Design and Optimization of Energy Systems

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Lecture No. # 26

Test for Maxima/Minima

So, we will continue the discussion on Lagrange multipliers. In the last class we have seen the test for maxima or minima. So, we went through very tricky mathematical derivation where we looked at the second order terms in a Taylor series expansion and we established the conditions required for minimum and I also told you that similarly there are conditions for maximum. There are also conditions for a saddle point or an inflection point. And first half of today’s class, we will solve one or two problems to reinforce our concepts whatever we have learnt with regard to test for maximum or minimum. To make matter simple, I am choosing some of the problems which you already worked out in the previous classes. Let us consider this problem. What will be the problem number?

Student: 27.

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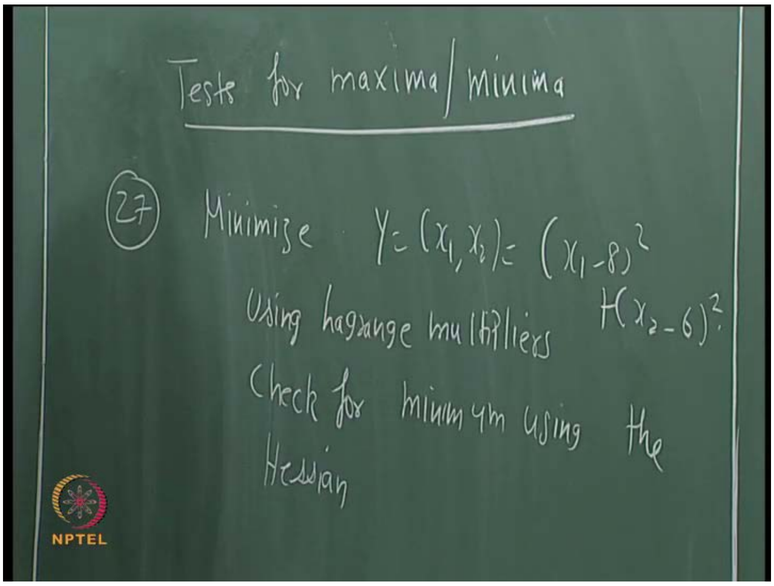

So, we are revisiting an earlier problem. The problem is like this, minimize Y is equal to x 1, x 2, Y is a function of x 1, x 2 equal to x 1 minus 8 whole square plus x 2 minus 6 whole square using Lagrange multipliers and check for minimum using the Hessian. This is an unconstraint optimization problem. We have already solved it dou Y by dou x 1 equal to 0, dou Y by dou x 2 equal to 0; it is pretty straight forward, you get the solution basically a circle. Right side Y is actually the r square the square of the radius. So, you know the minimum occurs at 8 and 6 when x 1 is equal to 8 and x 2 equal to 6 which correspond to a point, alright.

Now please evaluate the second order derivatives, get a 11, a 12, a 21 and so on and then compare it to the conditions which I gave in the last class and then get yourself convinced that the stationary point you have obtained is indeed the minimum. Alright, I will give you 5 minutes. So, let me just get the solution for you.

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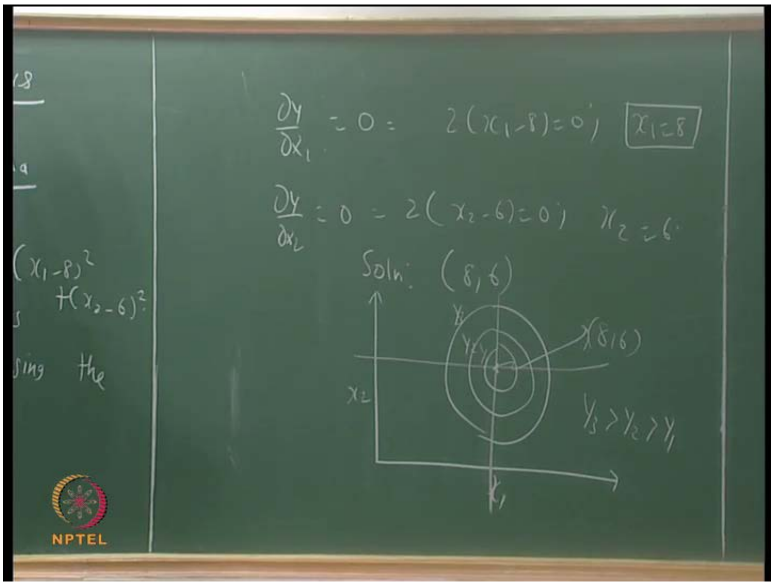

Allright, so graphical depiction of the objective function. Now the second order derivatives are pretty straight forward.

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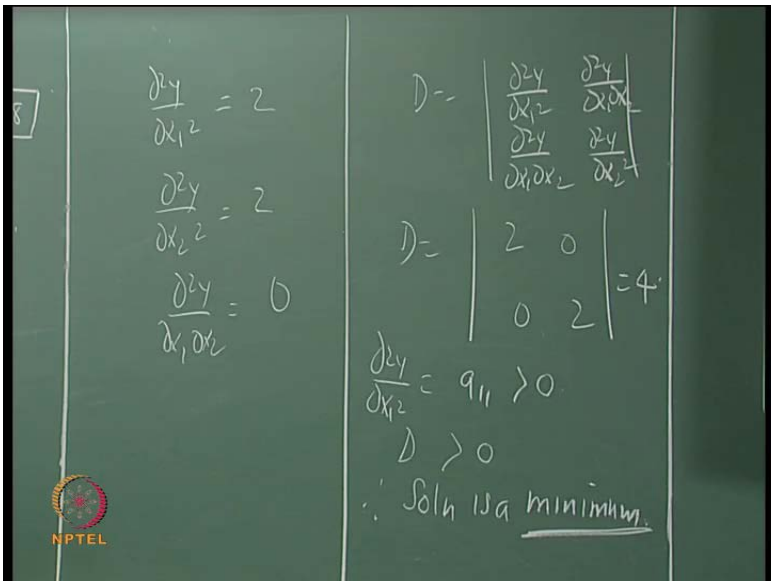

Is equal to 2, it is 2, 0, okay. So, the solution is a minimum pretty straightforward, okay. See for many problems the minimum is not so obvious, because for too many variables you do not know how it is behaving and all that. Therefore, it is good for us to have a condition like this which is very rigorous, mathematical and systematic. Only thing is if you have more number of variables, getting the determinant of the Hessian and all that is going to be extremely painful, but there is no other way. Is it so hopeless that there is no other way? I keep telling in most engineering problems, it is reasonably well known whether we are, say, heading towards the maximum or minimum but for mathematical exactness, if only if we want to be rigorously, mathematically established it as a minimum or maximum all these are required. For most engineering problems, it is obvious whether the stationary point is a maximum or minimum. Yeah.

Student: For the 3 variables what will the Hessian look like?

It will be 3 by 3.

Student: 3 by 3 and first will be a 11, a 12 and a 13?

Yes dou square dou a by dou x 1 square dou square y by dou x 1 dou x 3, okay.

Student: a 11, a 12, a 13.

a 11, a 12, a 13, a 21, a 22, a 23, a 31, a 32, a 33. Suppose you have 10 variable problem you will be sunk; you have to write a separate program for calculating the Hessian, already there is a pain in solving the Lagrange multiplier equations and all that, okay fine. So, this we will solve one more problem. Is this clear, over a simple case where the objective function is actually circle; the circle of radius x 1 minus 8 whole square plus x 2 minus 6 whole square root of. It is a radius; so, the solution is like this.

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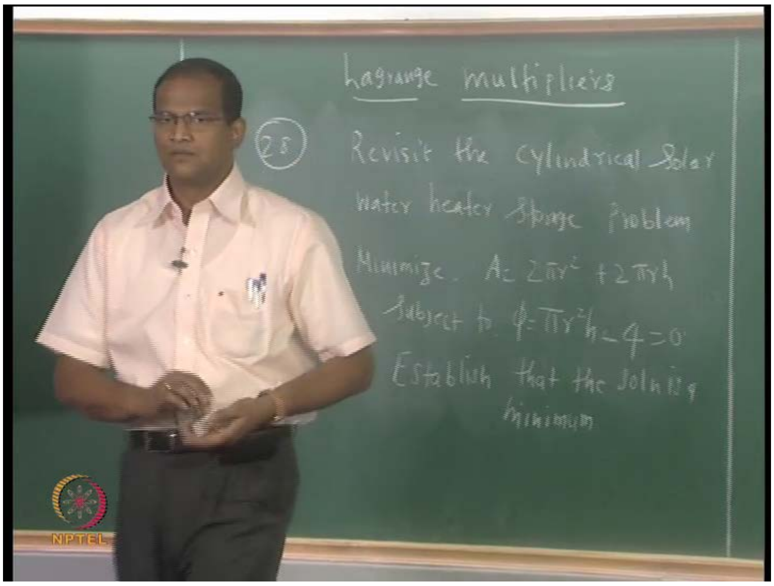

So, problem number 28; please revisit the cylindrical solar water heater storage problem, okay. Minimize pi r square h is no, no.

Student: It is 2 h multiplied by 2 pi r 2.

Okay, pi r square h equal to 4. Revisit this problem and treat it as a constrained optimization problem, do not try to make it constraint and make it a single variable optimization problem, alright. So, the equations are pretty simple.

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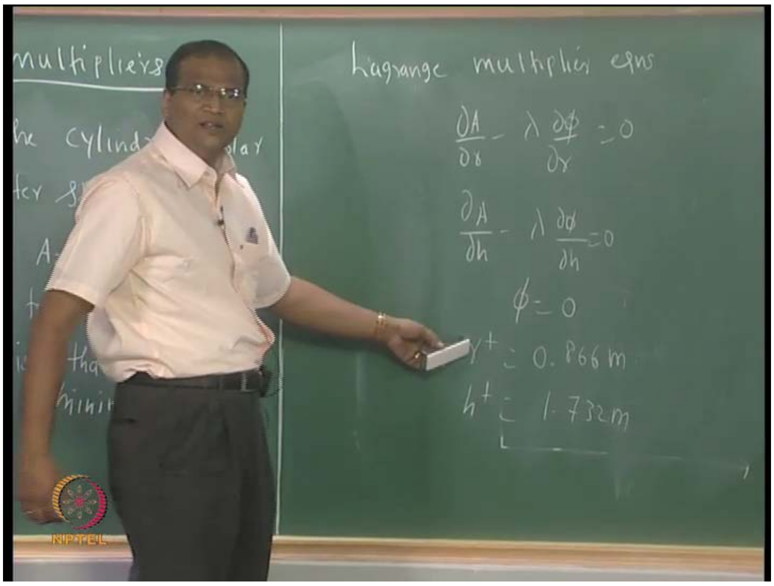

This can be call it phi or v, does not matter. Yeah, what is the r plus? 0.866? What is the answer, Deepak? 0.866 is r plus. So, the problem is the problem starts now. Establish that the solution is a minimum; needless to say, establish that the solution is a minimum using the Hessian. You can convert into one variable problem and prove that the second derivative is greater than 0 which we did in one of the earlier classes. Now we formally use the Hessian. This solution everybody has the solution, right; we have already solved it in one of the earlier classes.

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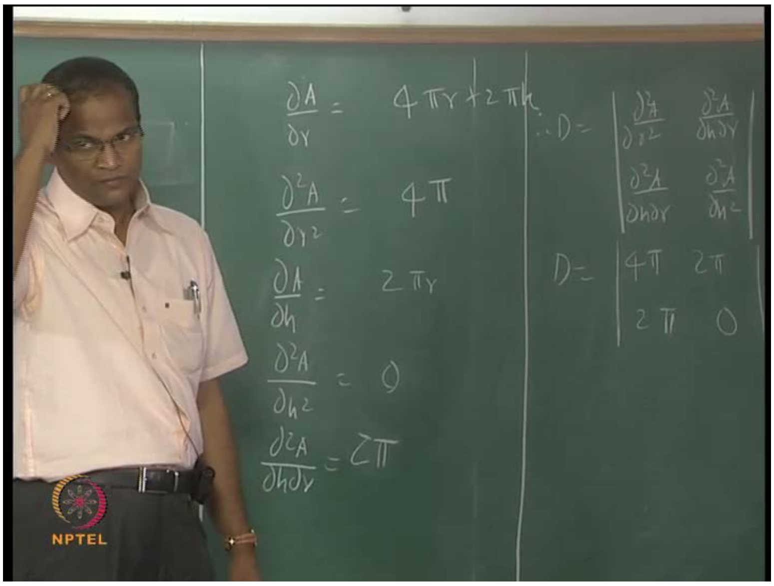

So, dou a by dou r is 4 pi, okay, 0. Therefore

Student: Second derivative of a with respect to h and 0.

Where?

Student: Dou square a by dou h square.

Yeah, yeah what are we getting now?

Student: Dou square a by dho h dho r exist.

Wait, wait, wait, wait.

Student: Dou a by dou r is pi h.

Dou a by dou r is?

Student: 2 pi.

I made a mistake here?

Student: The last one.

2 pi h, that’s right, okay.

Student: The last one is 2 pi.

Let us do it, hang on.

Student: a 21.

2 pi, fine.

So d is 4 pi, 0, correct? It is 0, right? Then, is it? D is negative, we will be in trouble.

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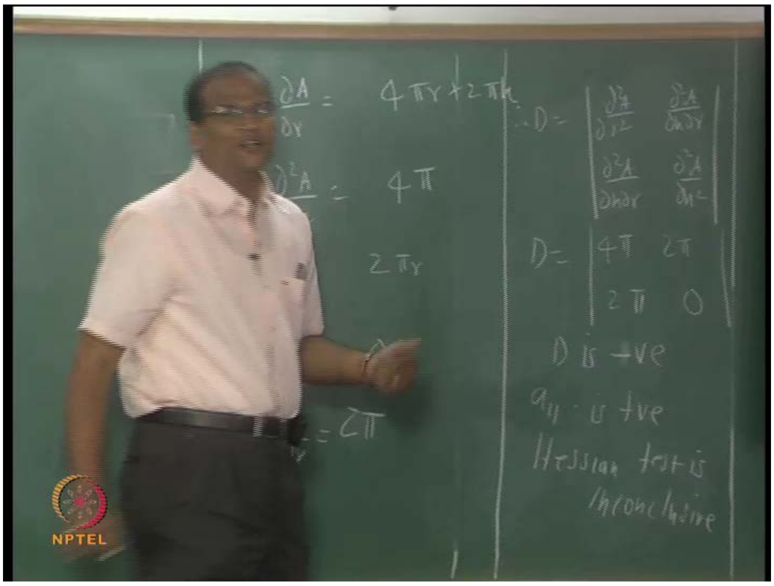

What about the, a 11?

Student: Negative.

Negative a 11 is?

Student: Positive.

So, what was the condition for us?

Student: a 22 is 0.

Yeah, here a 22 is 0 that is the problem because it does not vary with the variable. Anyway, according to the rules it is inconclusive or what is it?

Student: It is inconclusive.

D has to be positive or negative?

Student: it is negative saddle point.

No, no what did you say?

Student: Is equal to 0.

No, no what did you say? Yeah, a 11 as a 11.

Student: And a 22 both are pretty equal.

See when I said a 11 is greater than 0, why special treatment only for a 11? It automatically implies a 22 if any of these dou square Y by dou r square or dou square Y by dou h square; any of these things become 0, then already the second derivative is becoming 0 of particular this thing, then automatically what is it mean?

Student: Inconclusive.

It becomes inconclusive, okay.

So, Senthil what happens is if D is less than 0 provided all the other provided none of this is 0, then whatever you are saying is fine, then it is the saddle point or all these. But now instead of starting the test with a 11, we could have started the test with a 2 2 also. When you say that then when I said that if a 11 or a 22 is 0, then it is inconclusive. So, we have to say that it is inconclusive. But however, if we convert it to a simple variable problem, we establish that this is indeed a minimum or you can exhaustively search, you can do all combinations of r and h, write a computer program and with a precision of point naught naught naught 1, 10 to the power of minus 5 millimeter, what about minus 4 meter, minus 5 meter, you will still not be able to get something which is having an area less than this.

So, we should not get carried away by Hessian test and all that. Sometimes, what commonsense tells you the Hessian may not tell, okay. Here, I hope there is no bug here; everything is fine, okay. Now we will solve a practical problem using Lagrange multiplier method before we close for this class, then at 2’o clock I will teach you the very important Kuhn-Tucker conditions; I mean wherein we will able to solve Lagrange multiplier, use the Lagrange multiplier method for problems with inequality constraints also. So now we will go to problem number 29. If you see the other problem also, Y equal to 4 x 1 plus 3 x 2, we solved right, Y equal to 4 x 1 plus 3 x 2 subject to.

That one also if you look at the second derivative it will all be?

Student: 0 0 0.

When the solution is so obvious, if you over do by doing Hessian and all that it will confuse you, okay. Everything will be 0, but it is so obvious the solution, alright. So, but if you have function like e to the power of x, e to the power of 3 x or sine x and sine h x then you can keep on differentiating, right. But if you have x squared, you can do only one step, 2 x and then the other next step you are sunk; e to the power of x you keep on going, right. So, it depends of the nature of the function, fine.

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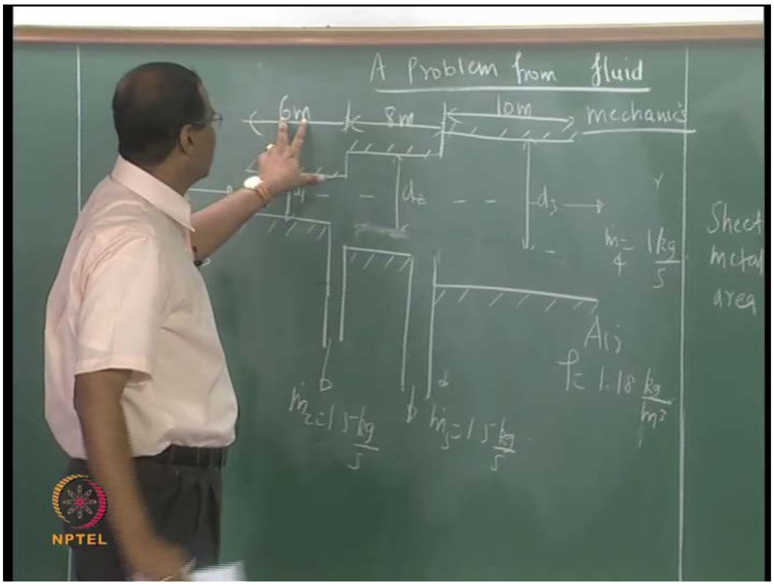

29, is it 29? I will also put it up on the computer screen but basically we have a problem like this. Basically, this is a problem from fluid mechanics, okay. Flow in a pipe network, we are trying to optimize it using Lagrange multipliers, is it okay. Where should this come? It does not matter, okay. So, I have a circulate duct whose diameter is varying, is not varying with x as a function or something, I mean it has got discrete three values d 1, d 2, d 3. Air is flowing in. So, this m 1 is 4 kg per second. Here, I am pulling out 1.5 kg per second; here I am pulling out another 1.5 kg per second. So, here under steady state m 4 has to be 1 kilo gram per second, right. So, it is basically air, rho is equal to 1.18 kilo gram per meter cube, alright.

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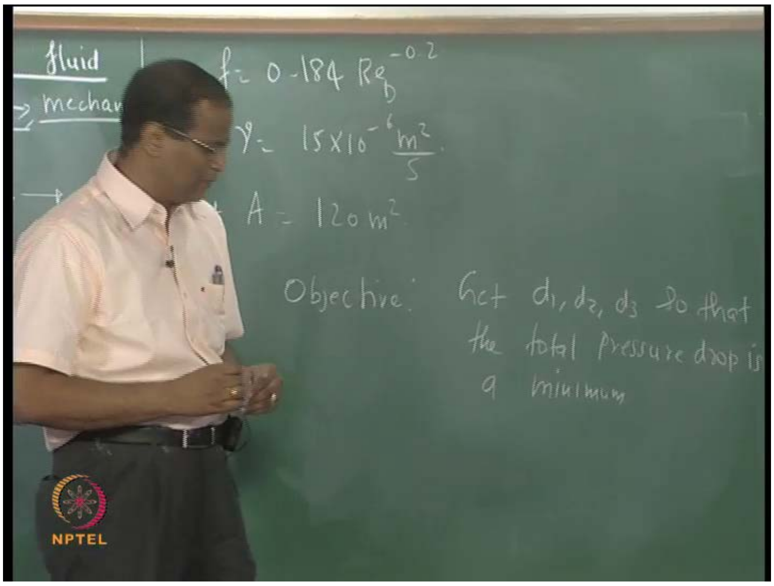

The friction factor for turbulent flow is given by; the kinematic viscosity of air is given to be 15 into 10 to the power of minus 6.

Student: Diameter side and both sides.

Which both side? No, no no I have not drawn properly. No, it has to be equal, right. Now this is tricky. Now again I am pulling it out. Okay, let us draw without the then I will pull out, is it okay? Now you say, here you are taking it out. Is there a mistake here it is alright, it is okay or? Yeah, okay the circular duct d 1, d 2, d 3, friction factor given is like this, mass flow rates are given, total area available, sheet metal area 120 meter square, total sheet metal area 120 meters square. Leave the sheet metal required for pulling out this. These are not important and then this is 6 meters. This is not the scale; this figure is not the scale. This is 8 meters and this is 10 meters; that is l 1 is 6 meters, l 2 is 8 meters, l 3 is 10 meters. The lengths are given, the mass flow rates are given, the properties of air are given and the formula for fiction factor is also given, the total sheet metal area is also given. So, what is the problem?

Student: Find the diameters.

Find the diameter such that the total pressure drop is minimized. This is a standard problem in fluid mechanic, okay. What is your problem? Objective, needless to say, it is a Lagrange multiplier method. Do not get confused, we are still talking about incompressible flow; do not treat it as total pressure drop. Okay, so total pressure drop. I am not talking about stagnation pressure. Can I have this screen now? Yeah, you can start working. This problem will come up on the screen also just in case. I also want to check whether I have made any mistake on the board.

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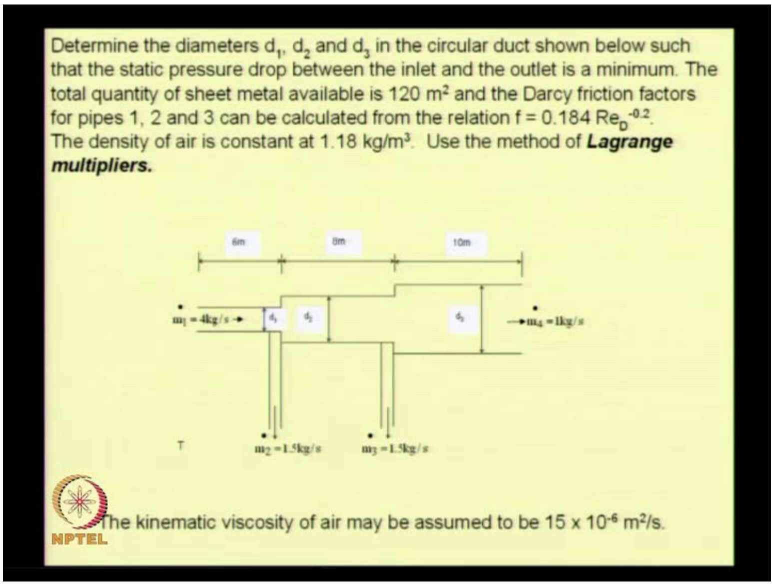

The first line is determines the diameters d 1, d 2 and d 3 in the circular ducts shown below. So, here it is given clearly 6 meters, 8 meters, 10 meters, 4, 1.4, 1.5, 1, d 1, d 2, d 3, viscosity is given. Set up the problem; first calculate the delta p 1, delta p 2, delta p 3 for each of this sections, add delta p 1, all the three are in series, right. Total pressure drop is delta p 1 plus delta p 2 plus delta p 3. That will be the objective function that has to be minimized. The constraints are given in terms of the sheet metal area the total area. I will take attendance then I will start working out. If everybody is through with this, we will take it off so that we can use the black board.

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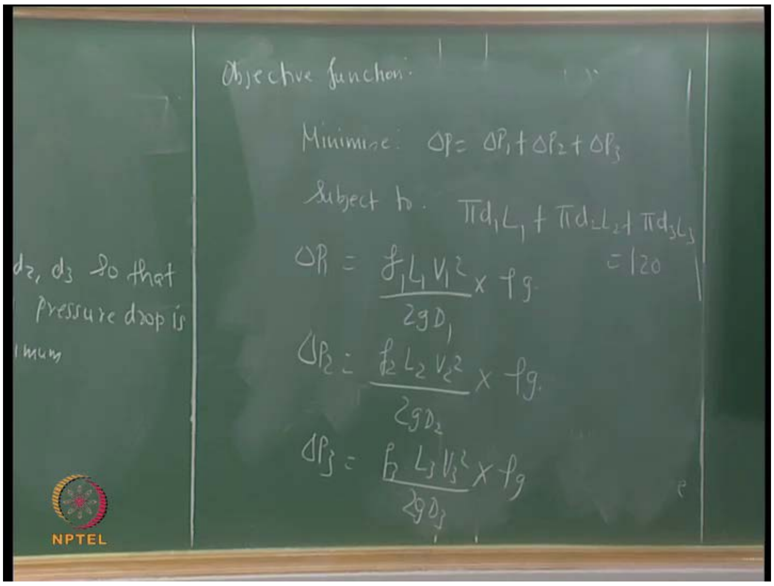

So, the objective function is subject to pi d.

Student: Pi d 1 2 pi r 6 pi d 1 plus 8 pi.

Yeah, yeah okay phi d 1 l 1, okay. Now what is delta p 1? So, delta p 1, correct, delta p 2.

Student: Kinetic head loss sir?

What?

Student: Kinetic head loss?

Expansion and other thing?

Student: Use the kinetics like d square.

You can neglect all those things, okay. So, after finishing the solution we can check if you want whether they are really contributing or whether it is friction losses, whether the f l v squared by 2 g d is much more substantial than the kinetic head loss.

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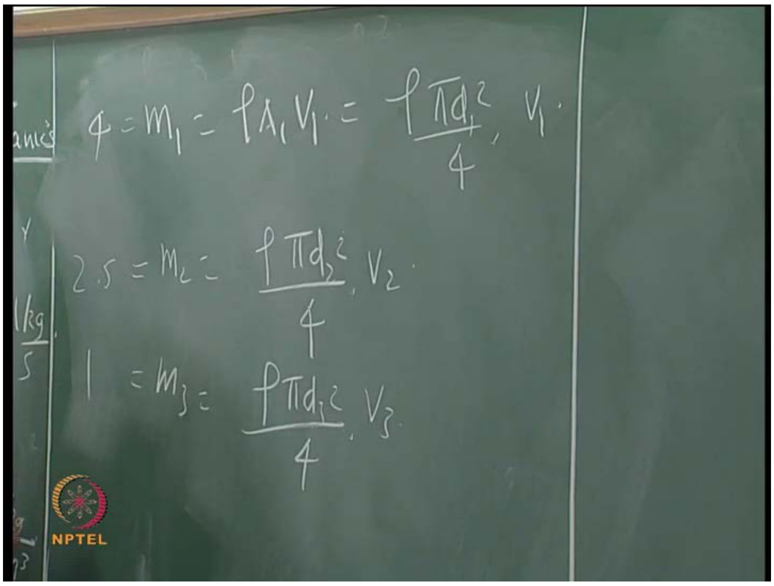

Now are there additional relations which are available? Are there additional relations available to us?

Student: Yes sir.

Yes sure, m 1, okay. So, you can substitute it for rho 1.18, m 1 is 4, m 2 is?

Student: 2.5.

2.5.

Student: 1.

1, good.

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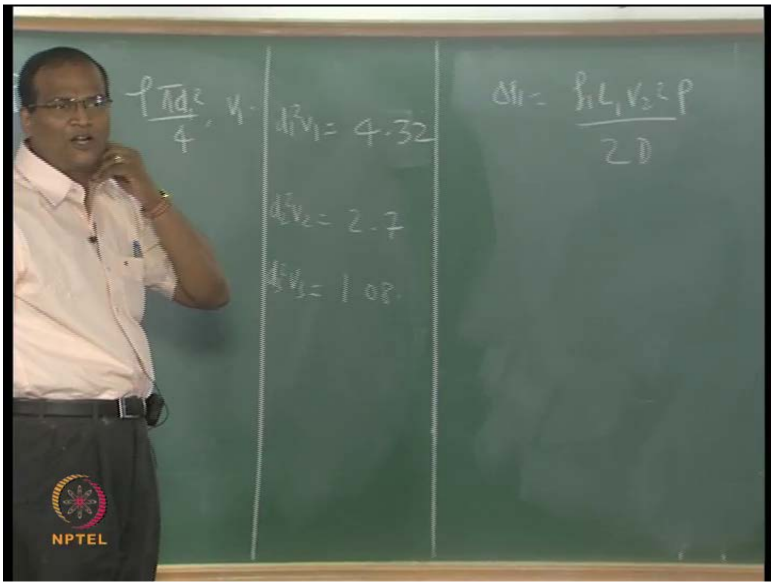

So, please tell me the relationship between d 1 square v 1 equal to? We will do some simplification. Can you tell me? Vinay, can you just tell me the values d 1 square v 1 equal to 4 into 4 divided by 1.182; you do not do the steps. Sureka, you have it?

Student: 4.319 sir.

Good, 4.32, then d square v 2?

Student: 2.6, 2.7.

2.7 okay fine. d 3 square v 3?

Student: 1.08.

1.08 fine.

Now we will work out for delta p 1. The same logic is applicable for delta p 2 and delta p 3 though the value of l d everything is changing. So, let us see. Now you are able to see that there is considerable effort in formulating the optimization problem. This is what is normally encountered in engineering. Suppose you are doing a project, nobody will tell you maximize Y is equal to x 1 minus 8 whole square, minimize Y is equal to x 1 minus 8 whole squared; that is okay for teaching and that is okay for if you are attending a mathematics course, but in engineering the problem will start like this.

You have to use your knowledge of fluid mechanics or thermodynamics or power plant or heat transfer or stress analysis or whatever. Then we will come to a stage and then hopefully after we have formulated, it is in a form which is amenable to solution by Lagrange multiplier. But since I already solved you can believe that this can be solved by Lagrange multiplier. Sometimes you think it can be solved by some method; after you have formulated you realize that it does not work or even after sometimes even after you formulated, you feel it will work and finally when you write the program it does not work. So, optimization is not so straight forward in real-life problems.

Here, you can see that there is considerable labor in formulating the problem, okay. Though we are not using any advanced concepts, there is lot of algebraic manipulations involving in getting the objective function. So, you want to develop a network for a big colony or you are a Chennai metropolitan development authority; you are looking at a piping and this thing at Chennai metropolitan water supply and sewage board. They are coming up with a new desalination plant; they have marked some areas in North Madras for this. It is going to be lot of work put diameters, this thing friction factors and what will be the pressure at the tail end of the pipe from the station, from the works; if you are looking at something which is 15 kilometers, 20 kilometers away, something like this we will do.

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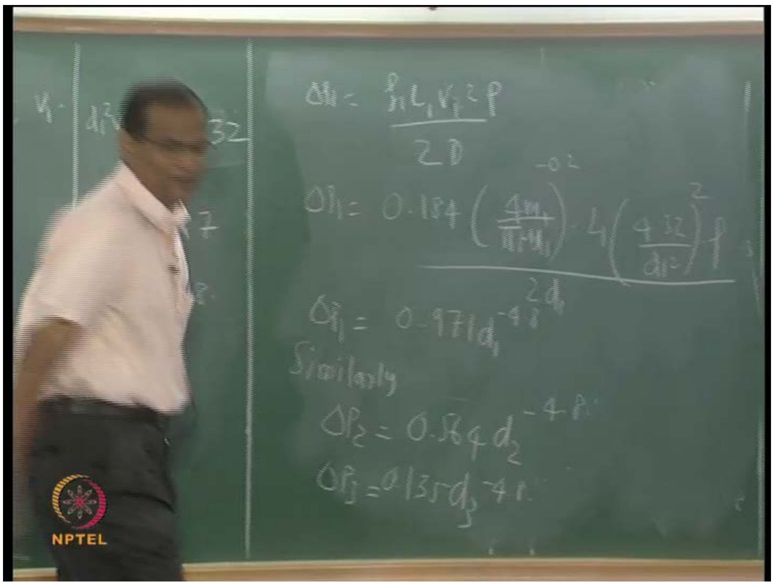

So, delta p 1 by 2 rho is there, right? The g got cancelled, right? Okay, so delta p 1. I have done several things here. Wherever we got, what is that, instead of velocity I have substituted 4.32 by d 1 square which I got from here. Instead of friction factor I substituted 0.184 Reynolds number to the power of minus 0.2. The alternative formula of Reynolds number in terms of mass flow rate is 4 m by pi mu d 1. Now you can see that delta p goes as d the power, this is what you solve for the pipe and for your assignment one, D goes as minus?

Student: 4.8.

Correct, D goes as minus 4.8 that you should never forget. Then, you can assemble all the other things and put it as a constant, can you tell me? So, you know all the other values, this is m 1, you know m 1, l 1, rho everything. So, delta p 1 is equal to. I have the solution, but I do not know whether it is correct. So, let us see. 0.971?

Student: 0.981.

Okay, it does not matter, so 0.971 d 1 then minus?

Student: d 1 minus 4.8.

Good. Similarly I hope all of you are able to follow these arguments, right. So, you have to substitute and it is just bull work here, right, so similarly delta p 2.

Student: Won’t there be any square term?

Where?

Student: In the bracket.

Sure, yeah yeah. You are able to see from there? You require a periscope. Vipin, I gave you attendance do not worry, you go. You cannot see anywhere, right; you require a periscope.

Student: I’m seeing the problem sir.

There will be transmission and distribution losses. So, similarly delta p 2 equal to Vikram you are very sleepy today.

What happened? The camera caught you. Okay, delta p 2?

Student: 0.564.

Good 0.564, whatever, 0.180.

Student: 0.135.

0.135, is it right?

No, all my answers are wrong. Okay so 0.135, you got it Anand 0.134, d 3? So, you can expect a problem of this difficultly level in quiz two, right. Not x 1 minus 8 whole square plus x 2 minus 6 whole square. To spice it up, we can have problems where u a l m t d is equal to m c p delta d and we have to differentiate log and all, correct.

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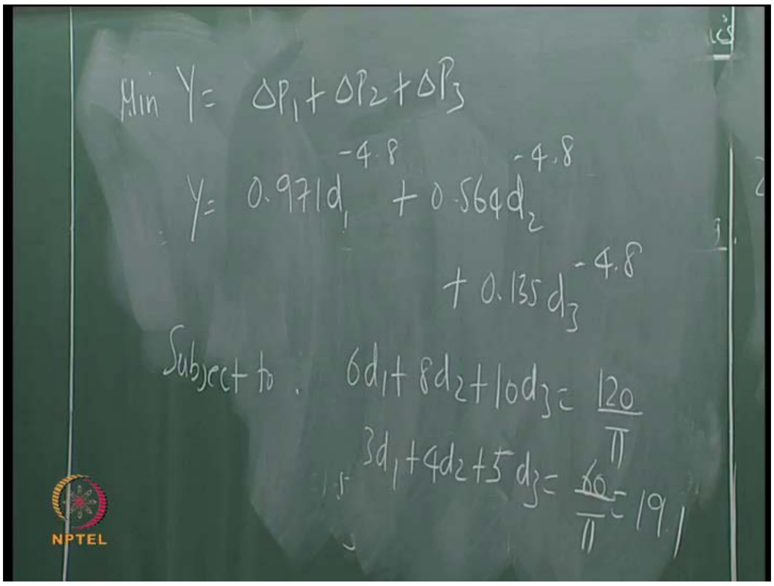

Therefore, mean Y is equal to yeah, now this is 0.971 subject to 6 d 1 plus 8 d 2 plus 10 d 3 is equal to 120 by pi, 3 d 1 plus 4 d 2 plus 10 d 3. What happen?

Student: 5 d 3.

So, what is this anyway? 19 point, 18 point, how much is it?

Student: 19.1.

Good. We have not solved the problem, now we have formulated the problem.

Student: This area we are not taking the kinetics this thing between.

Yeah, yeah we leave all that. I am making some approximation here to make the problem tractable, okay. You can write a program and solve it if you want, I will look at it. Then as he said we have not considered the other kinetic losses and all that kinetic energy loss, fine. Now we have to use the Lagrange multiplier method.

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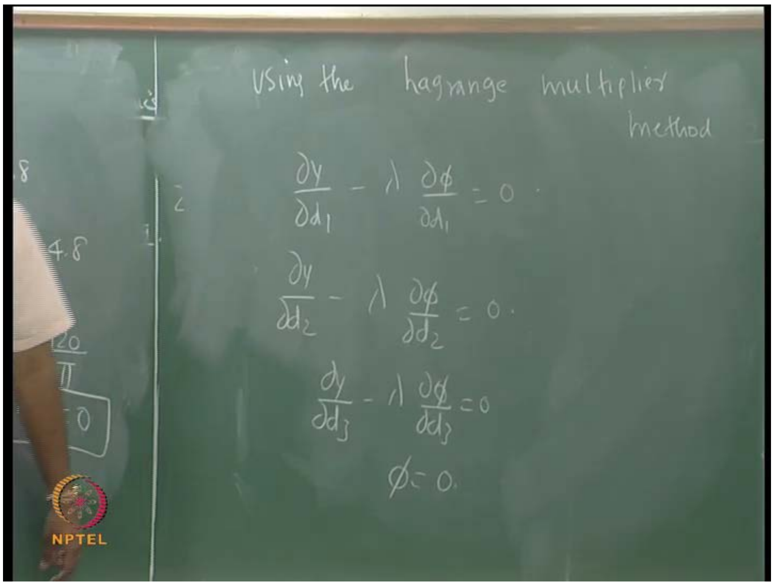

So, this is phi here, correct alright and lambda. Any problem, why?

Student: Both of them are negative sir?

No, lambda can be negative, lambda is unrestricted in sign. You assume lambda is negative; therefore, the root is possible in all that do not worry about that. Actually this is a how many unknowns are there?

Student: Four unknowns.

Four unknowns but I mean at first sight we will feel oh, it is not solvable but actually the resulting Lagrange multiplier equations are pretty straight forward, very simple, you can solve, right. So, shall we do that? See there is a problem root of minus something is coming, that does not matter lambda is minus we can take or we can take this whole thing as d by d 1 plus lambda, I mean lambda is unrestricted in sign. We saw that the only requirement was del Y and del phi must be collinear. I asked this question in 2009 end sem, so it is not coming for quiz two and end sem, okay.

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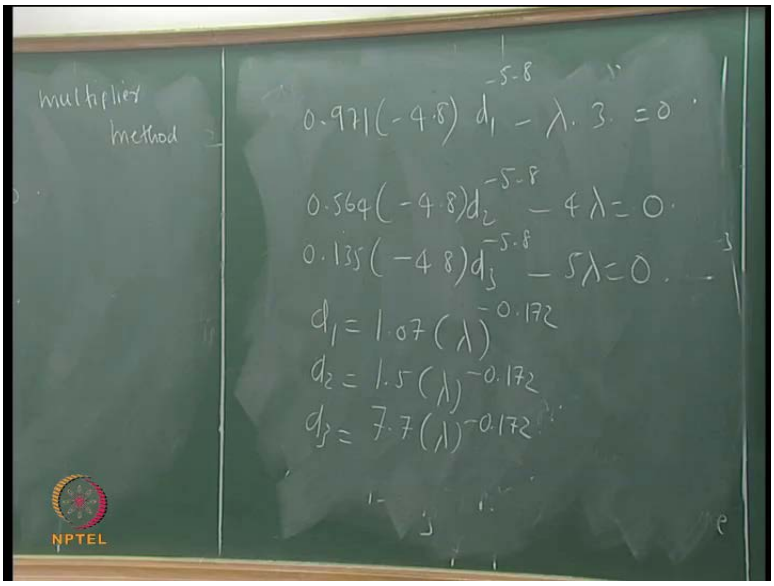

Okay, phi equal to 0. Now what is d y by d 1? Rohith, wake up, wake up Sid. Dou Y by dou d 1 is 0.971 minus 4.8 d 1 minus 5.8 minus lambda, oh this so nice, into 6 into 3, 4, 5, right, 0.564. So, d 1 is or lambda is or d 1 is in terms of lambda; we put d 1 in terms of lambda, d 2 in terms of lambda, d 3 in terms of lambda and then substitute for d 1, d 2, d 3 here, you are home, okay. So, d 1 equal to; please tell me, Vikram d 1?

1.07 lambda raise to

Student: minus 1 by 57.

0.172. d 2? d 2, how much is it? 5.64 into 4.8 divided by 4, 1 over that. 1.5 lambda to the power of minus 0.172, right. d 3?

Student: 7.7.

7.7 lambda to the power of minus.

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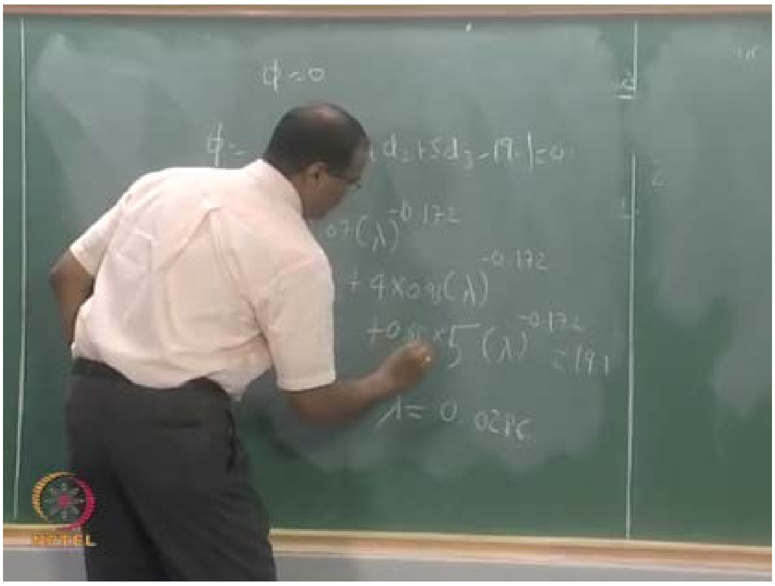

Therefore lambda should be negative, you have already made the correction? How much is lambda? How can d 1 be negative? You adjust lambda. Some adjustment is there. Is this correct, Deepak? Do not lead me into the wrong path. What is lambda finally? Why is everybody blinking? Lambda is how much? Point naught something?

Student: 0.286.

0.286, good, lambda is from this based on this, okay.

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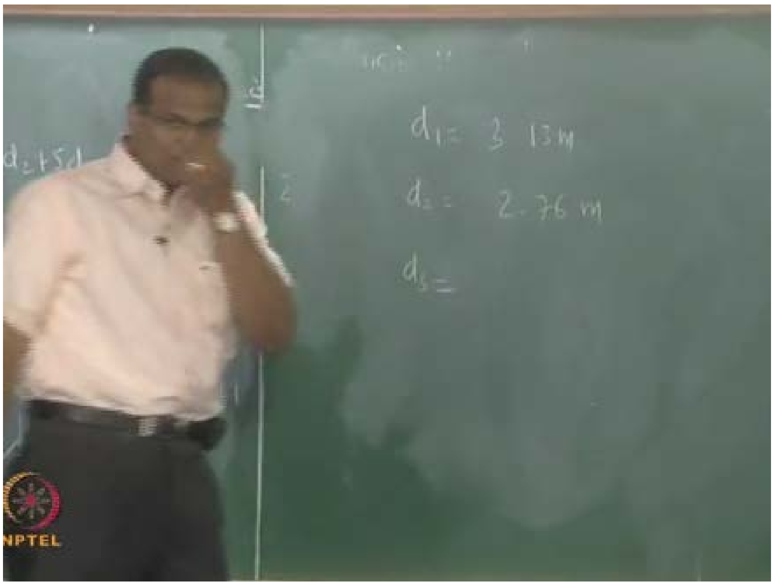

Yeah d 1, d 2, d 3, Abishek, you did not get?

Yeah you can substitute there. Kaustubh, what is d 1?

Student: 3.13.

3.13. d 2?

Student: 2.76.

d 2 is how much?

Student: 2.76.

d 3? large numbers. Which one?

Student: d 1 is 1.07 times not 1.5 times sir.

Lambda to the power of minus 0.12, lambda is less than 1.

Student: Both are common sir.

Student: d 1 should be greater than d 2.

d 1 should be greater than d 2 should be greater than d 3, alright. What is d 3?

What is d 1 finally?

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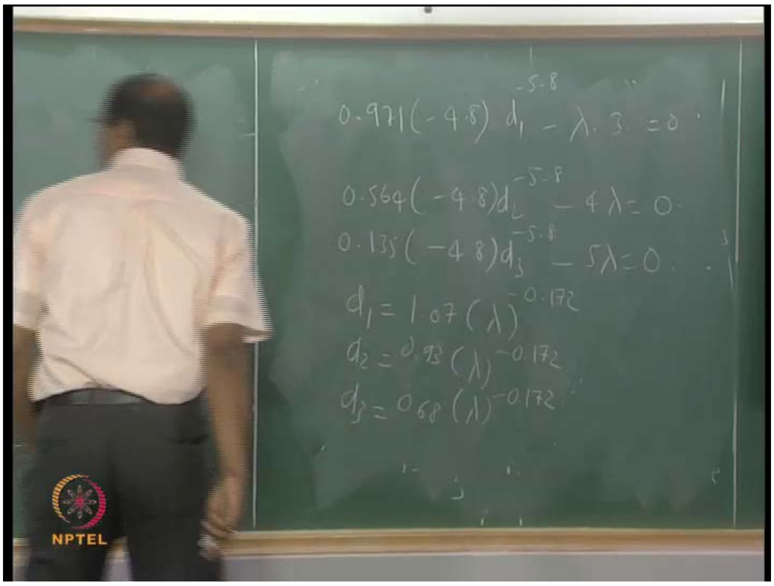

Student: Sir the coefficients over there are 1.07 is the first one, second one is 0.93, 0.68.

Yeah, so there is a mistake here.

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Now tell me d 1, d 2, d 3. Vikram now you will be happy. Why are you struggling?

Student: 1.97.

Okay, why are you struggling? 1.97.

Student: 1.69.

Third one?

Student: The last one is 1.279.

Okay, the way the values are given, d 1 has to be greater than d 2 which should be greater than d 3 but the way the figure has been given in the problem, I can now say that I gave that figure to mislead you or whatever. But after I gave the question I realized that d 1 was greater than d 2, but I thought I will leave it, okay. So, do not get carried away by, that is just a depiction d 1 is different from d 2 could be different from d 3. So, we can complicate this by having several streams which are coming in or going out; we can have several d 1, d 2, d 3 up to d n and so on. Simple fluid mechanics Darcy-Weisbach equation f l v square by 2 g d can lead to so much. Suppose apart from fluid flow where heat transfer also there is a heat exchanger, there is a hot fluid, cold fluid, so several combinations are possible.

Student: How did the coefficients be same?

How did the coefficient is arithmetical error?

Student: Previous solution what we got?

No. Because I have of course I made a mistake in delta p 3; I got values which are close to this for my diameters were 1.951, 1.69 and 1.336. I just worked out before coming to the class. It is basically that minus 5.8 and all that; it is all tricky, just work the algebra patiently. It cannot lead to the lambda should be very small; lambda must be negative, is not it? No, you have adjusted here, okay.