Design and Optimization of Energy Systems

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Lecture No. # 24

Constrained optimization problems

So, we will continue our discussion on Lagrange multipliers. In the last class I introduced Lagrange multipliers. Basically it is a technique which can be used to solve multivariable constrained optimization problems. Only requirement is that the constraints must be equalities and both the objective functions the constraint must be differentiable. So, you get to solve m plus n equation simultaneously where n is the number of variables and m is the number of constraints. So, you have m plus n variables, x 1 to x n and lambda 1 to lambda m, where lambdas are the Lagrange multipliers. Through a problem we also figured out what is the physical interpretation of this lambda. This lambda is basically the sensitivity coefficient. If it is a one-constraint problem, it is the sensitivity or the change in the objective function to a change in the constraint. So, in operations research it is also called as the shadow price.

If you have several constraints the particular Lagrange multiplier, so if you have m constraints lambda 1, 2, 3, up to lambda m, each of this will represent the sensitivity of the objective function to that particular constraint. So, in fact some people towards the end of the last class they came and asked me, sir second time when we increase the volume of the tank from 4000 liters to 4500 liters, I am getting a new value of lambda and all that. Of course lambda will change, but the original interpretation is for the tank volume of 4000 liters if you have a lambda; that means when you change from 4000, what is the answer? Suppose for 4500 you get a new value of lambda; that means from 4500 liters if you change to a new value, how is the area going to respond?

So, we will solve some more problems and through this problems I will also introduce to you what is the physical interpretation of Lagrange multiplier, is there a mathematical proof for this Lagrange multiplier, how do we figure out whether the stationary point that is the extremum we have obtained is indeed a maximum, minimum or a we have not checked that so far. If you are solving it as an unconstrained optimization problem in one variable, it is possible to take the second derivative.

If it is less than 0, it is a maximum; if it is greater than 0, it is a minimum; if it is equal to 0, it is an inflection point. But somebody asked me this question, I told you that we have to look at the Hessian matrix and all that. So, we will go through a very tricky derivation where I introduced the concepts of positive definite matrix, negative definite and all that, because these are all important in order that you are able to figure out whether the stationary point have reached is indeed a maximum, minimum or a saddle point; saddle point as an inflection point.

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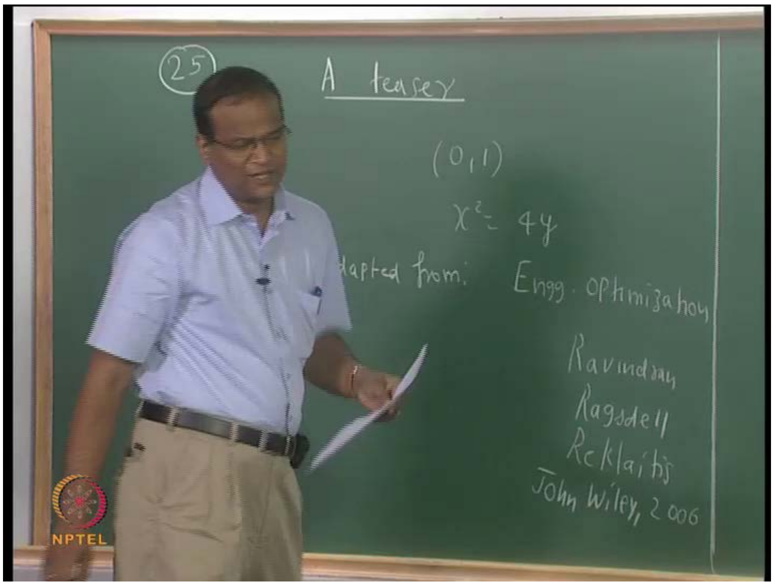

Now, what is the problem number?

Student: 25, 24.

24 or 5; No the tank was two problems, 24?

Student: 25.

Yeah, it is a simple problem but still let us do this.

Determine the shortest distance from the point (0, 1) to the parabola x squared equal to 4 y by a) eliminating x b) Lagrange multiplier technique. So, explain why the approach a) fails to solve the problem, why b) does not fail. So, this I will give the credits Ravindran, Ragsdell and Rcklaitis.

So, Ravindran is the author of hugely popular operations research, operation research by D. T. Ravindran Philips and Ravindran. Have you heard of this book? Operation research by Hamdy A. Taha that is like bible a very good book, D. T. Philip Ravindran and they are all industrial engineering professors, Ravindran and Philip is also a good book. This guy this Rcklaitis has done lot and is a stud guy, he has thought about several optimization problems how certain techniques can come, he has done all those things. He has written lot of papers about how to establish the robustness of various techniques and so on.

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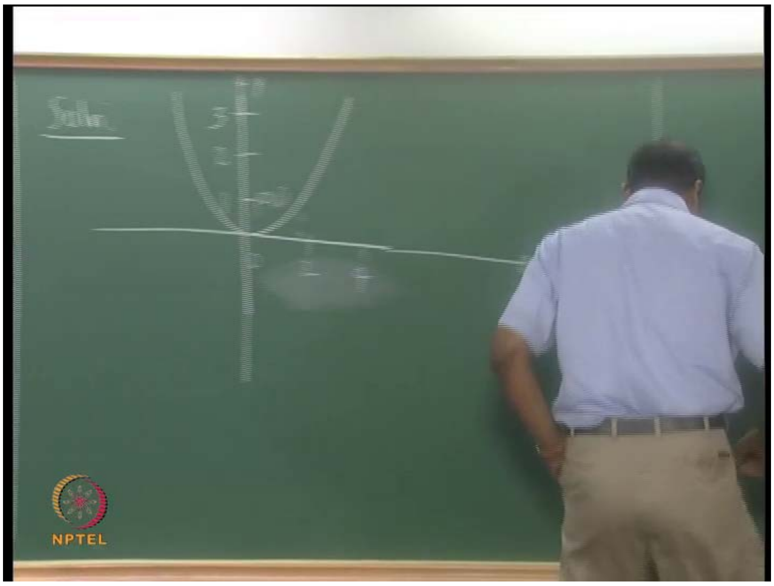

Anyway now the parabola is how does it look like? Yeah, please solve. Varun Joshi what happened; thinking hard? Where is this point (0, 1) anyway? Yes. Let us say y equal to 1, x equal to 2, right. Y equal to 1, x is plus or minus 2. So, I will change this scale now, now we are fine? No, approximately and the shortest distance is so obvious. So, the Lagrange multiplier should also give the same answer, right? What is the shortest distance?

Student: 1.

One, shortest distance is this. Are you getting the one? You can start with b first and then proceed to a. With b also some people may get struck.

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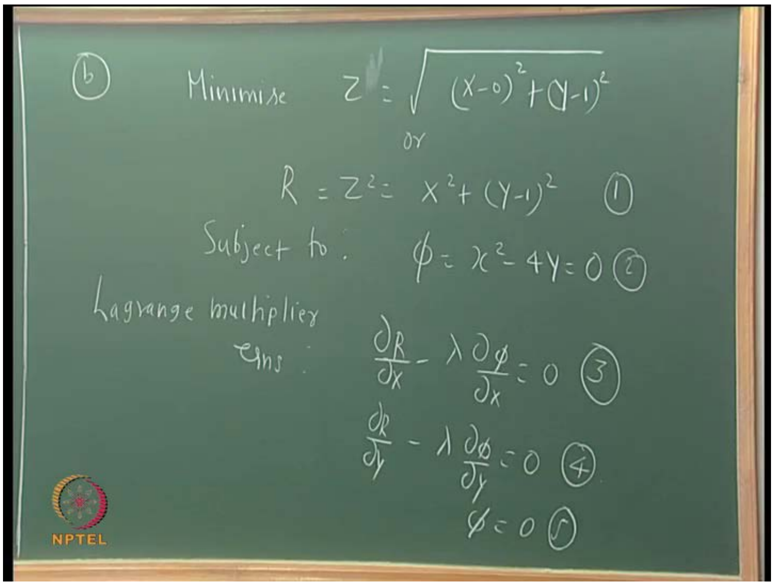

Okay, let us start with b. Minimize some R, I will take call some R; it does not matter. I can minimize Z square or Z, subject to?

Student: Phi equal to x square minus 4 y.

Phi equal to, alright. So, using the Lagrange multiplier lambda, 1, dou R by dou Y; I am writing the constraint again for the sake of completing. So x, y and lambda are the three unknowns, there are three equations. It is possible for us to solve three equations to determine the three unknowns.

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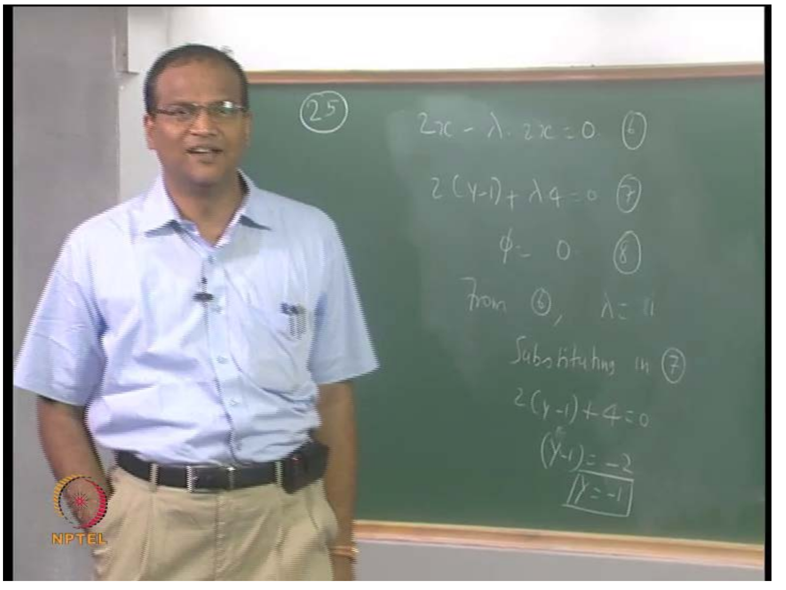

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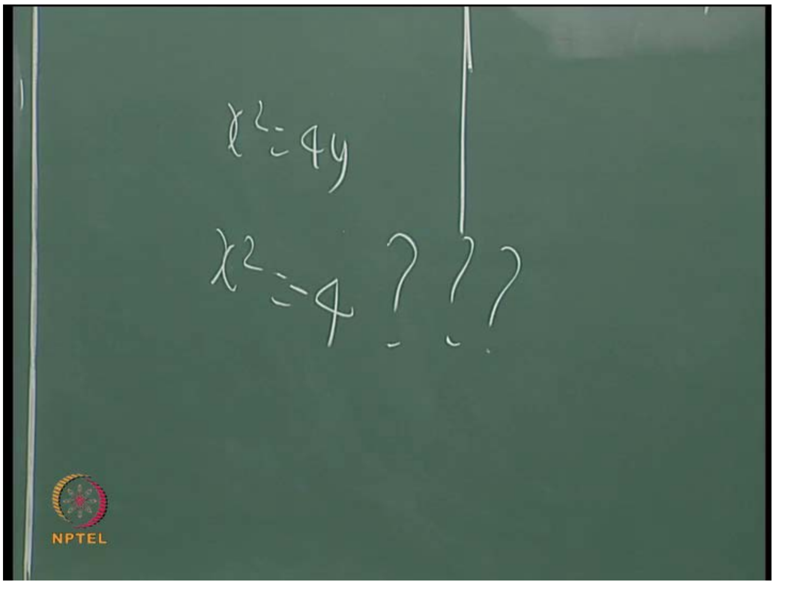

Okay, let us do the first one dou R by dou x is 2, this is also 2 x. 4 or minus 4, dou phi by dou y. So from 6, it is such as simple and stupid problem you are not able to solve.

Student: Sir y is minus 9 is all to the power 3.

Exactly, I think you are away.

Student: Sir what for equation 6 we get another solution x equal to 0.

Yes that is the point. When you cancel the x, it implies that x is not 0 but unfortunately x equal to 0 is the solution, right. That is the catch in this problem. You cannot cancel them because x equal to 0 is the solution.

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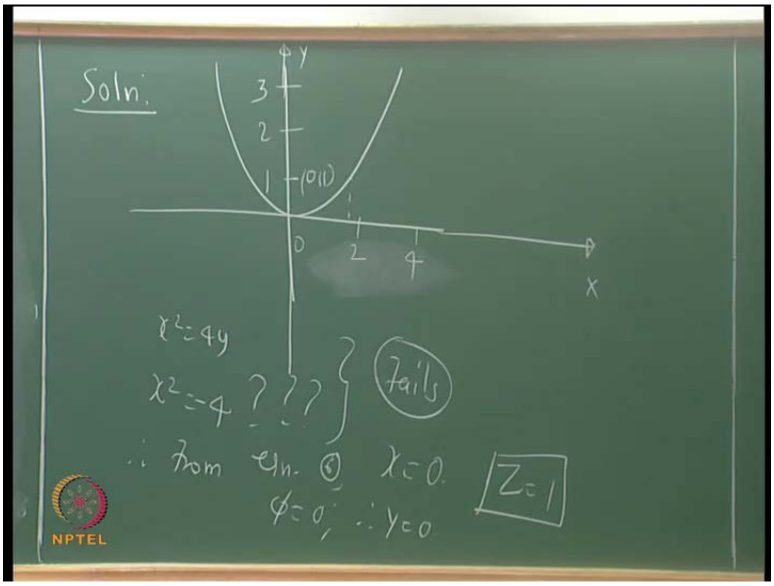

So, even Lagrange multiplier fails. We have to be alert; therefore, from equation 6 x equal to 0, Vikram is it okay? When x equal to 0 now you can phi equal to 0; therefore, y is equal to 0. Now Z will be 1. The x square is equal to minus 4 is forbidden and as I said, Y equal to minus y is outside the parabola, it is not the point in the parabola. So, sometimes we have to be alert. Of course in engineering x, y are all variables; you cannot put 0, 0, 0, everything is 0 is not but I was looking at various problems I thought this is an interesting one. So, I thought we will work it out. Now what about the other method is it totally fails. You have no clue what is it going on. Let us do that.

No, that gives some Z equal to 1.

Student: It gives distance equal to y plus 1.

Distance equal to Y plus 1, it does not give any more information. Let us work it out. See for those people, for those doubting Thomas's who said who thought that all the constraints can be put back and then you can make it an unconstraint problem, there is no need for Lagrange multiplier. This simple problem gives you trouble. So, it is not always it is not wise to convert it into unconstraint optimization problem and attempt. Even when you are attempting the full constraint optimization problem, Lagrange multiplier can give trouble if you are not alert.

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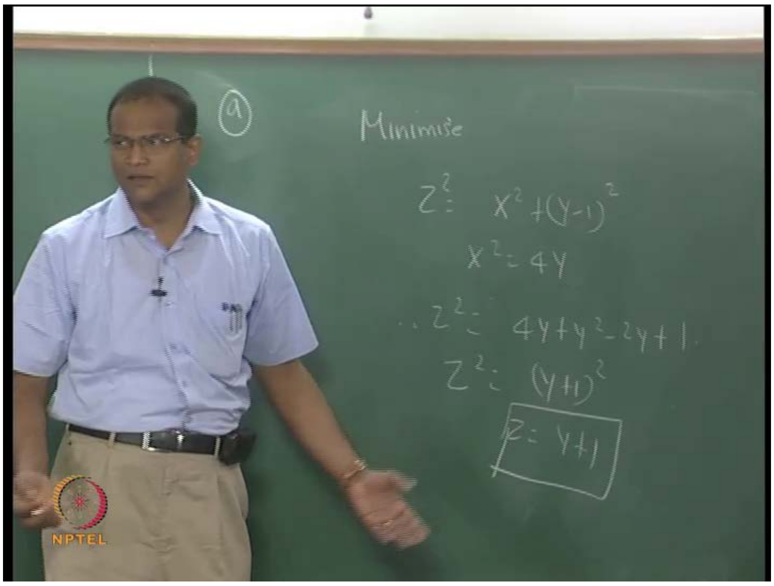

Let us do a part. That is all, after this you cannot do anything.

Student: Sir but we know the range of y equal to 0 to infinity.

No, all that is for the Lagrange multiplier method you do not have to do. Now you are intervening, right, you keep on intervening, but the Lagrange multiplier is without the intervention of the analysis it automatically gives because when we solve this equation x is equal to 0 is direct possibility. Here now you are going to say no, y is not a point you have to do, you have to give additional arguments in order to get the answer. Therefore, this is quite inferior compared to this. So, you just give Z equal to y plus 1. That is also correct but who will give us the value of Y.

Student: Sir can’t we find it using calculus method?

Which calculus?

Student: Using two variables.

Here also I am using calculus.

Student: Yeah.

No, see logical thing is x square equal to 4 Y you want to substitute in that equation and do, it does not work out; that is what I am saying. This just serves to illustrate the point that you cannot convert everything into an unconstraint problem and hope to get the solution. Fine, is this clear? If there are any other interpretations, you tell me outside the class or tomorrow.

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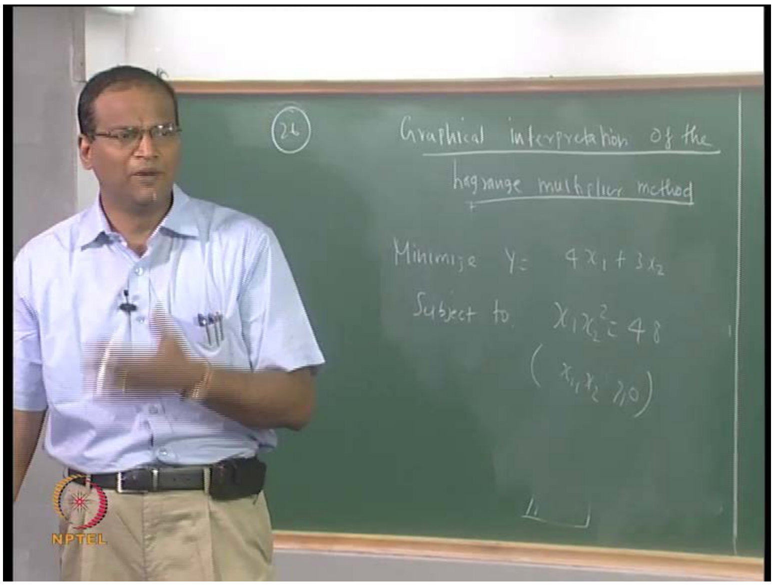

Now let us do a graphical interpretation of the Lagrange multiplier method. So, the methodology I would like to use for illustrating this graphical interpolation of the Lagrange multiplier method is as follows. We will take a two variable one constraint problem. Using the regular Lagrange multiplier method, we will get the solution in 10 minutes; let us say we will get the solution in 10 minutes, so that you know everybody knows for sure there is a solution and it is not a fictitious problem and all that. Then I have brought graph sheets; I will circulate the graph sheets, then you will plot whatever is required and then you try to interpret from your solution of the Lagrange multiplier and your plotting, is there a correlation between these two or is there a correlation between the equations I have given and what you are seeing in the graph and so on. Then I will sum up the discussions so that you remember for life what is interpretation of the Lagrange multiplier method.

Let us take a simple problem. Now I do not have to say adapted from and all that; it is my own problem. Minimize Y equal to 4 x 1 plus 3 x 2 subject to. Of course minimize Y equal to 4 x 1 plus 3 x 2, it could be minimization of some cost subject to some criterion, right. Now use the Lagrange multiplier method; do not try to convert it in to unconstraint problem. Solve it as a constraint problem because we want to know lambda and other things, get the value of lambda, get the value of x 1 and x 2 at the optimum, get the value of Y. In the meantime I will take attendance. Once you are through with the solution after 10 or 15 minutes and I will distribute the graph sheets, then we will plot this and let us see what is actually going on in this problem. Fine, so let us work it out.

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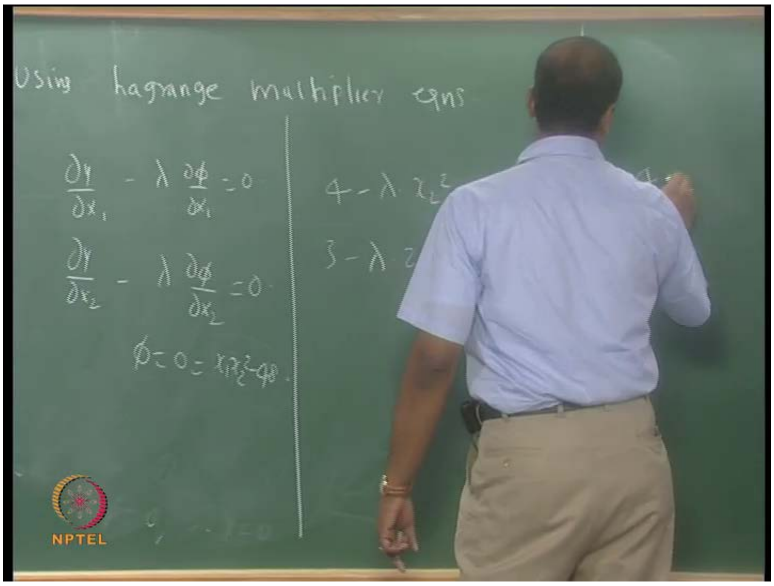

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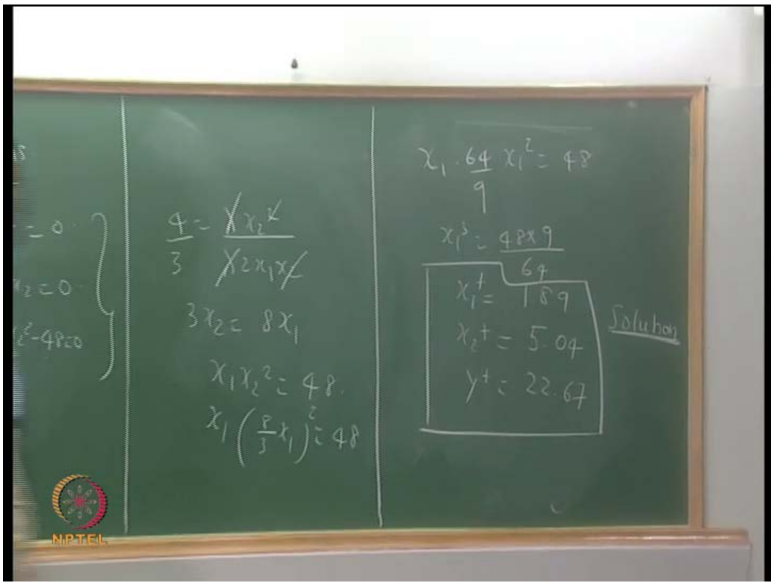

So, dou phi by dou x 1 equal to 0, 3 dou phi by 2 x 1, x 2, right. Is it okay? 1.89, this is correct. X 2 plus is 5 point something, 5 point?

Student: 04.

Y plus 6, Yeah at this stage we do not know whether it is a minimum or a maximum of. One possibility is to convert it in to an unconstrained optimization problem in x 1 and take the second derivative or unconstraint optimization problem in x 2 and take the second derivative if you do not have enough knowledge; otherwise you can go to the Hessian and all that which is too premature to discuss in today's class; tomorrow we will come instead of 10'o clock it is 9'o clock. So, it will be better. So, it is somewhat advanced mathematics is a very tricky derivation. So, we will do that derivation in tomorrow's class.

But now I will distribute the graph sheets. So, we need volunteers. Deepak will distribute for these two rows, the other two rows you do. So, do not worry about y, take x 1 in the X axis and x 2 in the Y axis. Plot the constraint and then think how to solve this problem using the graphical method.

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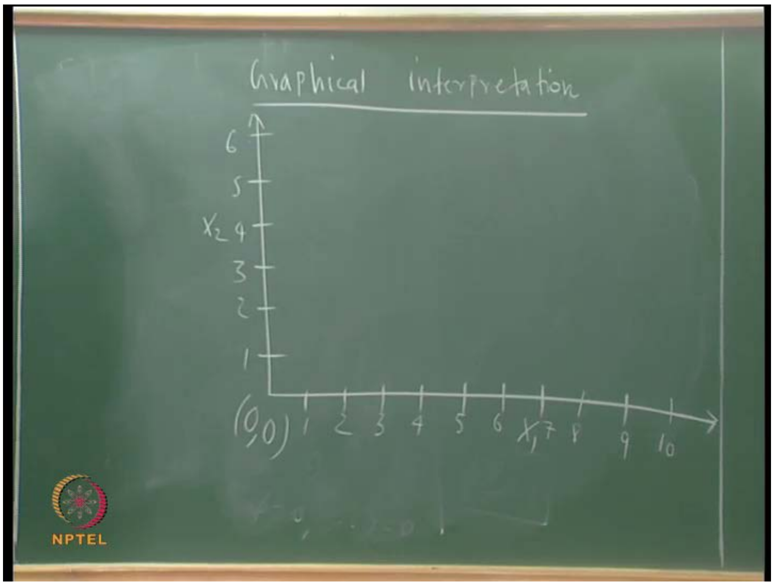

So, the graph sheet is given to you. Take the two axes as x 1 and x 2. It is better to do it on the board or using the computer, I will do it on the board? We have to organize; otherwise the first five minutes in tomorrow's class we will do it on. So, tell me the divisions. So, you want to expand to vary from what? See some 48 is coming, so you have to be smart. There are only 20 divisions of x and 20 divisions of y.

Student: Take x 1 on the y axis and x 2 on the x axis.

That is all right but if you are ready not to get confuse do that; you can take y axis to be x 1 and this, whatever, you are free to explore whatever, but I stick to the conventional this thing. So, what are the divisions you want 0 to 10?

Student: 0 to 10.

X 2 is also 0 to 10, but x 2 is more restricted; is not it because its square root is coming. So, let us say 1 2 3… So, please note that first we are plotting the constraint, so better to drop a table.

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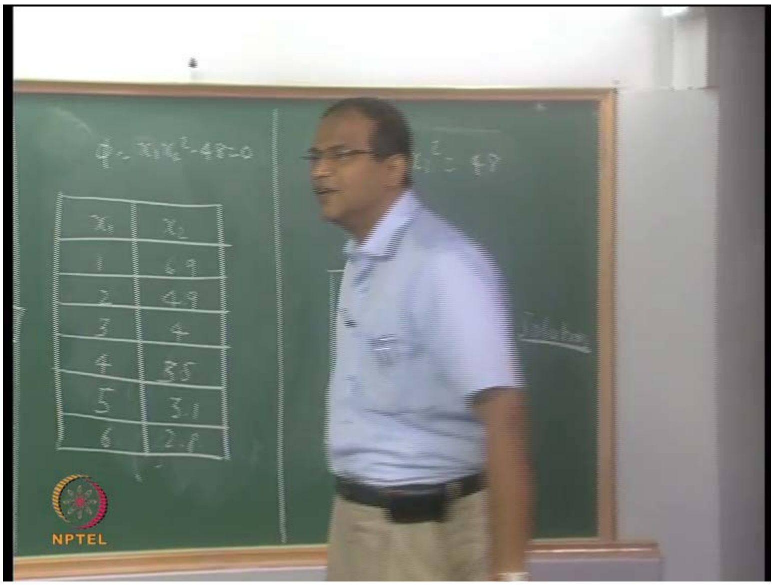

1. What you get 6.8 for x 1 equal to 1?

Student: 6.9.

Good okay. 2?

Student: 4.9.

3?

Student: 4.

4?

Student: 2.5.

Okay. 5?

Student: 3.1.

6?

Student: 2.8.

Yeah, now check these values once; it is possible that Sampath has made a mistake.

Student: Sir 4 is 3.5.

4 is 3.5 yeah, yeah it cannot. It is okay; then check these values once and then go ahead and plot.

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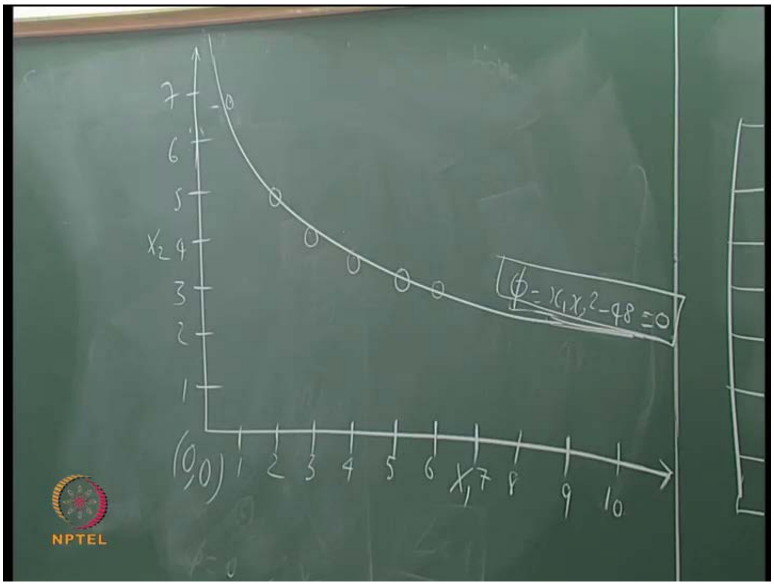

Okay for 1 6.9, so I have to increase this guy. So, 2 is about 5, 3, 4, 5, 6 is 2.8. So, it is going very or we will say, got it? So, some curve it looks like this at least qualitatively. Okay, what next?

Student: Draw lines parallel to it.

You have to draw several lines now. Each of it represents?

Student: Different values.

Different values of y. So, you can take y equal to 4 x 1 plus 3 x 2, y is equal to 10 it will give one line, right, 4 x 1 plus 3 x 2 equal to 10, you plot that line; that line may not meet the curve it is all right but that line is a genuine objective ISO objective line, because it represents 4 x 1 plus 3 x 2 is equal to 10. Now you will say take one more line 4 x 1 plus 3 x 2 is equal to 20, 4 x 1 plus 3 x 2 equal to 24 but if you take 4 x 1 plus 3 x 2 equal to 22.67.

Student: It becomes tangible.

It becomes a tangent to this. Do that, take three lines. Do not be lazy, we have another fifteen minutes. Take three lines.

Let us take Y as 4 x 1 plus 3 x 2 equal to 10, 4 x 1 plus 3 x 2 equal to 15 and do not take 22.67; take 25 so that it goes in to the curve. You will realize; therefore, if you take a scale and move this Y equal to c lines close to the constraint, one particular line will just touch this constraint, okay, that is where you get the solution. Is that okay?

So, let us draw some ISO-objective lines. We will draw at least two lines.

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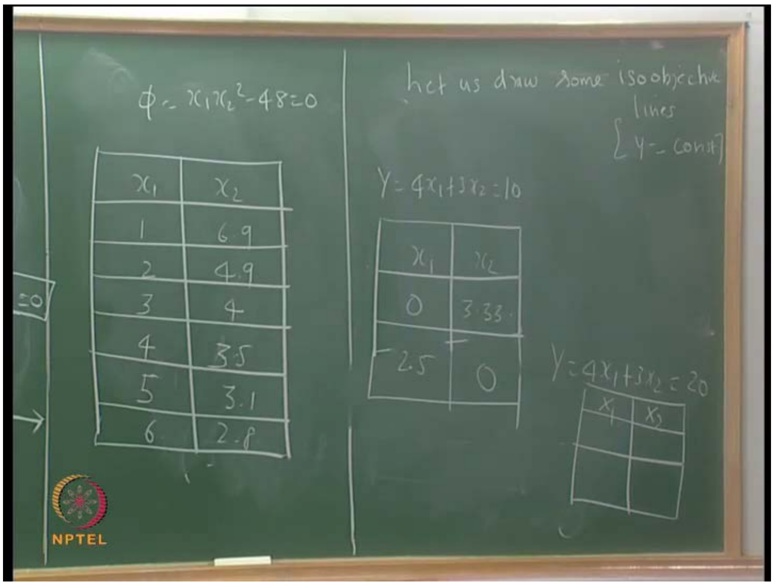

What is the ISO-objective line, Y is equal to constant. Let us say two points are enough, right? Okay, so two points are enough. Are you able to cover this? And you thought that graphical method is simple; Lagrange is better, right. So, let us take x 1 equal to 0, right, x 2 equal to 3.3, x 2 equal to 0, 2.5, correct. So, x 1 equal to 0, x 2 equal to 3.33.

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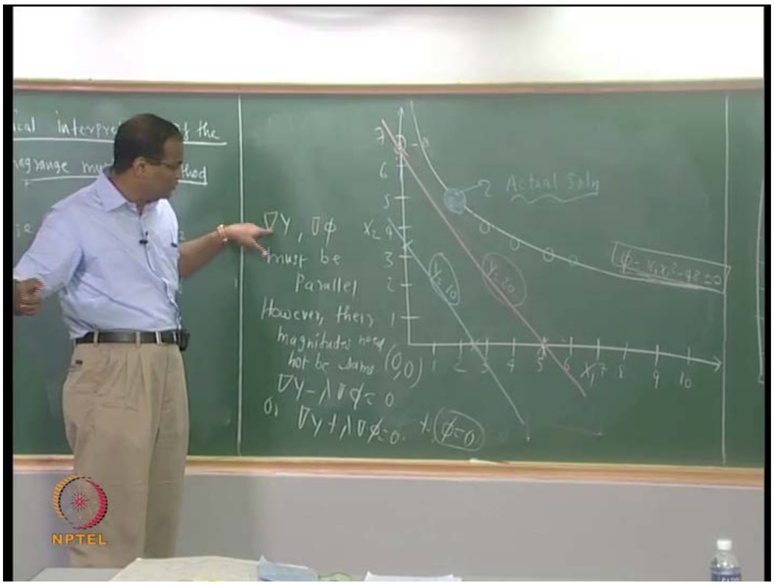

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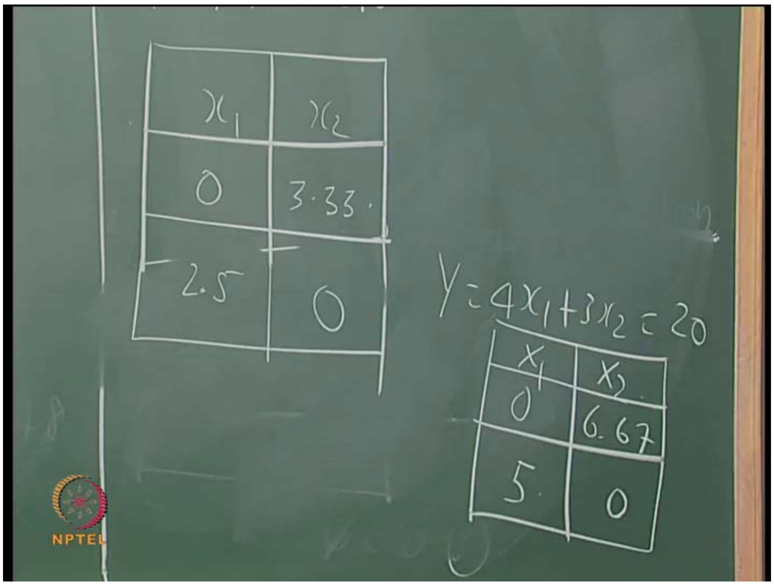

X 2 equal to 0 is 2.5. So, that is Y equal to 10 line. For Y equal to 20, so x 1 equal to 0, 6.67, this is 0, this is 5. Is it right, it is going off. What is the actual solution you got?

Actual solution is somewhere here, right. What you mean by actual solution; the solution we obtained using the Lagrange multiplier method.

Student: Sir I think now what we are doing is whatever we are drawing there is inferable solution.

No, no I have not drawn that, I am just trying to figure out a way of getting the solution because I do not know the 22.67 now. Suppose I do not know Lagrange multiplier method, but it is a two variable problem. Suppose I do not know this advance calculus, I do not know lambda and all that, I want to still solve. So, how I will do this? Okay, is that fine? So, but when we move from Y is equal to 10 to Y is equal to 20 we are approaching close to the curve; that means we are near the answer but not quite there.

So, what is the procedure we are trying to follow? We are trying to draw ISO-objective lines; that is lines representing y is equal to constant where this constant keeps on changing. So, we are trying to move this y is equal to constant lines which are basically parallel lines because they all have the same slope. We are trying to move such that one of those lines becomes a tangent to the constraint. When it touches the constraint at that point you will get the solution because the ISO-objective line is meeting the constraint and any point on the constraint is a valid solution to the problem, because the constraint cannot be violated.

Now if you move further, again the constraint will be violated. Therefore, only if such a line such a y is equal to constant line where that constant is something which is the ultimate value for objective function; that is y optimum, when it just touches the constraint, that is the solution you are seeking. Now mathematically what does it mean?

Student: Sir at least to go further the constraint will still be met in the point of intersection. The only thing is it won't be a minimum solution.

In this case it will be met; it would not be a minimum solution; when you cross that it would not be a minimum because again the Y will increase. See so which means when your left of the curve, you can get values of Y which are very low. If Y is the cost you can actually cheat, you can try to cheat, you can report very low values of Y. Because Y is equal to 4 x 1 plus 3 x 2, you can take some arbitrary values of x 1 and x 2 which satisfy this and say that that is the solution, but your x 1 x 2 squared is equal to 48 will not be met. Okay, right.

Therefore, when it exactly cuts this that represents the minimum cost to the problem at which the constraint is not violated. Now, what does this Lagrange multiplier method do? So which means the tangent to the constraint equation at the optimum solution at the optimal point and the ISO objective line, they are parallel to each other. An alternative way of saying that these two curves are parallel to each other is saying an alternative way of saying this is the gradient vectors will be parallel to each other. If the gradient vectors are parallel to each other, I am not saying that the gradient vectors will have the same magnitude, but I am saying that the gradient vectors have to be collinear; they can even be pointing in the opposite direction.

So, if you want to say that the tangent to this curve on the ISO-objective line are parallel to each other, it is akin or analogous to saying that delta Y and delta phi must be parallel; however, their magnitudes need not be the same, they can even be pointing in the opposite direction. Therefore, mathematically if you want to say that delta Y and delta phi must be parallel, but their magnitudes can be different. The only way of doing that will be delta Y minus lambda phi is equal to 0, if you do not like this or delta y plus lambda phi is equal to 0. Therefore, we are saying that the gradient vector the gradient del of Y and del phi must be collinear vectors.

Okay, but now the thing is, is that all? Though the directions are satisfied by this, the solution must be on the constraint itself; therefore, the constraint equation is also satisfied. Therefore, del Y plus lambda del phi equal to 0 is not the only thing, this in conjunction with phi equal to 0 because anyway the final solution we are seeking is a point on the constraint equation itself. Therefore, if you come up with the set of equation del Y plus lambda phi equal to 0 or del Y minus lambda phi is equal to 0 and you solve this set of scalar equations in conjunction with equal to 0, then you will get a solution which satisfy the constraint which satisfies all these properties. So, this is the graphical interpretation of the Lagrange multiplier method. Is that okay, fine?

So, this point is very important, right. Ultimately this point is very much a point on the constraint. So, the test for maximum or minimum we have a few minutes but I think we do not have to start it today; we will do it tomorrow how to do this Hessian. Any doubts I can answer now. So, del Y and del phi must be collinear vectors.

Student: How did phi equal to 0 come?

Phi equal to 0 came anyway the final solution is a point on this. We are not seeking any solution which is not on the constraint, right. But this is not so definitive; this only says that this should be those on the tangent and this must be in the same direction that is all. Anyway, the point must be on this itself, alright. Yeah, you think about it; it may be little bit not clear now, but once you think about it for some time, it will be, it should become clearer, alright.