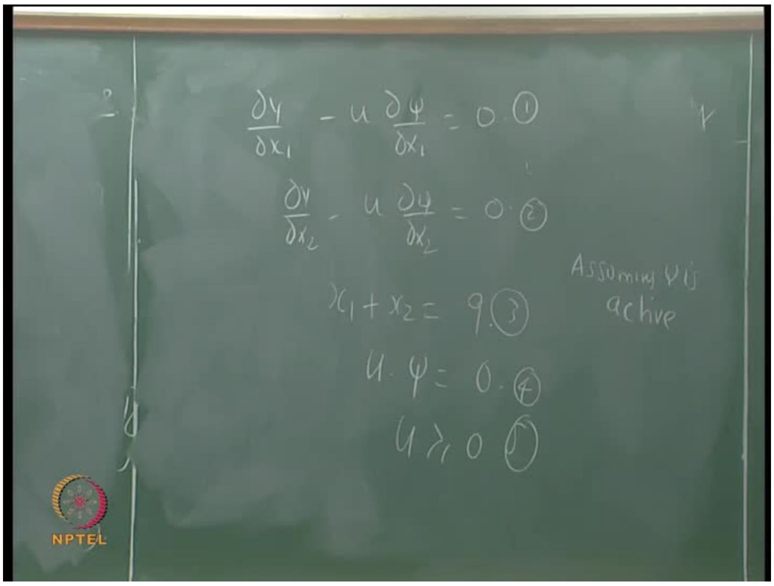

Let us start with the KTC for this dou Y, okay. Now assuming psi is active, this will be equation 3. Then u into psi equal to 0, u greater than equal to 0. These are the five equations. So, this gives you the KTCs.

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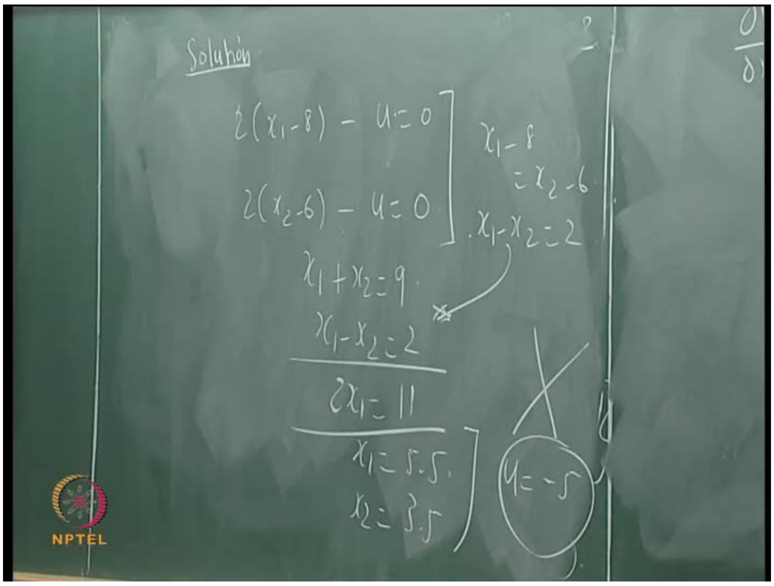

Now solve now solution. So, what is the solution? 2 x 1, that is all right, equal to 0. Am I doing all right? Sampath, what is u?

Student: U is minus 20.

Totally wrong, u cannot be negative, that is the KTC. Therefore, your original assumption that the psi is an active constraint is incorrect; therefore, psi does not affect the solution to this problem, okay. This is also graphically beautifully seen.

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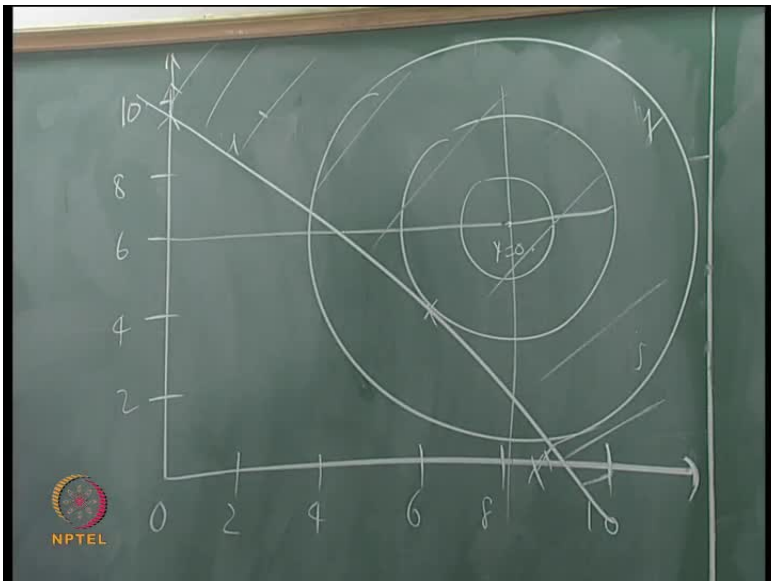

So, the center point here Y equal to 0, where is this fellow x 1 plus x 2 equal to 9? He is here, right, okay somewhere. So, you are talking about this region. When it cuts you get some other solution, you get this solution for which y is equal to 4 or something. How much is the value of Y? How much is the value of Y for this?

Student: 6.25.

6.25 plus 6.25.

Student: 12.5.

6.25 plus 6.25 12.5. I have a solution for which y is equal to 0 which does not violate x 1 plus x 2 is greater than 9. So, it is absolutely wrong. So, the KTC has helped me identify that by putting x 1 plus x 2 is equal to 9 as an active constraint; I made a major mistake, all right. Let us look at it, is it clear to everybody now? How many of you still have issues with KTC? Hari is it okay.

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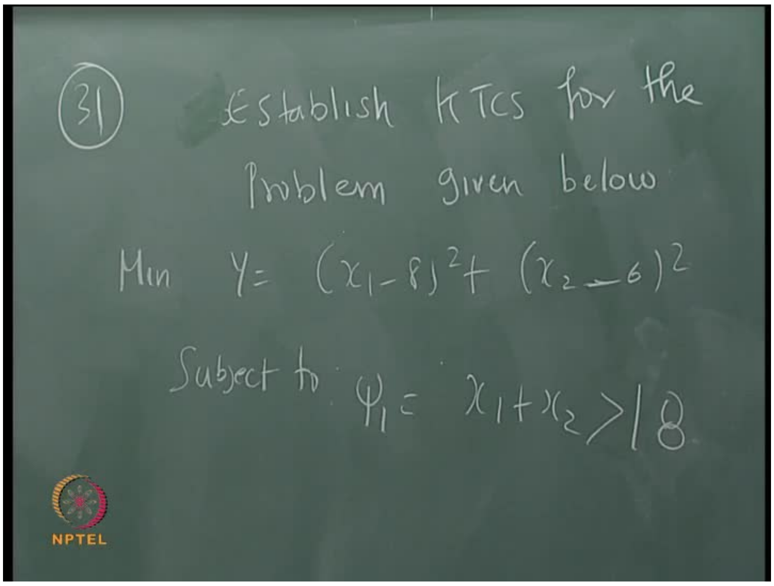

Now let us rework the problem for x 1 plus x 2 greater than 18.

U comes out to be 0 but psi equal to 0 equal to 0.

Student: U is negative and psi is zero by the other condition u r psi are equal to zero.

If u is equal to 0 it is not at all acting on a binding constraint. There is no sensitivity of the objective function; there is nothing more you can talk about it.

If u is 0 it means Lagrange multiplier is 0 means what are you saying? See some people call the u also Lagrange multiplier, because when you actually find the value of u you are treating it as equality; so strictly it is also Lagrange multiplier. If u is equal to 0, there is nothing. It does not cause any disturbance to the objective function.

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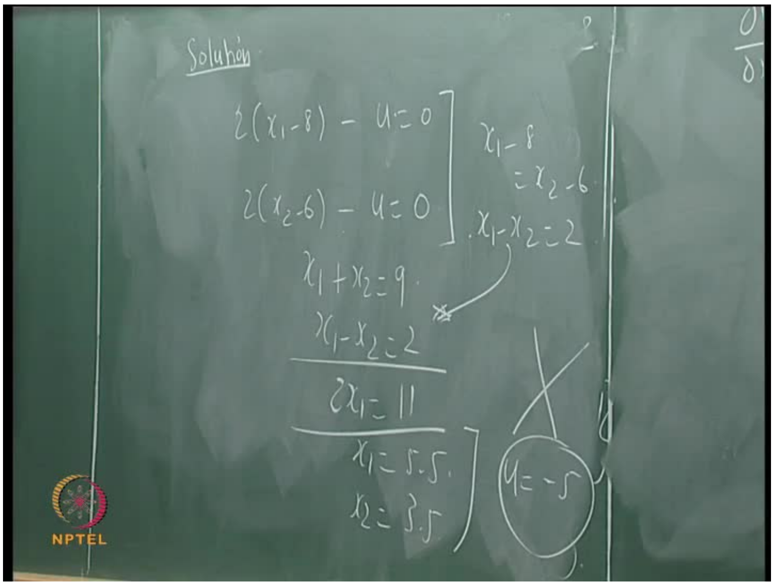

Now please rework the problem, all the other things is pretty straight forward; up to this the steps are the same, right. Abhishek, you got it? You got the idea of the whole thing.

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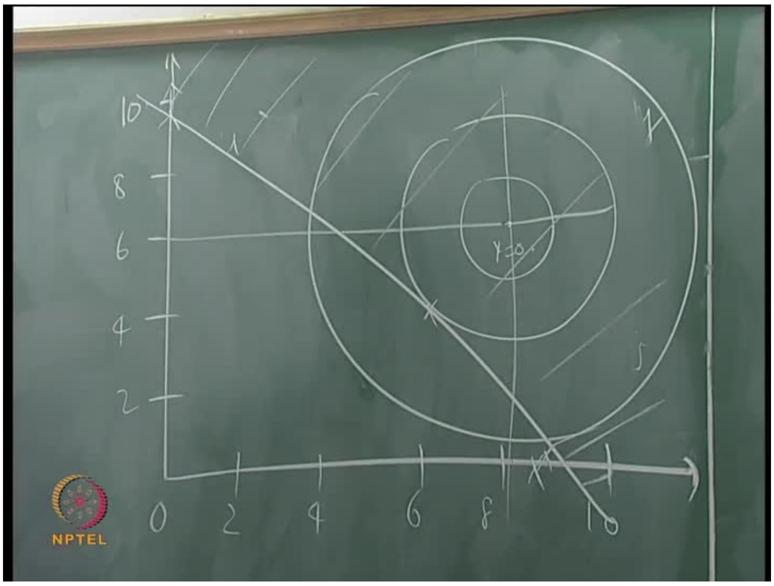

Now this will be so evident. I think what we should do now if you are smart what we should do is for 31 you should assume that it is not active and get a solution and then prove that you made a mistake; therefore, you have to consider, that is all right. Otherwise, take it as binding and you will get a value of u which is positive; that means your original assumption is correct, fine.

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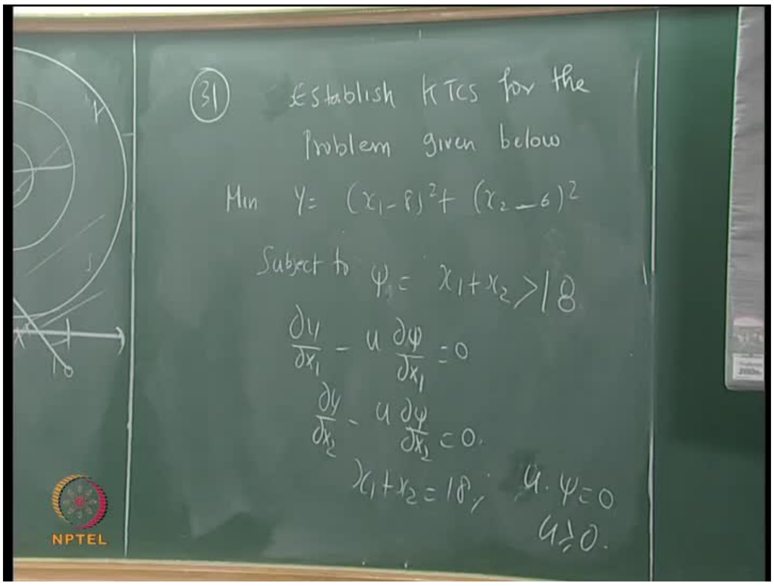

Now, let us do this. So psi, then psi u r u into psi is, correct?

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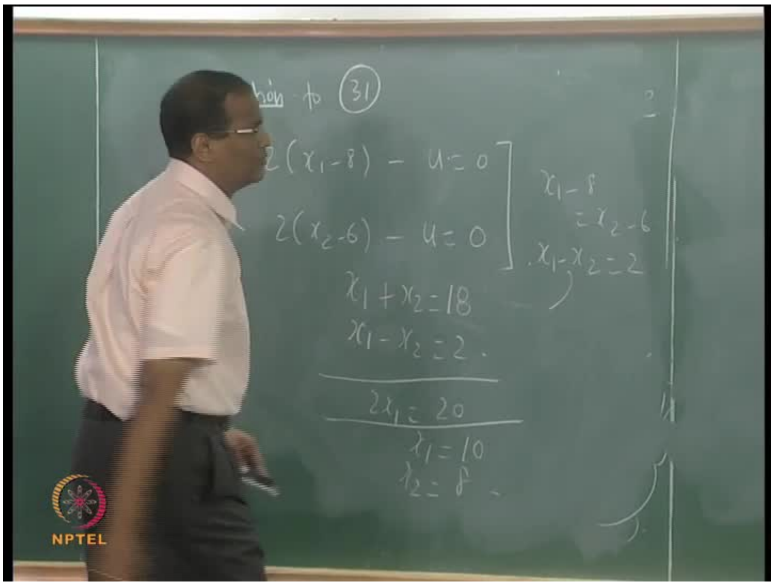

x 2 is 8.

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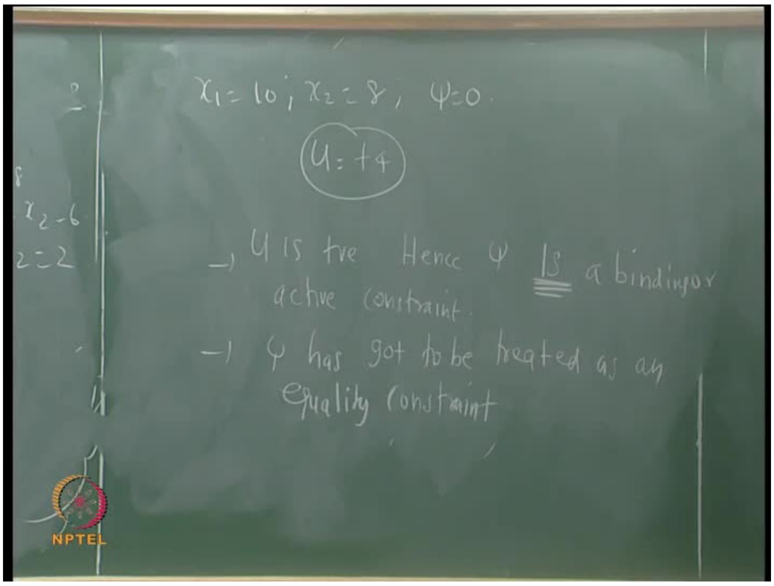

Now, I think we have to redraw this. So, you get x 1 equal to10, plus 4. That is the logic. So Kuhn and Tucker, they nicely put common sense in a formal mathematical way and then established this.

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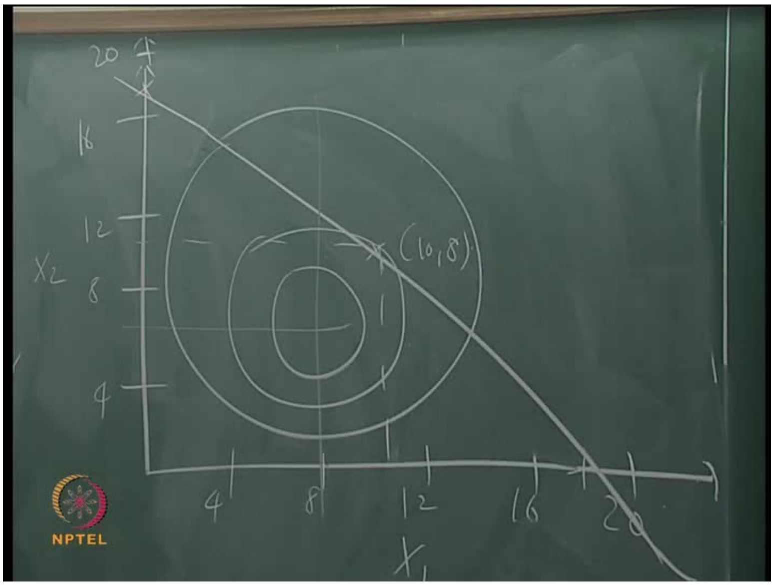

So, we have this 8 6 here, right. So, this is somewhere it cuts, 12? Okay, 10 somewhere.

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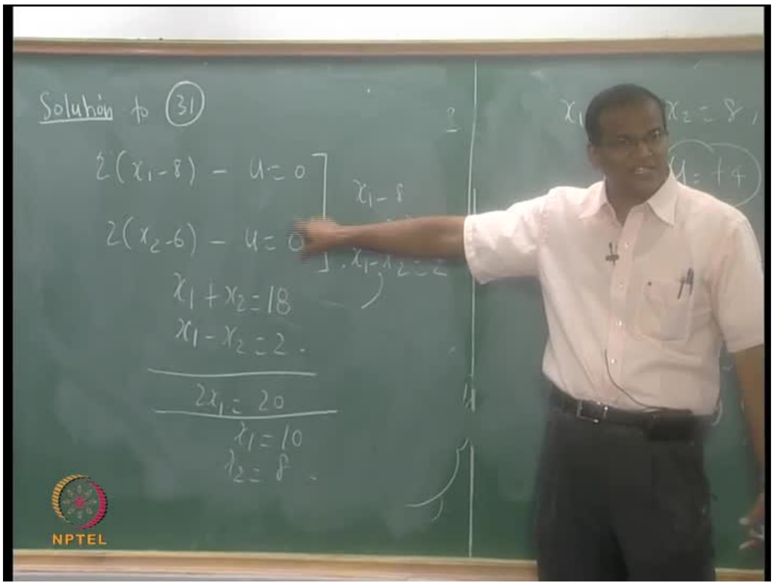

So, now the two approaches alternatively you can disregard the constraint and find out whether if you disregard the constraint what is the solution you are getting? 8 and 6, if you substitute 8 and 6 this will violate that constraint and then what is the value of u you will get?

No, if you do not consider the constraint. Minus, therefore it is incorrect, therefore your assumption that the constraint is not binding is not correct.

Student: How do you get here minus sir?

You did not catch the point. There is one more way of doing it, what I am saying is if you disregard the constraint you get the solution 8, 6. Proceeding with 8, 6 if you try to calculate the u and other things, the u will lead to a negative answer u which is again incorrect, right.

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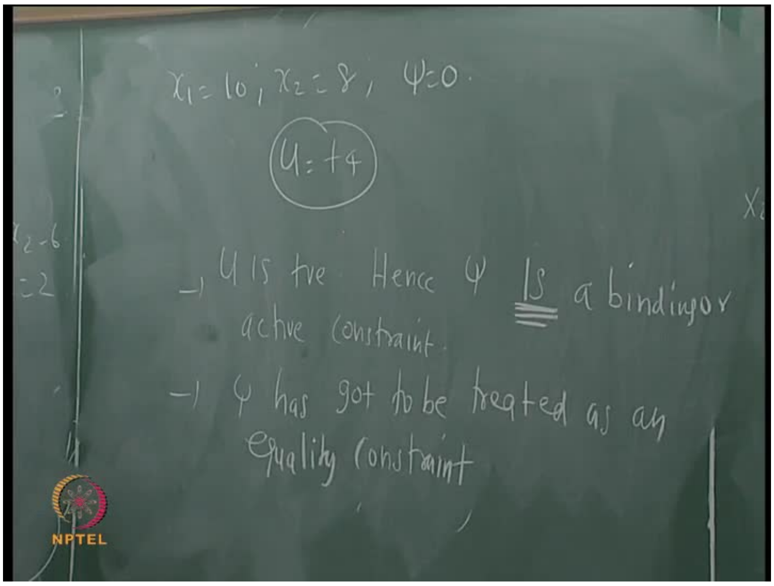

Therefore, for that since u is negative what is incorrect; your assumption that the constraint is not binding, okay. Now before closing this discussion, so there are two approaches to handling in-equality constraints. Number one, ignore the constraint and see whether it is violated. Now we ignored the constraint you get the solution as 8 and 6. 8 and 6 if you substitute here, x 1 plus x 2 greater than 18 it is violated. Therefore, the constraint is absolutely violated if you ignore. Therefore, you made a mistake by not considering the constraint; this is a very crude way of handling this.

The second is formally apply the KTC, treat it as an active or a binding constraint, take psi equal to 0 and get the value of u. If the value of u is positive, your assumption that it is an active or binding constraint is indeed correct and in the next iteration subsequent iterations if your problem involves many iterations you have to treat it as an active constraint, replace the in-equality by an equality. If on the other hand it is not binding, you can omit that constraint from the problem. For example, x 1 plus x 2 greater than 9, there is no need for it to consider to that in subsequent iterations. One last point and then we will close.

What new value?

Student: When this constraint is applied 8 and 6 will not do the solution.

Yeah, it is not a solution.

No, I do not get your point.

Student: When we are considering.

Have you got the new solution 10 and 8 what is the problem? We got the new solution10 and 8.

If you assume this to be they are considered to be inactive you will get 8 and 6 but 8 and 6 we will violate the constraint x 1 plus x 2 must be greater than 18. Therefore, it is wrong. So, you cannot live with that 8 and 6 for this problem.

Student: If there are more than one?

Each of the u you have to find; each of the u's have to be positive.

Student: If there are three?

Everything, all the u's has to be positive.

Student: We may have to check, 2 power n what will happen?

Yeah, your computer will check; computers do how many million floating point operations per second; you do not have to do it in the classroom.

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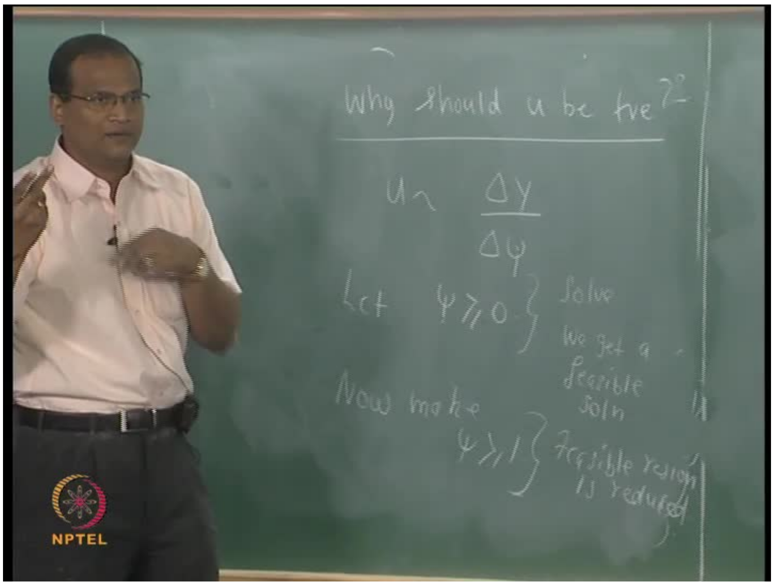

So, one last point, conceptually why should u be positive? All this discussion hinges on the assumption that while the Lagrange multiplier lambda in unrestricted in sign the u is restricted. Why are we partial to this lambda and why do we orgy this u? It is very simple. u is also delta, correct. It is a sensitivity of the objective function with respect to the particular constraint; watch I am do not using del phi I am doing del psi because u is u corresponds to the in-equality constraint.

Let psi be greater than equal to 0, solve, we will get a feasible solution, okay. Now what does it mean? Now I make psi greater than or equal to one and solve the problem again. So, what I have done is I have essentially made del psi positive. When I have done that the feasible region is reduced. Any solution which satisfies psi greater than equal to one will automatically satisfy psi is greater than equal to 0, but the converse need not be true. Any solution which satisfies psi greater than equal to 1 will satisfy psi greater than equal to 0, but now I have a reduced feasible region.

There is no chance for my Y to improve; at best it can be in this portion of the curve, I cannot expect Y to improve, Y can be the same or Y can only decrease. Therefore, while psi increases, why has to necessarily increase for a minimization problem. Therefore, if del psi is positive del Y is also positive; I cannot hope to get a new value of psi which will be less than what I got when psi is greater than equal to 0 when I apply psi greater than equal to 1. Therefore, u is positive, okay. It is just plain common sense.

Student: What will be the changes in maximization problem?

No, generally we define for minimization instead of putting words in my mouth, I will say treat minus of Y and consider it as a minimization problem that is it. Are we through, right? So, this closes our discussion on Lagrange multiplier. From the next class onwards we will start our journey of search methods, okay. Thank you.