Design and Optimization of Energy Systems
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Lecture No. # 19
Least Square Regression contd…

In the next two classes, we will try to wrap up our discussion on regression. In the last two classes, we have solved two problems. One on fully developed flow in a pipe; where, we are trying to regress the Nusselt number as the function of Reynolds number. The other was the unsteady convection problem in which lumped capacitance system is losing heat. And, it is a first order system; which means theta is equal to theta e to the power of minus t by tau. So, both the forms are actually not linear regression. But, they can be reduced to linear regression if we take log. So, we took log and reduced it to linear regression; and, we did e n sigma x i y i minus sigma x y whole square by… – all those formula we applied. We also found out how we got this formula using a linear least square and trying to minimize the S, which is sigma r square; where, you try to minimize the residual in a least square sense.

I also told you how this is linked to the Gaussian distribution; how you are trying to maximize the likelihood. So, this is actually the maximum likelihood estimate. These two problems I want to demonstrate through Excel. There are several ways of doing it, you can just plot xy and then go to Excel and then draw a line or a scatter plot and then enable the equation, add a trend line, and get the regression. That is one level. The other level is used; what is called the data analysis tool pack. I searched for this before coming to this class. Unfortunately, I am not able to get it. You can use data analysis tool pack, which will give you lot more information; information in the sense, what are the errors associated with each of these exponents; what are the errors associated with these components and all these. Errors associated with the constants, the exponents; and, you will get additional information. Then, you can get the standard error of the estimates. So much of… So much additional information can be obtained. If you have Office 2007 and if the data analysis tool pack is enabled, you can try these problems out. There are other tools available. For example, like data fit and so on with which… Or, you can write a code in matlab or something with which you can do all these.

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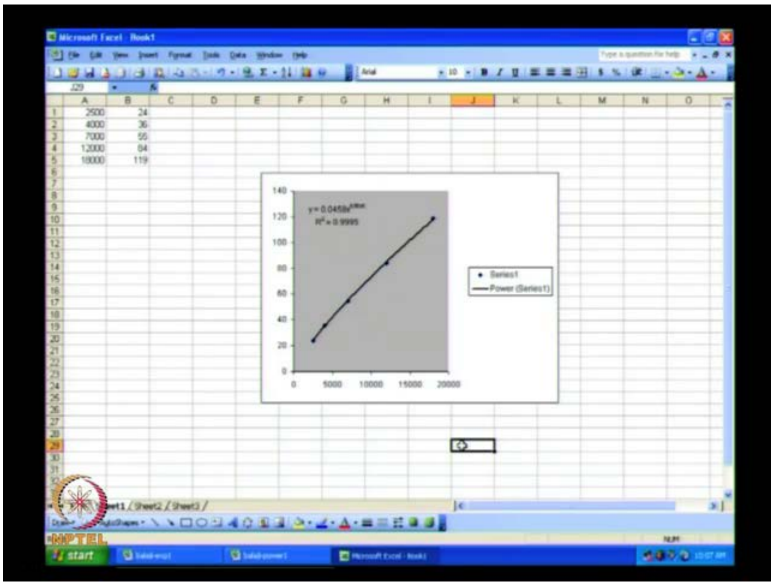

Now, I will do a simple demonstration of these two. The data are given on the board. We already have the answers to this. The first is fully developed flow in a pipe. So, 2500.

Student: 18000

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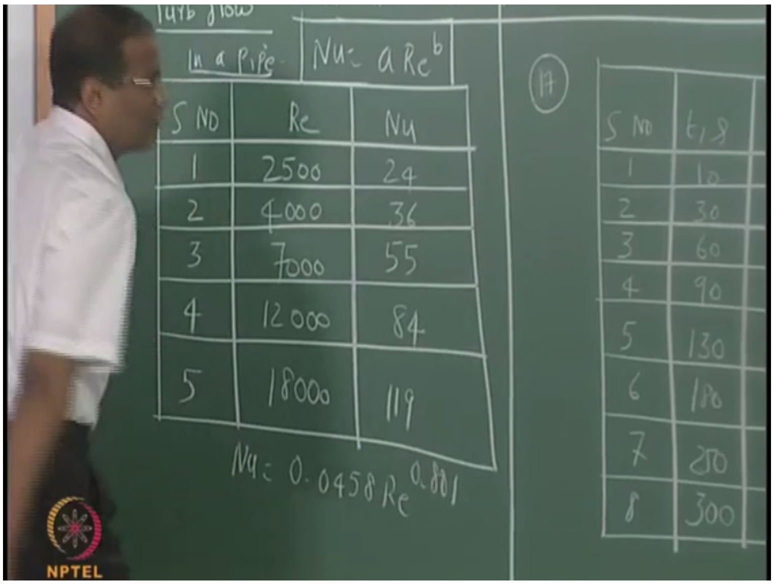

Student: 24, 36, 55, 84, 119

You can (Refer Slide Time: 02:51) convert it into line and then do it or you can directly fit. So, you just take these two. You can also draw a line, whatever. So, this gives you an idea. Now, we can add a trend line. What is the power... We want to use the power-law form; power. We can go for options; display equation on chart; and, display r square value; that is it. So, what did you get? 0.0456 into Reynolds; 0.801; very close. What is the use of doing all these? The goal is not to use Microsoft Excel; the goal is now… Some of you should be able to do something, which is better than this. Anyway, this is how it does; can do it very fast. But, if you have this data analysis tool pack, you can put y equal to a. So, I can have additional columns. For example, here into Prandtl number, into aspect ratio, into viscosity correction factor [FT] So, you can have all these. Then, it will give you values of a, b, c, d, and so on; that is, y is equal to a into x 1 to the power of b into x 2 to the power of c and so on. And then, you can do other things; is equal to 58? Point?

Student: 0.0015

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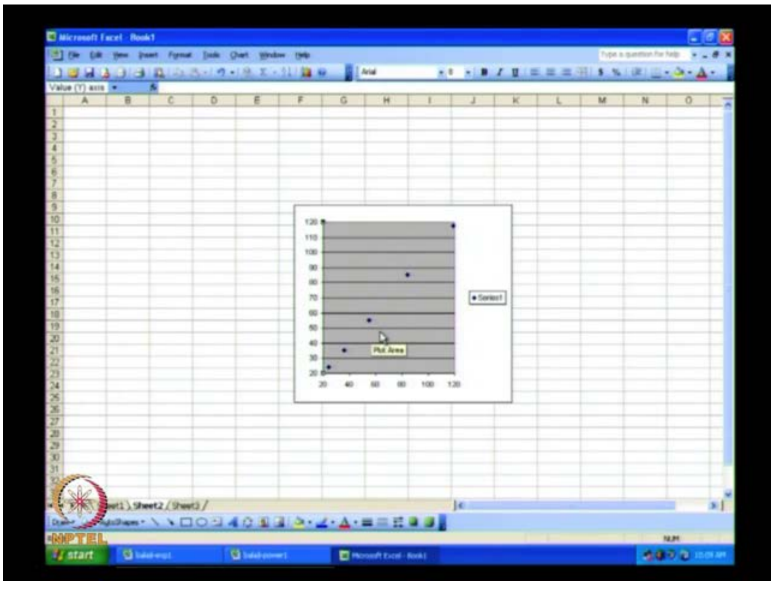

What is (Refer Slide Time: 05:32) this? I can change this. What is the lowest? Minor axis, major… This is basically called the parity plot in which everything is rid. We started with the regression and then I changed it and so on. These are not based on the experimental data; experimental data will show lot of scatter. If you get something like this and report your papers; invariably, if I am the reviewer; I will doubt whether an experiment was conducted at all or… because it is too good to believe. There will be lot of scatter, because lots of uncertainties associated with the measurement of temperature, bulk mean temperature, measurement of velocity and so on. Everything is associated with an uncertainty. In fact, the Dittus-Boelter correlation – 0.023, Reynolds – 0.8, Prandtl – the 0.4 for heating, and Prandtl – the 0.3 for cooling, is associated with (( )) plus or minus 30 percent. But, there is no problem; this plus or minus 30 percent hardly matters at all, because invariably, what ultimately decides is overall heat transfer coefficient, because m cp delta t of cold is equal to m c p delta t of hot is equal to u a l m T d. So, you have to calculate the overall heat transfer coefficient.

If the turbulent flow inside a pipe; and then, you have got natural convection or convection over air on the other side; even if you have fins, heat transfer coefficients inside will be 2000 or 3000 watts per meter square per Kelvin; the outside will be 150 or 200 watts per meters per Kelvin. 1 by u will be 1 by 1100 plus 1 by 3000. Who cares whether it is 3000 or 2700 or 3300, because this is not what is called the controlling resistance. The controlling resistance is the one, which is got a lower heat transfer coefficient, so that one over the lower heat transfer coefficient will have more power in deciding what eventually the overall heat transfer coefficient is. That is why fins are not kept on the inside. There is no point in improving 2000 watts per meters square per kelvin to 2400 watts per meters square per kelvin. You should know where to put your money, because the fins are always kept on the side on which the heat transfer coefficient is very low. So, this is how you can keep on doing this.

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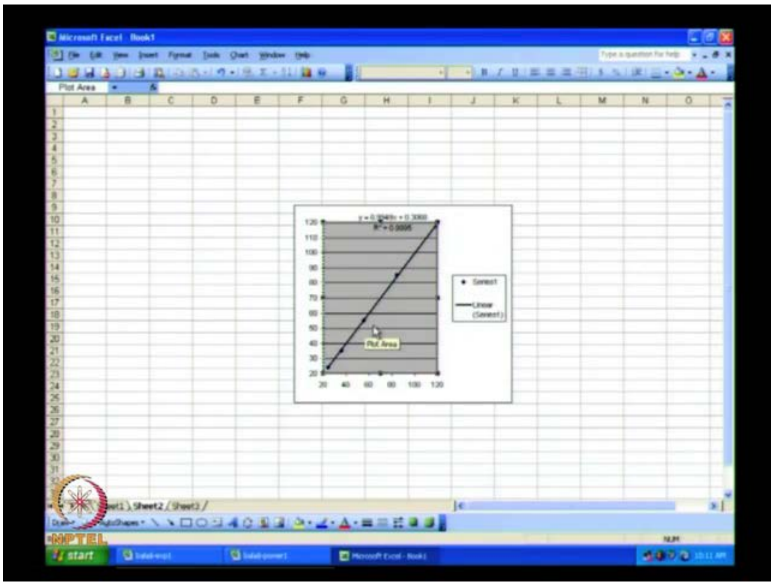

In fact, you can put a trend line to this also. That looks very silly, but we will do this. Linear? So, very linearly correlated – 9949. So, it is where without… So, it is without bias and all that.

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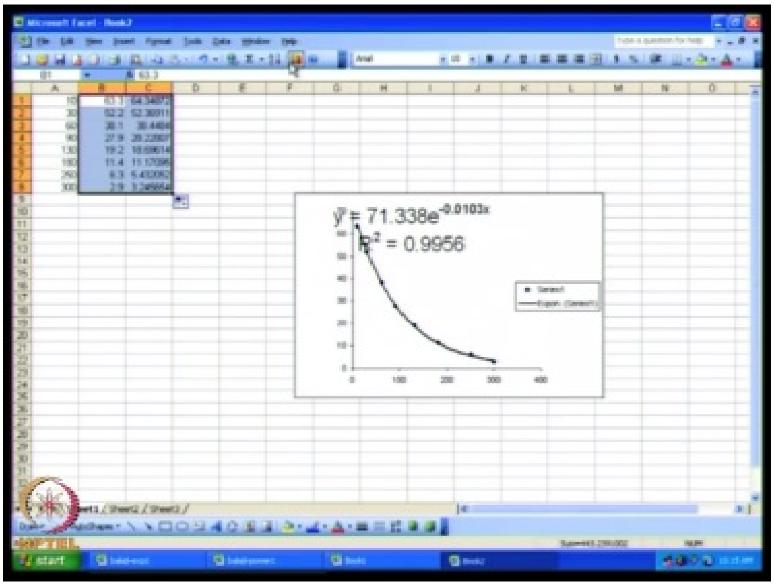

This is problem number 16. You can use data analysis tool pack and do multiple linear regression. But, with the Excel, you can do only linear regression. I am go to talk about non-linear regression, which you cannot do with Excel. So, you have to do… You write your own code or you have to use some advanced this… because Excel even housewives can use and all that. Excel is basic simple; it cannot be used for scientific work, but to limited extent you can use it.

Now, 10?

Student: 30, 60, 90, 130, 180, 250, 300.

That is good. These are the time. So, this is a cooling of a first order system. Column A gives the time in seconds; columns B gives the temperature excess, that is, temperature minus 30 centigrade, where 30 is the ambient temperature. Please recall.

Student: 63.3, 52.2

52.2, is it?

Student: 28.1, 27.9, 19.2, 11.4, 6.3, 2.9.

So, there is theta versus the t; t versus theta or theta versus t. So, we will do the same thing. I am trying to get rid of that grey color; why is not going? Pink color; what is the problem? I removed the grid line; all right.

Student: Format plot area.

Format plot area; none?

Student: Area – none.

Scale? Scale is giving trouble? It is ok. Now, it is an exponential; that is, scale is seen; add trend line; exponential, options. So, what did you get? 71.33 e the minus. So, this was what? Nu is equal to 0.0458 Re to the power of 0.801(Refer Slide Time: 11:04). Theta is equal to 71… 0.01? Are you able to see that? Is it possible to increase the size? It is obscenely big. Anyway; 0.0103. So, what is the tau you are getting?

Student: 100

100. It is not quiet 100. 100? Now, you can do the… No? Is it correct?

Student: (( ))

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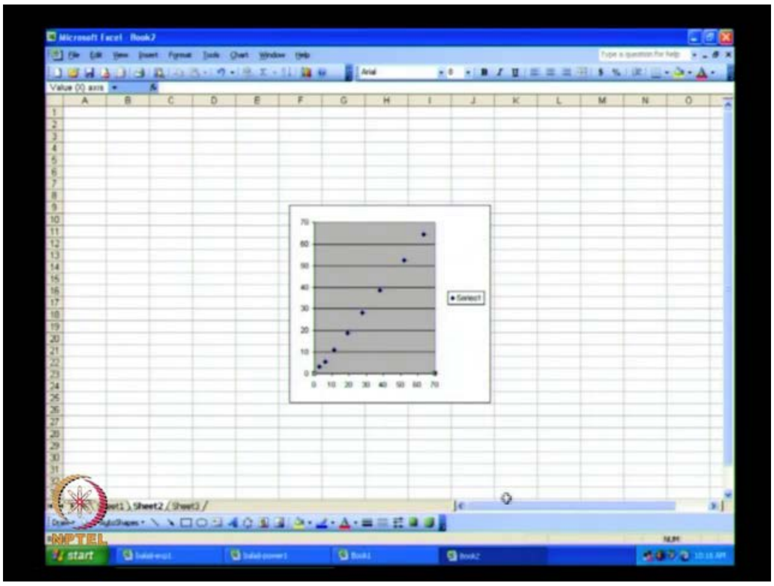

Make it a square plot. Why? This axis? What is the maximum?

Student: 70

This is what is called the parity plot or scatter plot or scatterogram. This is the standard way of… After you get research result, these are called post processing of the result. This is how you post-process the results and present the results when you are writing a project report or a thesis or when you are presenting your paper in a conference or when you are writing a journal article; all these are… You have to… Once you propose a correlation, you have to prove that it is the… You have to establish the so-called goodness of the correlation by looking at r square by the scatter plot. In fact, if you do this what I told you – I was telling you data analysis tool pack; it will tell you 71.33; what is the error associated with this, what is the error associated with this, what is the error associated with this and so on. What is the confident level? Plus or minus 3 sigma and all that. It will give additional statistics, kurtosis, skewness; you can keep on higher order moments. Variance is the second moment, because it is already something whole square, difference whole square and so on. So, additional statistical parameters can be generated. So, this is …

In so far as least squares is concerned, now, I will tell you how least squares can also be explained or done with the help of matrix algebra, because this establishes the link between linear least squares and non-linear least squares. Then, I will establish a case, where a standard thermal system or heat transfer problem; we get a final form of the model. So, this is basically the model (Refer Slide Time: 14:25). This is basically the model governing the phenomenon. This is the model governing the phenomenon. All these are amenable to linear least squares. But, I will tell you there are cases where they are not amenable to linear least squares; you are forced to use non-linear least squares. So, we will establish the case for that and stop. And in tomorrow's class, we will do this non-linear regression technique, Gauss Newton algorithm, which is very powerful algorithm. The extension to which is the LMA – the Levenberg-Marquardt algorithm; LMA – which is one of the most powerful and widely used techniques for solving optimization problem. This… Actually, we are solving an optimization problem already, because you are minimizing some least squares. So, you can minimize a cost function. So, whatever technique I am teaching you is also an optimization technique. Least square is basically minimization. Least square minimization is basically an optimization technique. So, we will close with this and then we will… I will take attendance now.

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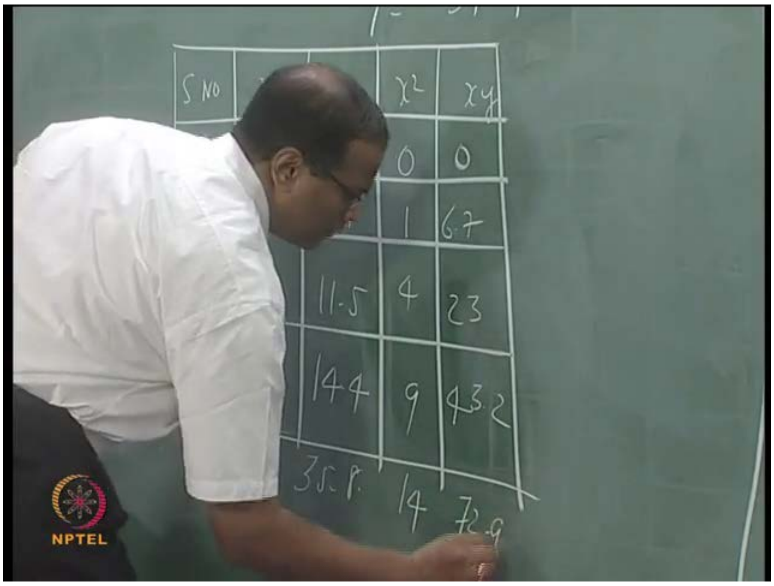

Let me introduce linear least squares with matrix algebra. And, to drive home the point that it can be written in very compact form; which will be very useful for us when you actually do the non-linear regression. Let us take a simple case. Y is equal to… I have worked out. So, I will use that. Let us say Y equal to 3 plus 4x. To make matter simple, let us take only 4 data points: serial number, x, y, x square, xy; 1, 2, 3, 4. When x is equal to 0, what you get? Let me put 3.2. When x is equal to 1? I will put 6.7. When x is equal to 2, 11; let me put 11.5. Please note that I am putting on either side of the line. This is about 14? So, 14.4. So, you will not get 3 plus 4x; you will get something close to that. Now, we can do the x square and all straightforward; 0, 1, 4, 9. This is 0; xy – 0, 6.7, 23, 43.2. Please work out the sigma; 6, 14, 35.8…

Student: 72.9

Good; 72.9.

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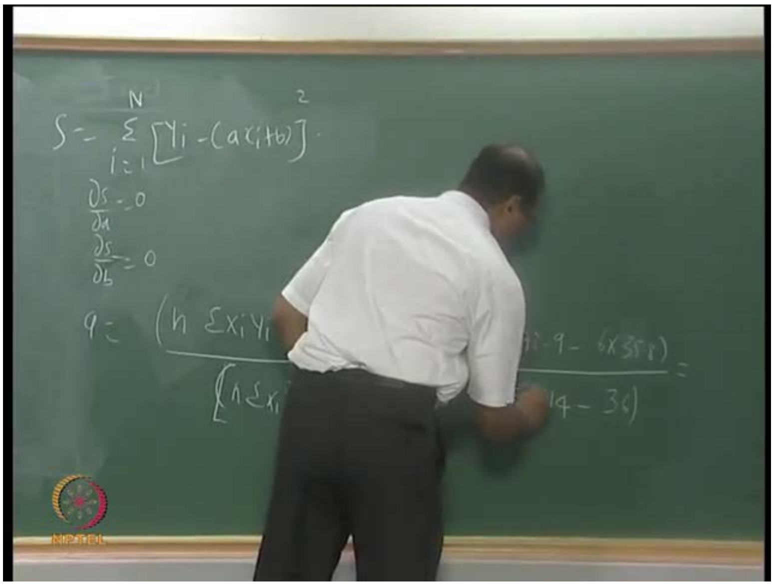

Now, you know this. This is from the theory of linear least squares. We already know this. And, we have the expression for a. What is a? Abhishek? Not very difficult; 4 into… You know what you have to get approximately. Correct; minus sigma x i… No? Into 35.8 divided by 6 into 14 minus 36; correct?

Student: (( ))

4; 3.?

Student: 14

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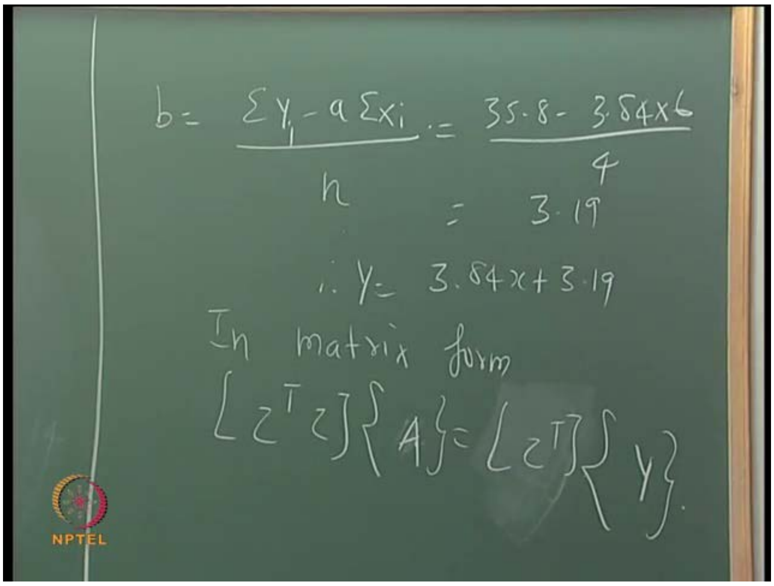

Then, b is equal to… You have a formula for b. Sigma Y i minus a sigma X i divided by n is equal to 35.8 minus 3.84 into 6 divided by 4 is equal to 3.19. Therefore, Y is equal to 3.84 plus 3.19. We are not able to get 3 and 4, because we tampered with data.

Deepak, you are present? Both the Deepak's are present.

This is nothing; it is silly; we already worked out tougher problems. So, what is the point in taking up the simple example? I want to demonstrate, how the same thing can be done using matrix algebra. So, the whole thing can be written as… You can work out the standard error of the estimate, r squared and all that. This is simple too. In matrix form, you write it like this; Z transpose T Z operating on A will be equal to Y. I will tell you what Z transpose A. And, Y you know; Y is basically Y i. Now, you may have lot of questions – how this is coming and all that; but, when you actually write it out, it is so evident. Let this be there.

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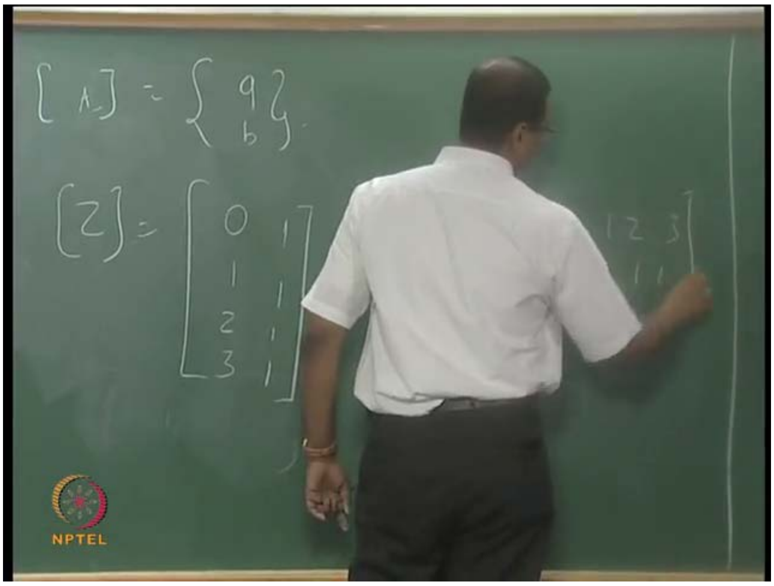

What is this Z now? What is Z now?

Student: Sir, what is A (( ))

That is a good point. What is A? If I say A is… Let me… It is (Refer Slide Time: 24:01) not Y; it will be Z operating all these. First, let me… A is fine. What is Z first? Hold on to that equation; what is Z? It should be Z transpose T on Y (Refer Slide Time: 24:27). So, what is Z now? Z is 0, 1, 2, 3; 1 1 1 1 is basically to… Something like an identity, because you want to make it into 2 cross 2. What is Z transpose? What you get actually now? Z transpose T Z. Z transpose T Z; what you get? What we get?

Student: Sigma (( ))

So, if you… Then, it becomes so evident to you.

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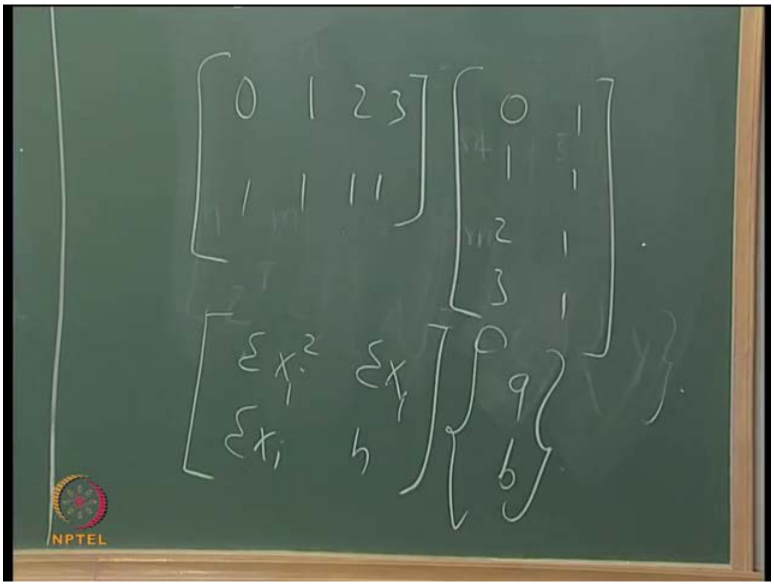

Now, if you do this Z transpose T Z into Z or… What you get? n. I told you a is a b (Refer Slide Time: 26:26). So, this is operating on a b. So, a sigma X i square plus b sigma X i is equal to what? Right side, what was the original equation? LSR? I wrote the equation here. Sigma X i Y i. Left side is clear? When you actually do d s by d a and d s by d b and equate it to 0 whatever you are getting; a portion of it you have already got by writing out a Z matrix like this and doing the operation Z transpose T Z.

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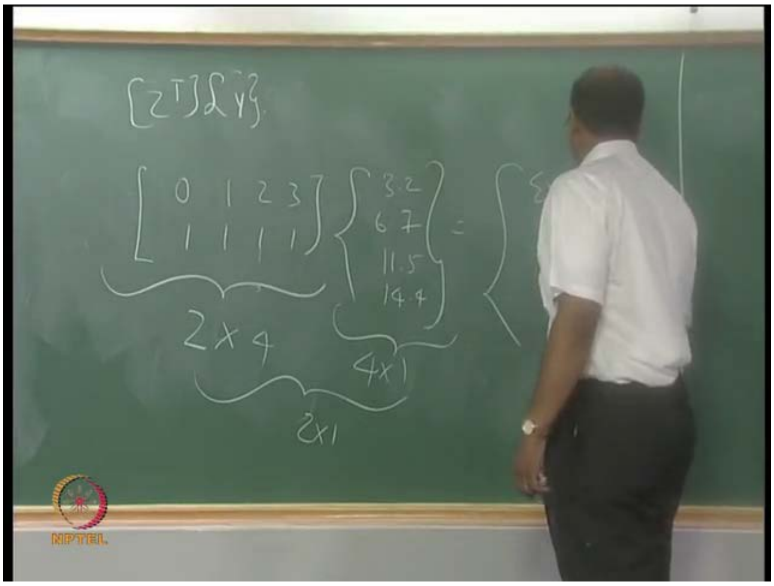

Now, what is Z transpose T Y? What is Z transpose here? Y is? 0?

Student: 3.2

Correct?

Student: 3.2

No; 0; 3.2, 6.7, 11.5, 14.4. Is that correct? Am I writing the right one? So, this is? It is a 2 by 4 matrix. Correct? This is a 4 by 1. When you multiply, you will get? 2 by 1. Correct? Imran? What is this actually? Sigma X i Y i. What is this?

Student: Sigma Y i.

Good. Sigma Y i. So, when you propose Z transpose T Z operating on a is equal to Z transpose T Y, you are very compactly, succinctly, concisely, epigrammatically, you have written the linear least square. Now, all your tools available for matrix inversion, this thing can be used; there is no need simultaneously solve like what we did. Are you getting the point? In the next 5 minutes, I will demonstrate, how by simply inverting the matrix, you would get the same 3.19 and 3.84; you can yourself do it. So, Z transpose T Z operating on a will be equal to Z transpose T operating on Y. The right side is what is called the forcing vector. Right side is called the forcing vector. It is same thing like… How many of you have learnt the finite element?

Student: (( ))

Stiffness matrix operating on? k times?

Student: (( )) equal to the applied (( ))

k times something is equal to f; where, k is the stiffness matrix; and where, the right side is the forcing vector. So, it is like this. So, the Z transpose T operating on Z, Z transpose T Z operating on A, will give you Z transpose T Y.

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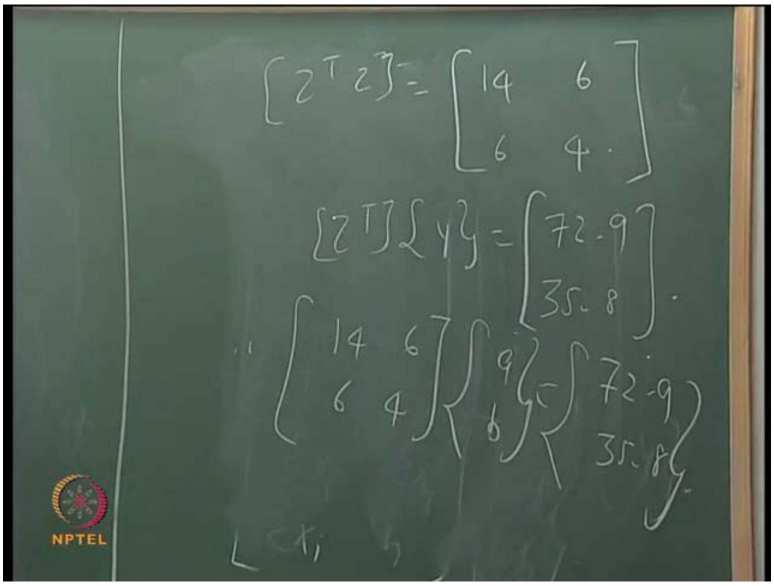

Now, we will solve this using the… What is Z transpose T Z? You already got it, no? Sigma x square…

Student: 14, 6, 6

14, 6, 6? 14, 6, 6, 4. Is it? Then, Z transpose T into Y?

Student: 72.9

72.9?

Student: 35.8

35.8…

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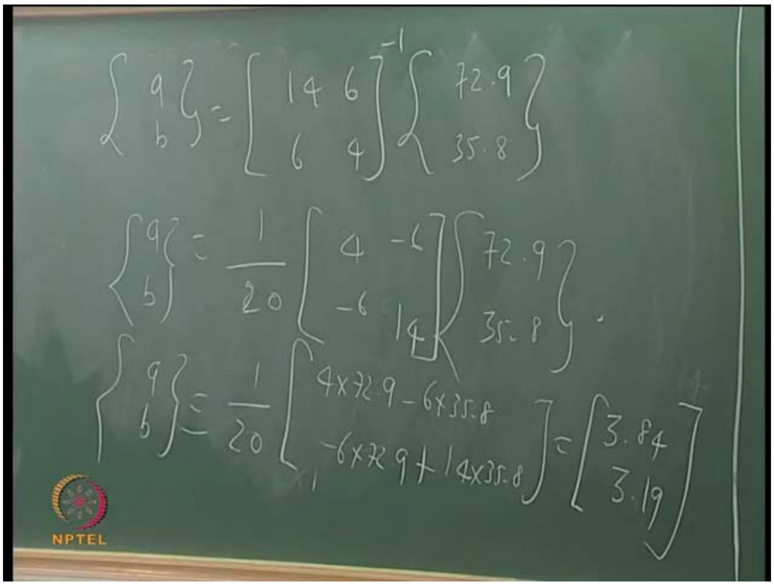

It is just a b, is the inverse of this matrix; inverse of this matrix. First, get the determinant. 14 into 4 – 56 minus 36; 20. So, change the two places and change these two. You have already inverted now; into 72.9 into 35.8. Therefore, a b… 4 into 72.9. What you get?

Student: 3.84, 3.19.

Good; that is it; we cannot get anything else. This is another way; this is very smart way of doing it; you can do 3 by 3, 4 by 4. Beyond this, you do not have to worry why… I mean you will be worried, you have to solve simultaneous equations, all that. You can use the power of matrix algebra, if you have several unknowns. No need to use Excel at all; you can write your own code in scilab or matlab and do regression. If you have data, which is coming out of simulations or coming out of the experiments, you can use this, you can use matrix algebra to accomplish this. So, this is the matrix notation.

But, now, let us go to non-linear least squares. First, we will establish the need, why non-linear least square arises or non-linear least squares arise; non-linear least squares arise in thermal system. Is this clear? That 1, 1, 1, 1 was basically obtained, so that you get a 2 by 2 system and all that. We cannot just work with column vectors. If you disassemble the (( )) of your desktop, you will see processor; on the top of the processor, you would have seen an excluded aluminum heat sink. Have you seen? How many are there?

Student: 2

More than one. How many fans are there in the normal desktop?

Student: 2

There are 2 fans. One is a fan, which is located near the exhaust, I mean, outlet, which is sending out; the other one is the dedicated fan – fan dedicated to the CPU. So, the second fan will turn on only after you… When you boot, the second fan will not turn on. After you run several applications, you do multitasking, you open all windows; then, suddenly one sound will come. So, when it exceeds certain temperature, the other fan will turn on. So, if you do this…

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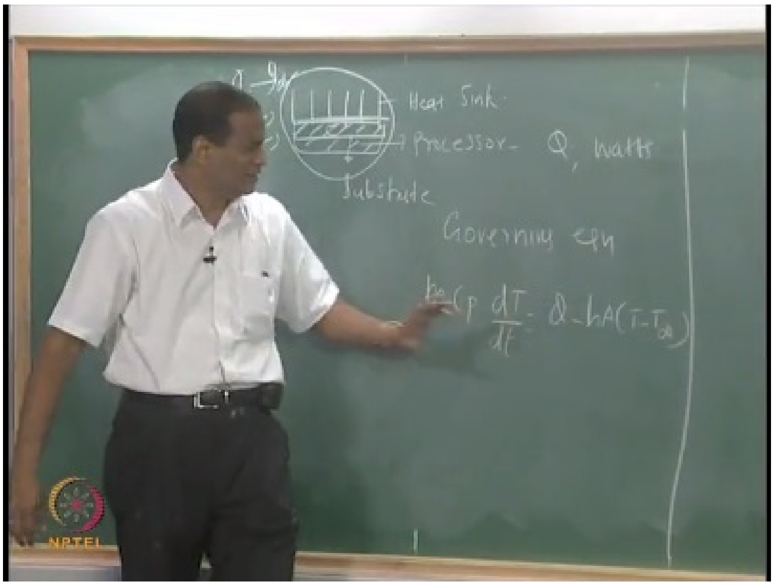

Basically, if you have a heat sink like this; you have something like this; I can say that, this is a substrate; this is the heat sink. Somewhere you will have a fan. Suppose the whole thing is such that, it is made of highly conducting material; and, the temperature – if it is going to be at 1 temperature, we can treat the whole thing as a lumped capacitance system. When the system is turning on, I want to find out the temperature response of this, because there is a processor. Let us say there is a processor, which is generating heat at the rate of Q watts; this whole thing is having a mass m; it is having a specific heat c p; and, the heat transfer coefficient away afforded by the fan is h; and, the surface area is

A. If you treat the whole heat sink and the processor substrate, everything put together to be at one temperature; which is an ok assumption to start with.

Suppose you want a very gross model. What is the governing equation for the problem? m c p dt by d tau; governing equation will be m c p dt by d tau is equal to Q minus hA into T minus T infinity. What is the story of this? Initially, when you start, this fellow is… Watch carefully, this fellow is constantly generating heat at the rate of Q watts. But, you are starting from cold. So, the T is equal to T infinity. When T is equal to T infinity, even though there is a h of 100 or 200, there is an area h A is available. The delta t is so less that the h A delta t is not able to compensate the Q. Q minus h A delta t is positive; it forces the system to higher temperature. The system temperature keeps climbing up. But, when the system temperature keeps climbing up, the Q is the same, because it is a constant heat generation for this processor. This delta t is going up. Therefore, this is becoming constant; this is going up; and, the time will be reached when this Q is equal to the h A delta t. Therefore, the left side is switched off; that means the m c p dt by d tau becomes equal to 0. So, the system approaches steady state. After the system approaches steady state, when you are trying to shut down the system, what happen is; no q equal to 0. Now, it will be m c p dt by d tau will be equal to minus h A delta t. So, it will follow a cooling curve, which is analogous to what we studied in the previous class.

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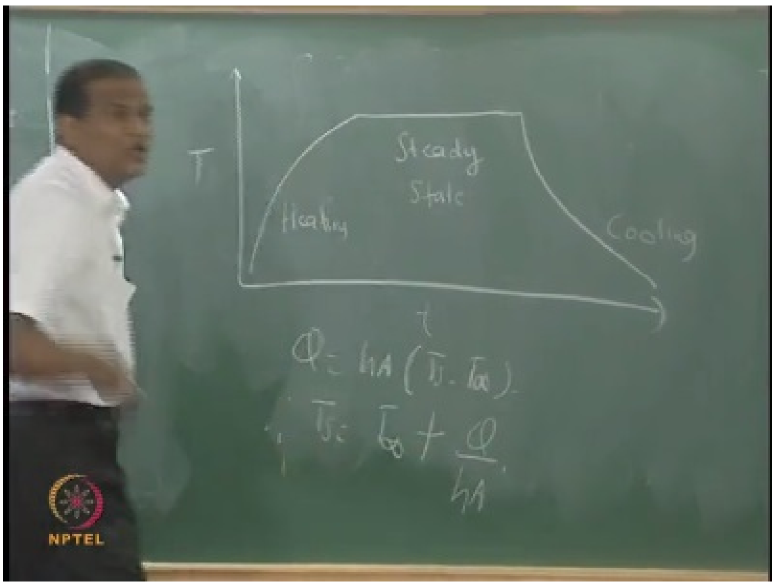

Therefore, if you plot the temperature of the time history for this problem; when it is starting up; when it is operating under steady state for a long time; and then, it is shutting down; the temperature profile will look like this. Correct? Heating, steady state, cooling. If you are actually a chip designer; before you begin to worry about the unsteady state; you will be worried about the steady state temperature. So, for steady state temperature, you do not have to solve all these equations. When equilibrium is established, Q will be equal to hA into T s minus T infinity; T s is equal to T infinity plus Q by hA. If there is a limit set by the manufacturer, the maximum permissible temperature of the chip – if it is 80 degree centigrade, its ambient is 30 degree centigrade, T s minus T infinity is fixed; and, the Q – you know what the processor is going to dissipate. Therefore, you do not have much choice. If there is a fan; you know for force convection it is only 100; then, it tells you what is the size of the heat sink; simple. This is the chip design; that is all. But, how long will it take for it to reach this; how long will take it for it to cool? Next time, you should again heat it and all that. Then, you will have to solve the differential equation. The solution to this differential equation is interesting.

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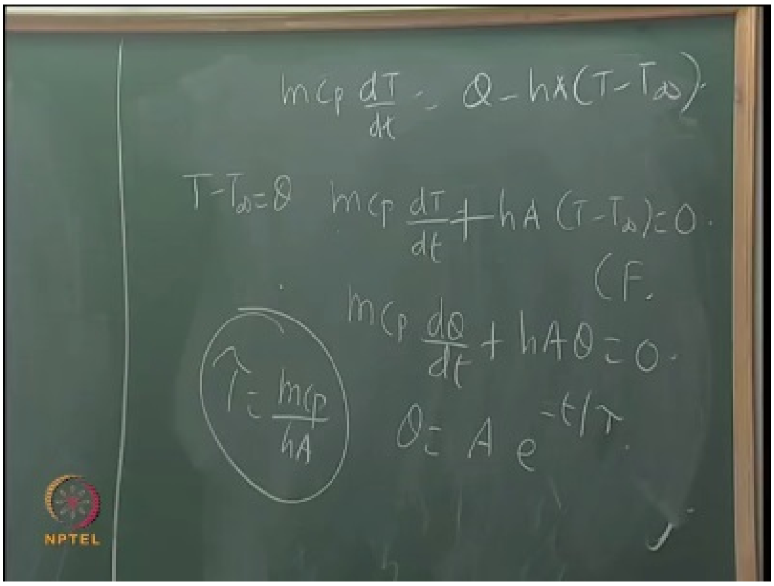

m c p dt by d tau is equal to Q minus hA into T minus T infinity. So, m c p dt by d tau plus hA into T minus T infinity is equal to 0. So, this is basically the complementary function. So, we can say that, T minus T infinity is equal to theta. So, this will be m c p d theta by d tau plus hA theta is equal to 0. So, the solution will be theta is equal to A into e to the power minus t by tau. So, this is the complementary function.

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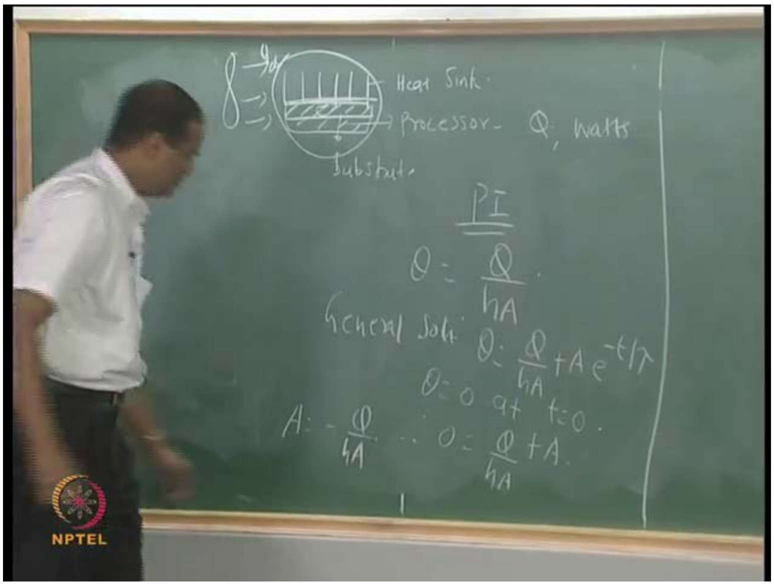

The particular integral will be theta is equal to Q by hA. A particular integral is one which satisfies the differential equation exactly. So, if you take the theta equal to Q by hA; if you substitute back into the (Refer Slide Time: 40:48) differential equation; it is not a function of time. Therefore, this vanishes. Now, if you substitute for theta here; hA into theta is equal to Q by hA. hA, hA, will get cancelled; you will get Q on the left side and you will get Q on the right side. You have to you have to do it here. So, m c p dt – this will be 0; hA into – this is Q by hA. hA, hA will get cancelled. Q minus Q is equal to 0. So, you will get 0 equal to 0. Therefore, this satisfies the governing equation exactly. Therefore, the complete solution to the problem will be the sum of the complementary function plus the particular integral. So, the general solution will be theta is equal to Q by hA plus A e to the power minus t by tau. That is a constant A, which has to be figured out. The constant A has to be figured from the initial condition. When you start at time t equal to 0, what is the temperature?

Student: (( ))

Theta equal to?

Student: 0

0. So, theta is equal to 0 at t is equal to 0. Therefore, 0 equal to Q by hA plus A. A is equal to minus Q by hA.

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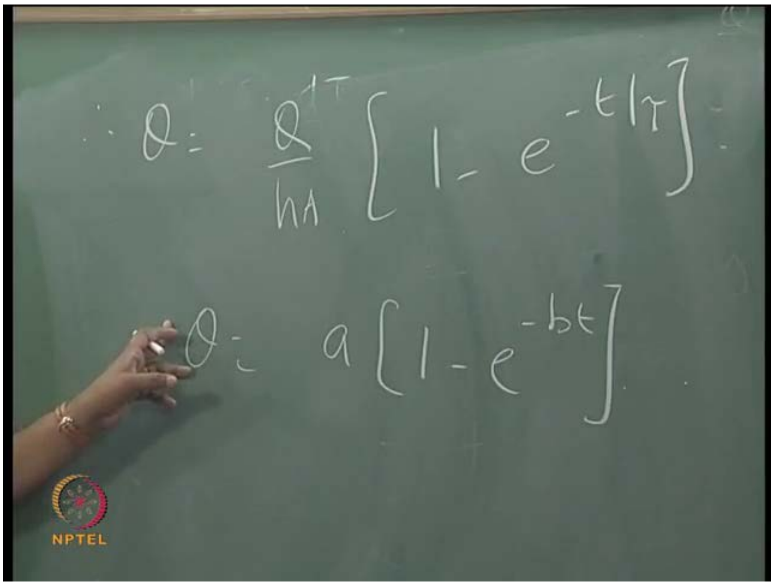

Therefore, theta equal to Q by hA into 1 minus e to the power of minus t by tau. So, I can say theta – a into 1 minus e to the power of minus bt, What is so great about this form, theta equal to a into 1 minus e to the power of t by tau – 1 minus b t. Whatever tricks you do, this cannot be reduced to a linear regression model. You do whatever – log or you take 1 by x c c; you turn upside down, you cry; whatever you do, it will not become a linear regression form; that is it. You have to surrender. You try; if you want, you try. a will be the problem; you can take a to the left side; you will take ln and all that; you can get theta by a. Who will tell you the a. Now, we have come to a stage, where a and b cannot be determined using linear least square. What is this so-called non-linear least square? You will assume with… You will start with the value of a and b, find out the residual; and then, see as you keep changing this a and b with new iterates, whether the residual is falling down. In one shot, you cannot get it, because it cannot be linearized. So, much easier way for me would be… There are several forms like this. For example, if you take a form like this, this cannot be reduced to linear least square. Therefore, you have to start with non-linear least square. But, I took my time out to prove to you that, this is not a fictitious kind of situation; there are situations in heat transfer, where you will get a form like this.

In tomorrow's class, we will consider this form; we will take a problem, where some chip is, heat sink is going to be… We will give appropriate value for all these; generate a data; and then, get a set of same time and theta. But, now, it is like this. Then, we will try to figure out how to use the Gauss-Newton algorithm to solve this. It is 10:52, we will stop.

Thank you.