Design and Optimization of Energy Systems
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Module No. # 01
Lecture No. # 12
Solution of System of Linear Equations

So, we will continue with the discussion on Solution to System of Linear Equations. Yesterday, we started off with a very simple system where we considered only two variables x and y. I told you there are umpteen ways of solving this system of equation when only two variables are present. The simplest would be to plot them on a graph and the point of intersection will give you the solution. This could be extended to three dimensions x, y, z; anything beyond three dimension, this graphical method will break down. It will not work; therefore, we have to take request for other techniques.

Matrix inversion is the powerful technique where you put the system as A x equal to B and then take the determinant of A, and rewrite this as x equal to A inverse of B, determine the inverse, and then multiply it by the B and get the values. This can be used for any number of components but beyond 100 variables, generally it is very very painful; matrix inversion is painful. So, in the subject of Numerical Analysis, people have developed several techniques to solve the system of linear equations.

So, I told you some of the methods: Gaussian elimination, Gauss Jordan actual as (( )) decomposition, (()) decomposition, upper triangularisation, lower triangularisation, Cramer's rule, so on and so forth. But the Gauss-Seidel method is a very powerful iterative technique which is frequently used by the thermal science people basically because eventually, one day or the other, many of you will solve some CFD equations.

So, we solve the Navier Stokes equation, the equation of energy which basically you convert all this different partially differential equations into finite difference or finite volume equation. Once you have finite difference equation, the information at one point depends on the information in the surrounding points. This is what is called as an elliptic system. So, information at one point is dependent on the information of the surrounding point. You connect all these through the physics and then you have got a relationship. You get a relationship one variable in terms of the neighboring variable. Likewise, a neighboring variable is a function of which neighboring variables and there will be overlap and so on. So, you will have a system of simultaneous equations which need to be solved at each time step, for example each iteration. So, there, the Gauss-Seidel method will be extremely powerful. People do not use matrix inversion and so on for if you have million nodes and so on; Gauss-Seidel method is extremely powerful, but before that, as I had promised, we look at this ill conditioned system.

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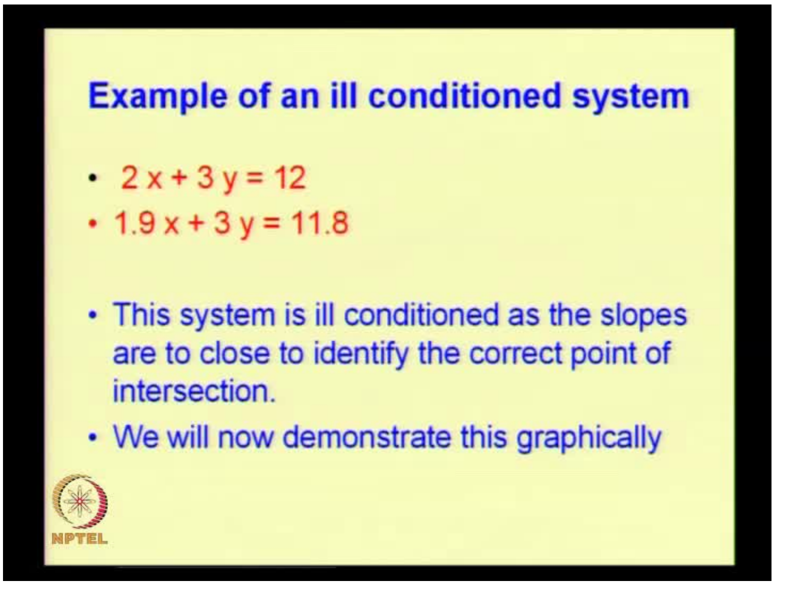

Yesterday, we looked at the system 2x plus 3y equal to 12 and 1.9x plus 3y equal 11.8. This system is ill conditioned because the slopes are so close so that it becomes very difficult for us to identify the correct point of intersection. So, the determinant is also very small. When the determinant is very small, since you get 1 by d, it leads to divergence; it leads to difficulty. So, will just plot this on excel and see.

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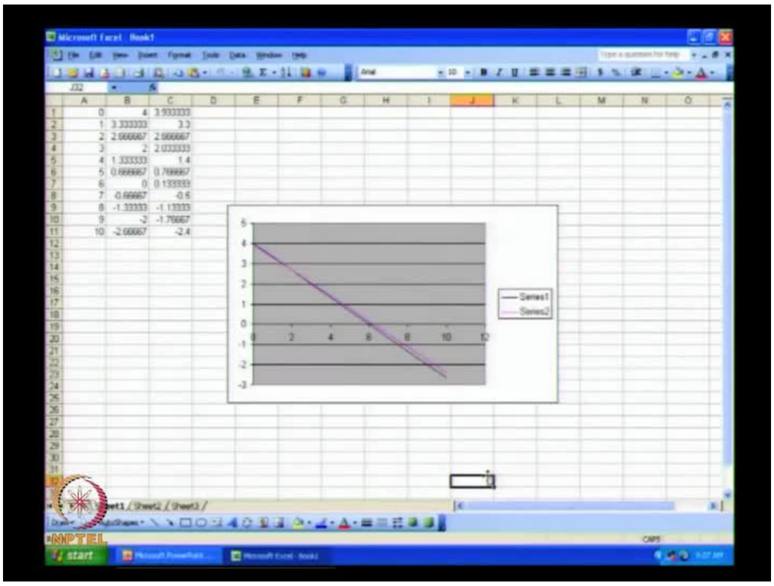

Where are the equations? 12 minus 11.9? 8. 11.8 is also very bad. You can hardly see anything. There is the plot. You want a line plot? This is also okay. We will do that also. So, you can see that; hardly indistinguishable. You cannot use graphical method. Suppose you take point 0.7 mm lead pencil, when you I mean it is all over the place we get the solution. So, Deepak is not happy with this. So, what do you want me do?

line

Student: Second

Here? here? You will like that.

Now, you can see that. Late comers you are still trying to look at how that the system we considered yesterday is ill conditioned. We can see the two lines will keeps flickering all the time. The two lines are indistinguishable. So, you do not know where they exact… Of course, you can solve it algebraically and get the solution. If you want graphical method, if you want to solve it by using graphical method, this is the problem.

Though you may do this algebraically and get the correct solution, this means that very small changes in the coefficients will lead to large oscillations in the results. So, the round off errors becomes very critical. This means, if you have a large system, if I am solving a partial, non-linear partial differential equation and convert it into a finite different approximation and you have a system of linear equations, all the coefficients are such that the system is ill conditioned. Then the round of errors become very very critical; that is why we ask all our students to use what is called double precision and so on. What is double precision?

So, we instead of 8 decimal places, we ask them to use 16 decimal place and so on. So, these are all very critical. So, this gives an example of a simple two variable system which is ill conditioned.

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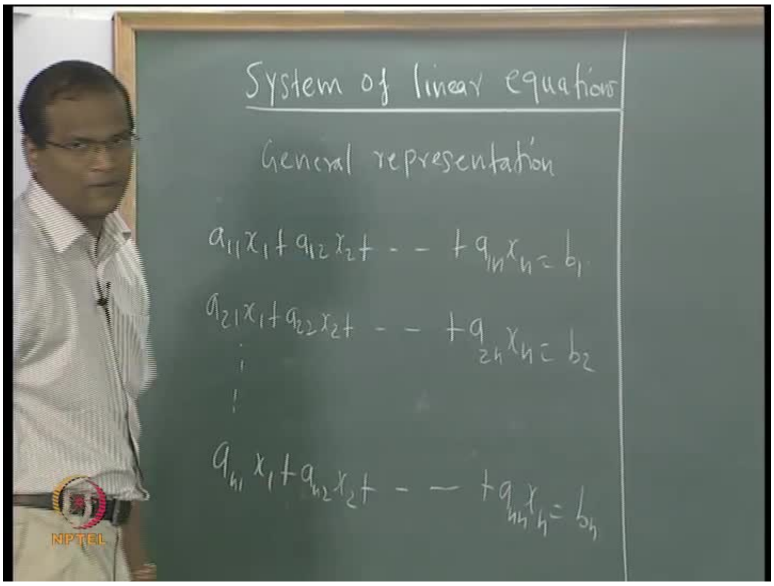

System of linear equations: Now, you are considering several variables; so, the general representation.

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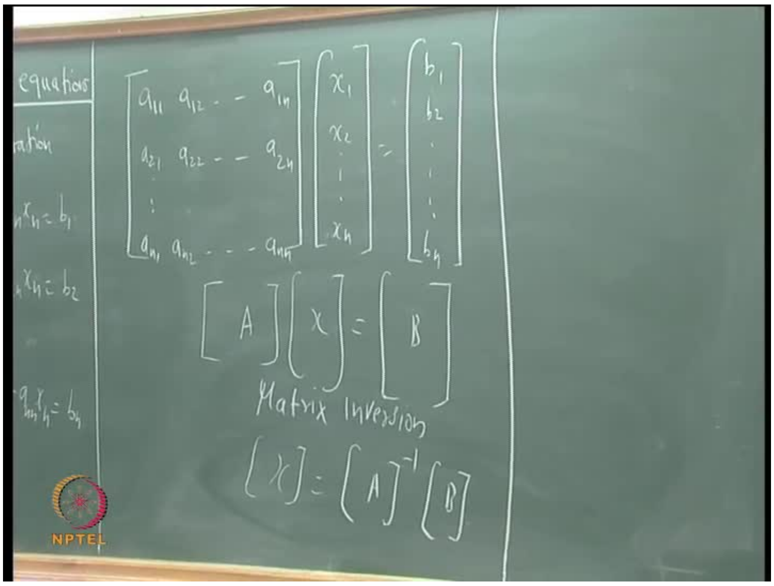

So, can also be written as… correct? So, this is the general representation. So, I told you the examples for such system are legion; I mean the many examples which are available fluid flow, fluid flow, chemical reactors, conduction heat transfer; it is also used in data analysis and so on, electrical engineering; the many examples where you have to solve a system of linear equations.

So, now, what is this Gauss-Seidel method? Of course, we saw in matrix inversion. In matrix inversion we write like this. You have to reiterate again that it becomes messy beyond a certain number of components and sometimes matrix is very difficult to invert, and so on.

Gauss-Seidel method is a iterative method. The matrix inversion is one short; I mean you will there is no need for iteration; invert the matrix you will automatically get x 1 to x n. The Gauss-Seidel method, you have to start with some values of x 1 to x n and then see how this progress and so you have to apply criterion of x i plus 1 minus x i divided by x i modulus into 100 is less than something; like that you have to look at all the components. You put a norm and you have to look at the propagation of this error and how this error reduces; whether it reaches an acceptable value, and so on.

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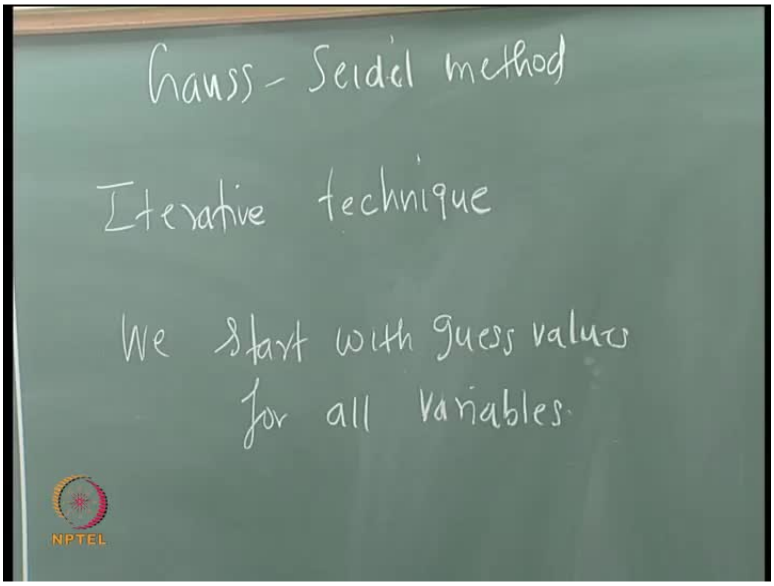

So, the Gauss-Seidel method is an iterative technique. So, we will start with guess values for all the variables. Lot of people take the guess, all the guess values to be either 0 or 1. There we will something which is non-zero which will start driving the system. If we can take 0, there will be non-zero value; then it will start driving the system. Then you have to see how these converge.

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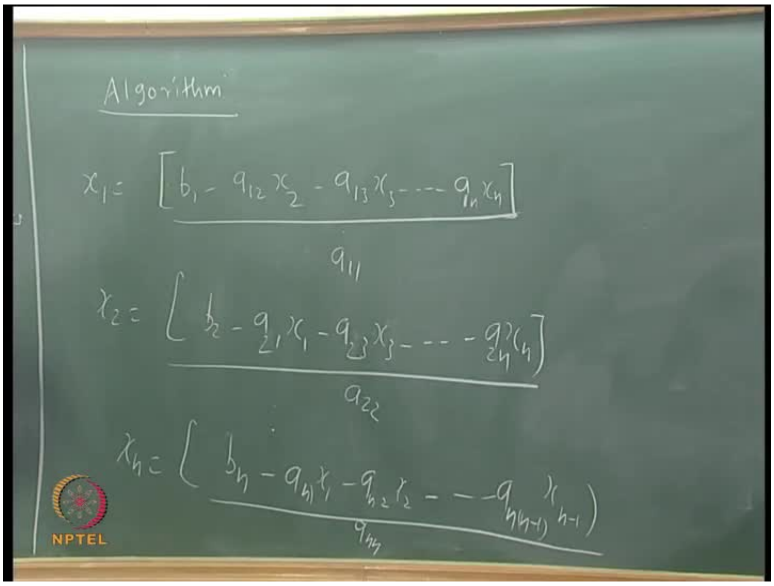

What is the algorithm itself? divided by a 11.

Good.

I am making a mistake.

by

Good a 22. Likewise dot dot x n, a n 1 x 1. What will be the last entry?

Student: a n n minus 1

Correct. a n n minus 1 divided by a

Needless to say, the system whether will cup if a i i is equal to 0; so, that is suicide. Why do you rearrange the system that a i is equal to 0?

Now when you are given a system of equations, it is important for you to rearrange the system of equation such that the a i i is the highest. Are you able to get what I am saying?

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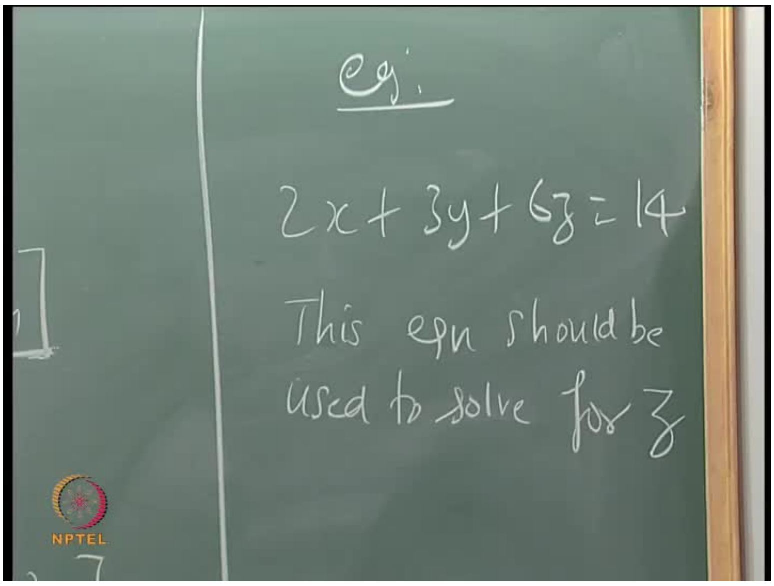

This equation should be used to solve for… two simultaneous equation are there; I am just taking one example. This equation should be used solve for… please be confident.

For z because it divided by 6; It is divided by the highest possible number so that the errors do not grow and you can rearrange the order of the equation; there is no problem. You can rearrange, you can rearrange the variables and all that. These algebraic manipulations we can do.

So, what you have to do is you start with guess values for all the variables. You substitute the guess values for all the variables and determine x 1; watch very carefully; you determine x 1; go to x 2. When you when you get x 2, for all other values expect x 1, you will use the initial guess value whereas for x 1 itself, you will use the latest or the updated iterate. When you go to x 3, you will use all the old variables expect x 1 and x 2 which will be the updated variable. Therefore, there is a dynamic updating of the variables in the Gauss-Seidel method. It is perfectly for you to start with these guess variables and do not dynamically update and get all the values and then do it. That is called the Jacobi iteration.

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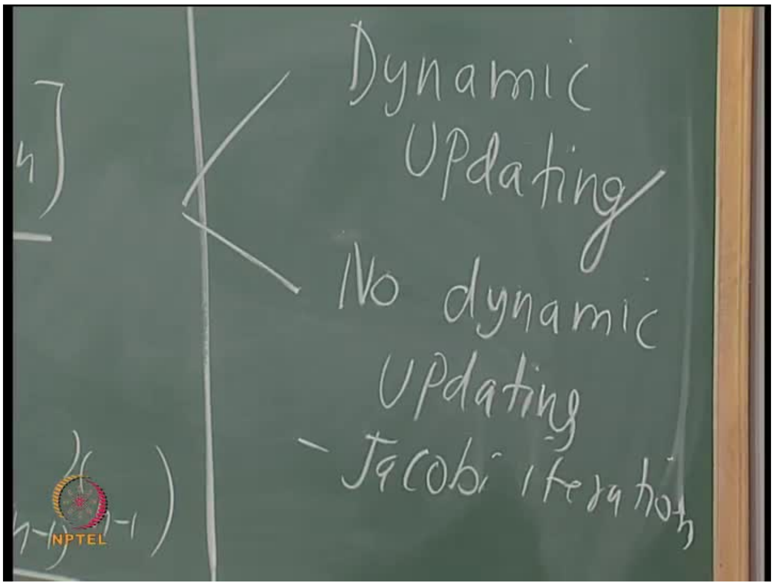

So, Gauss-Seidel, so basically the key point is dynamic updating. Why would somebody want to use Jacobi iteration? So, for Jacobi figure out then Gauss-Seidel (( )) and came up with something which is superior compared to that. First you wanted to keep things simple. So, you figured out the Jacobi method. Then you have something which is got accelerated convergence. What should be stopping criteria? Is the algorithm clear? So, for the class room environment, two variables is too trivial. So, we will have 4 variables is difficult; so we will restrict ourselves to 3 variable problems, whether it is a class room or the quiz; so, 3 is the optimum. What about stopping criterion?

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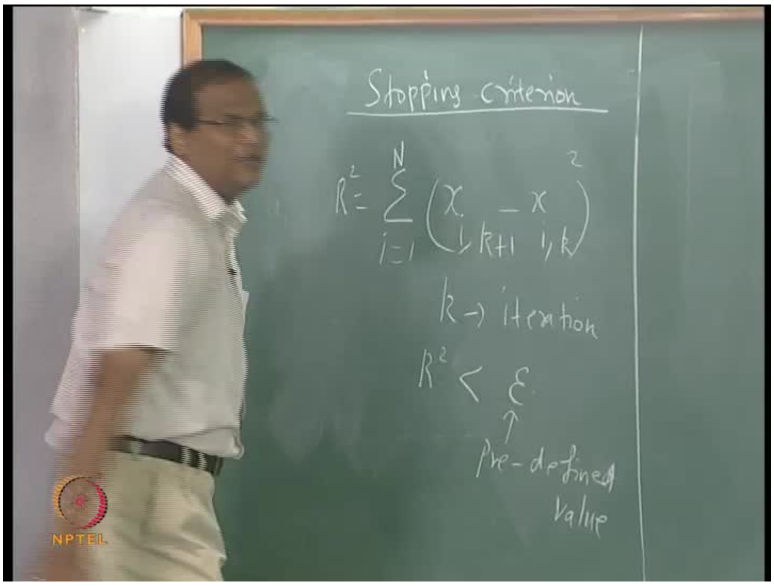

Student: Sigma equal to 1 by R

So

J? No notations.

Student: x k fifth iteration

Is that ok? Where it is the kth iteration, R square less than some epsilon; predefined; 10 the minus 3, 10 the minus 4, whatever.

So, this is only the stopping criterion, but we do not know whether it has converged. When I say whether it has converged to the accurate solution and so on, so there are some additional criteria which have to be satisfied.

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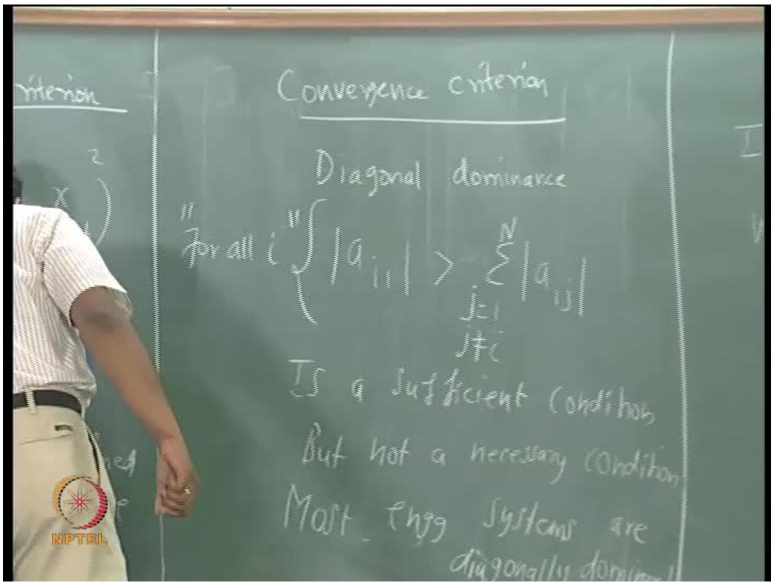

So, convergence criterion: This is from a numerical point of view. From a funda of point of view, what is it that guarantees convergence in a Gauss-Seidel system? Stopping criterion is numerical. It is left to you, left to us; we can put it in various ways.

And I can also say that x i k plus 1 minus x i k divided by x i k modulus should be less than epsilon; this epsilon should be the same for all the variables and… but what is the convergence criterion is? We are talking about a deeper funda; I mean will all system converge?

Before up front before starting the iteration, is there an idea whether the system converges or not? Not be singular but for that you have to get the determinant then you might as well… not singular is ok.

(( ))

Modulus

So, you have already learnt linear algebra. How many of you have learnt linear algebra? So, it will be familiar too. It has to satisfy the property called the diagonal dominance, diagonal dominanace. This is the very crude representation of diagonal dominance. Diagonal dominance does not mean that this equation can be used to solve for z. Diagonal dominance means this coefficient 6 must be greater than the sum of all the coefficients of other variables in modulus.

Mathematically, we can show. Mathematically we can show that this is indeed the thing, but again, this is not a numerical methods course. It is available in most books. So, the diagonal dominance is like this; a i i modulus must be greater than sigma a i j, i is equal to 1 to n, j equal to 1 to n, j not equal to i right j equal to…

So, you can rearrange the equations such that this diagonal dominance is satisfied. This diagonal dominance condition is satisfied. That is, if you have an equation 2x plus suppose I gave this equation as 3y plus 2x plus 6z, what are… or 6z plus 2x plus 3y, you can rearrange such that you push this equations to the third one so that this equation is actually used for solving for z.

Now, there is a deeper mathematical funda. Diagonal dominance is the sufficient condition but it is not a necessary condition. The absence of diagonal dominance does not mean that the system will not converge. There are many systems which are not diagonally dominant, but may converge. However, if diagonal dominance is there, convergence is assured. So, it is not a necessary and sufficient condition. It is a sufficient condition, not necessary. Fortunately, most engineering systems are diagonally dominant. So, we need not have to agonize about this point more. We will just close it by saying most engineering systems are diagonally dominant sure for all i.

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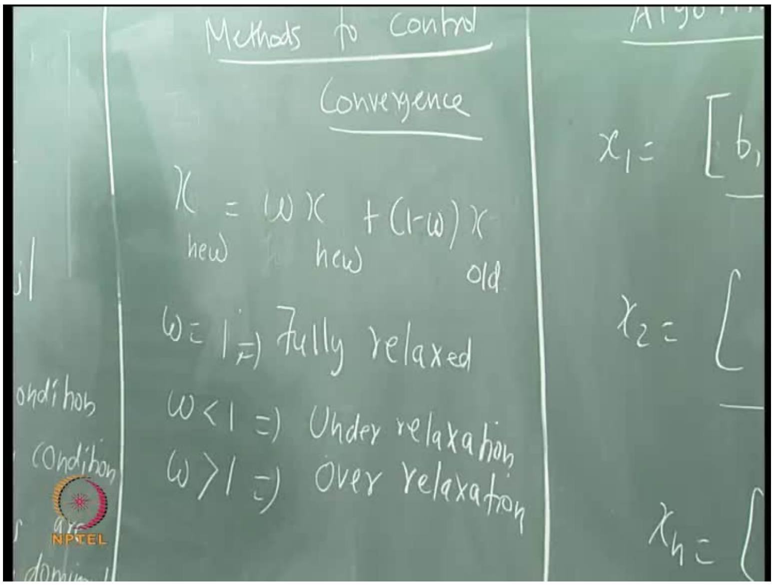

Methods to control convergence: Once these are satisfied, you may ask me, sir why do you want to control convergence? I want to always accelerate convergence; not necessarily. Many times if you try to accelerate convergence, it will diverge. Particularly CFD systems and all that because non-linear equation. What is the non-linear equation and equation which has a term u dou u by dou x? So, the derivative is multiplied by not x or y, the derivate is multiplied by the variable which you want to determine; that is a non-linear equation. Radiation epsilon sigma t to the power of 4 is highly non-linear equation.

Radiation itself is very non-linear because of t to the power of 4. Some people loosely try to linearize it by having a radiator heat transfer coefficient. You are thought this. You know what is called radiator heat transfer coefficient. You learn that what it is? What is the radiator heat transfer coefficients? No. what you do is basically epsilon sigma t to the power of 4 minus t infinity to the power of 4; you write it as t square minus t infinity square into t square plus t infinity square; the t square minus t infinity square is rewritten as t minus t infinity into t plus t infinity; the t plus t infinity and t square plus t infinity square are together combined to make it as 4 t infinity q; therefore, it becomes 4 epsilon sigma t infinity q into t minus t infinity.

When you have something multiplied by t minus t infinity, what precedes the t minus t infinity, prefixes the t minus t infinity can be called as equivalent a transfer coefficient, so that you can have a convectary transfer coefficient h c and this you can call it as radiator heat transfer equation h r. It is possible for you to combine h c plus h r, but this can be very fatal in many problems, but for a starter it is all right. Somebody does not know anything about radiation can go ahead and use 4 epsilon sigma. I can show in my conduction radiation course. I will show many examples in this Ferry law. I will just fore warn the students, under what condition these have to be used and so on.

This is simple linearized. So, linearization is required if people want to get a quick solution to the problem. So, these non-linear systems, we have to sometimes slow down the convergence in order to get a solution to the problem.

So, what we can now do is so if x is what you are seeking, x new is equal to… you heard about convergence? I mean relaxation factor; yes, anybody heard about relaxation factor? You want to write like this, x equal to 1? I am going through an iteration. At the end of the fifth iteration, I have got a value of x and at the end of the fifth iteration, I have got a value of x at the end of the fourth iteration. It is possible to use straight away the new value and put omega, sorry it is possible for you to put omega equal to 1. That means you do not care about the old value. You have so much faith in the new iterate; I want to use 100 percent of the new iterate; go ahead and do it. That is called a fully relaxed. It is called a fully relaxed numerical scheme.

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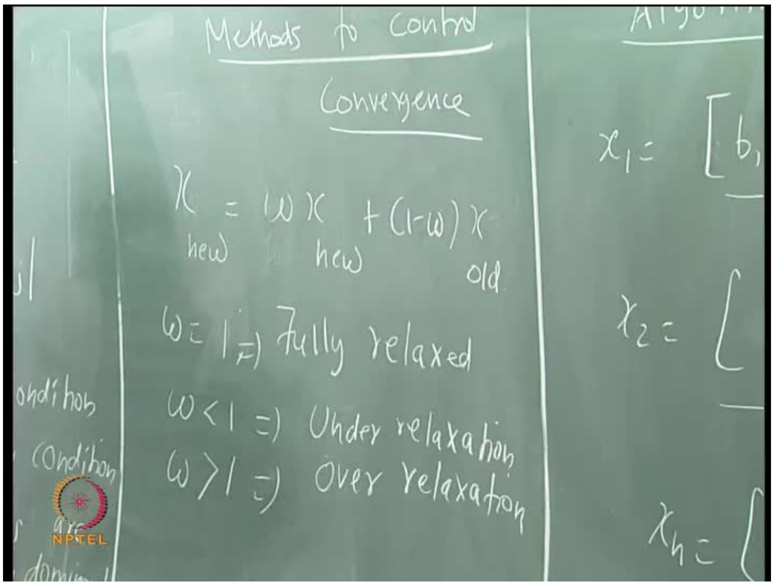

Suppose omega is 0.5, that means you are giving 50 percent weight to the new iterate and 50 percent weight to the old iterate; so, you are under relaxing. Why you want to do that? Already I struggle so hard to which I mean I am proceeding with iteration; why sir you want me to do under relaxation? Sometimes, there will be the lot of oscillations; you want to dampen out the oscillations; the system is going haywire. Therefore, we deliberately slow down the system so that convergence is guaranteed. So, that is called under relaxation. It is possible for you to have omega greater than 1. You can give omega equal to 2; that means you are giving too much weightage to the present iterate; that is that is called over relaxation or SOR, successive over relaxation. So, omega equal to 0, fully relaxed.

1? Sorry; omega equal to 0; you have to go home; there is nothing there right; omega is less than 1, what did I say? Under relaxation; whether to choose under relaxation or over relaxation or fully relaxed scheme, what is the value of omega I have to choose? I am solving electrical engineering problem; I am solving Helmholtz equation as an electrical engineer or as the mechanical engineer I am solving the fluid dynamics CFD equations or I want to solve the Pennes bio heat transfer equation. I want to model cancer; I want to solve some bio heat transfer equation; cancer in the breast or cancer in the eye or cancer in the liver or whatever; what is the value of omega? Sorry. I mean nobody will tell you what is the value of omega. You have to experience; you have to suffer and then by experience, you will I mean through you through the hard root, you realize that there are certain values of omega which will work; there are certain values of omega which will not work. They are very specific to a problem and it has to be, that value has to be determined only by experience; bad experience more importantly than good experience.

Of course, since we have all gone through all that and we have seen many students work and when some students come to us, we will say, for momentum equation - under relax; for the energy equation - over relax; so, we can give some general guidelines because under relaxation will slow down the convergence. Under relaxation means it will take a long time, but what is the point in quickly diverging and saying that, now you remember, some mathematical over flow or it gives some orbit answers?

Sometimes you take longer number of iteration. We take more number of iterations, alright, but if you do under relaxation, it will converge. Sometimes it is terribly under relaxed. So, turbulent flows you may have under relaxed; we use relaxation factors of 0.2 and 0.3 and so on. That means 80 percent I use the old value, correct, and only 20 percent I use the new value. Now, let us solve some problems. Solve a problem; let the algorithm remain. Where do you encounter this? In thermal system is a energy system.

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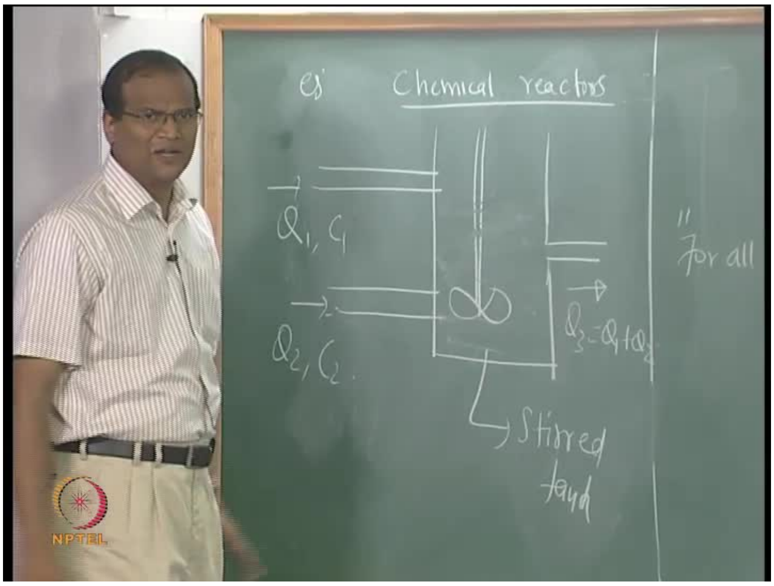

For example, one example where linear systems are… Suppose you have got two streams which are entering a reactor vessel; so, this has the flow rate of Q 1 and the concentration of C 1; this as the flow rate of C 2 and concentration of C 2; Q 2 and concentration of C 2 and I am specifying it is Q 3 equal to Q 1 plus Q 2. From the point of view of an information flow diagram, should I specify C 3? If I specify Q 3 equal to Q 1 plus Q 2, the fate of C 3 is sealed.

So, I can put or chemical engineering prof. will call it as stirred tank. They will call it as stirred tank. Some chemical, some petro chemical, something is there. Now, the key point in chemical engineering is you know, in all these stirred tanks and so on is from this tank, this output will go to some other tank; from for that tank third tank, output will come; they are trying to mix and trying to get some appropriate mixtures. They are always obsessed with mixtures.

Conservation of mass is critical in chemical engineering. Conservation of mass with appropriate concentration with appropriate florid, they mix and max; so many things. Chemical engineers also chemist and these all linear systems. Q 1, so you will take Q 1 into C 1 plus Q 2 into C 2 is equal to Q 3 into C 3. And then, under steady state Q 3 is equal to Q 1 plus Q 2. Like that if we have multiple reactors and the input of 2 or 3 are becoming the output of 1 or 2 of these reactor is going to third reactor the output of third is going to fourth and fifth and so on. You can come up with the system of equation and you solve for the concentration of the various reactors.

Apart from getting the concentration because we have declared that Q 3 is equal to Q 1 plus Q 2, it is under steady state, this C 3 which is coming out will also be the concentration of the chemicals within the reactor itself. So, this is an additional information we are getting consequent upon the fact that it steady state and it is well mixed. So, with this background, we will try to solve the problem. People who came early would have seen me desperately trying to do something before coming to the class. I was trying to set up some problem; it is realistic. Let me see. I hope this works.

Problem number 13, 14? What is it? Problem number 13. The mass balance for three chemicals x, y and z in a chemical reactor is governed by the following equations. So, minus 6x…. x, y, z are in appropriate units. x, y, z are in appropriate units. x, y, z are in appropriate units. Solve this system of equation using the Gauss-Seidel method, Solve this system of equation using the Gauss-Seidel method, Solve this system of equation using the Gauss-Seidel method to determine x y z.

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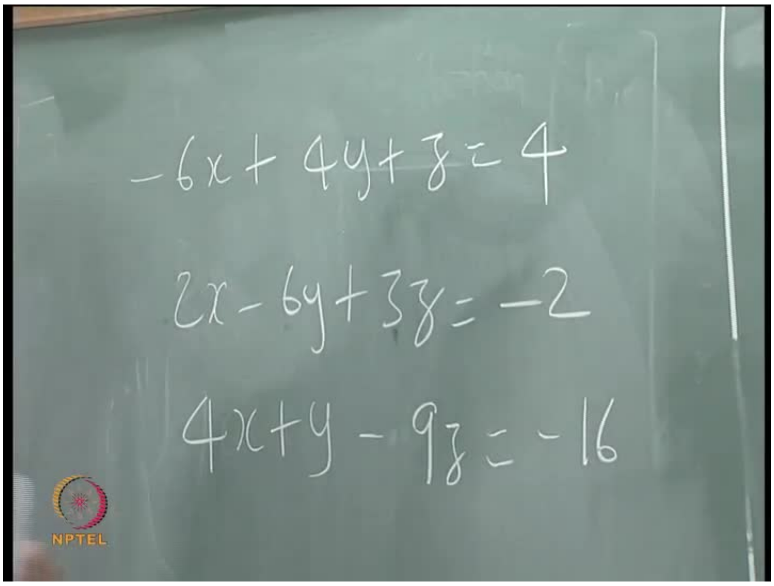

You are allowed to rearrange the equations, rearrange the equations, if required. You can take it down; rearrange the equations if required; do you expect convergence for this problem? You can write on all these. You can put it in telegraphic language; a rewrite rearrange equation a; b is convergence guaranteed; c initial guess. x y z equal to 1; x equal to y equal to z equal to 1.

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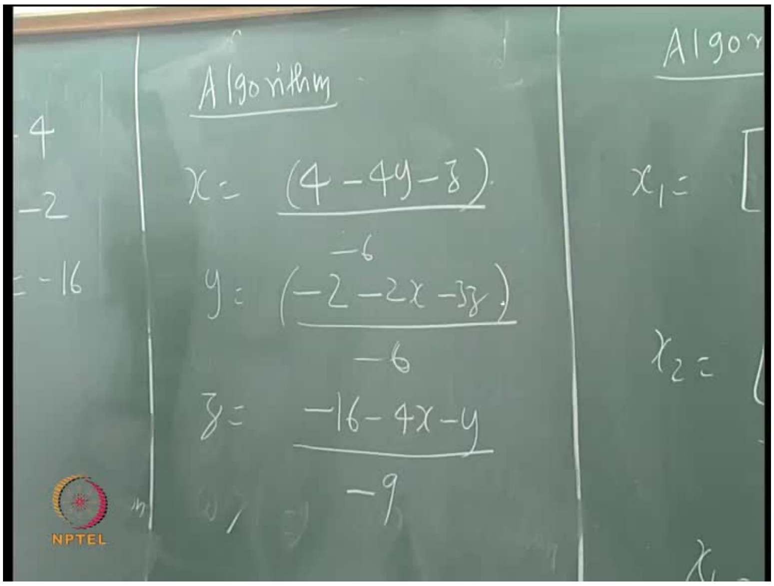

I will take attendance. Meantime, proceed. So, we should draw tabular column; write the algorithm first. I ensure that you do not have to rearrange. I could have twisted that but it is okay. Algorithm, so everything has become minus. Is it okay? Does this satisfy diagonal dominance?

(( ))

Sure? No? Who says no? You are free to say no, but say why. It does not depend on minus or plus; somebody asked that silly doubt if it is modulus.

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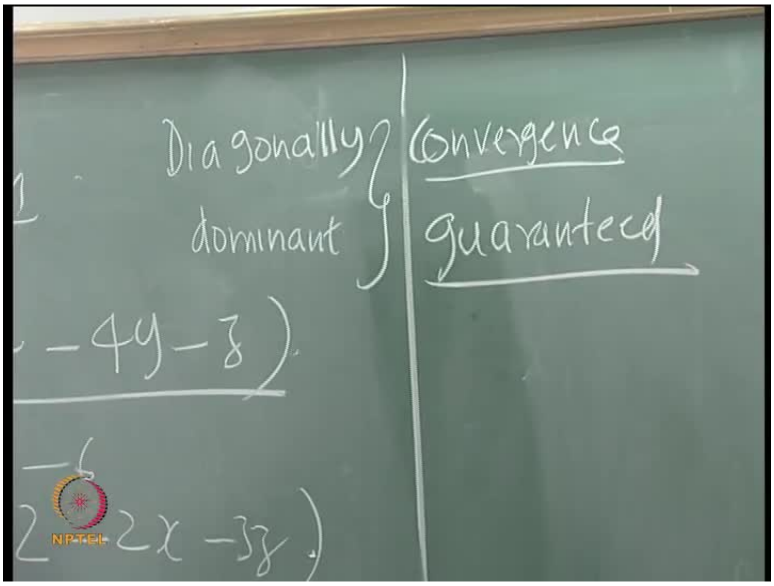

So, the convergence for this problem is guaranteed.

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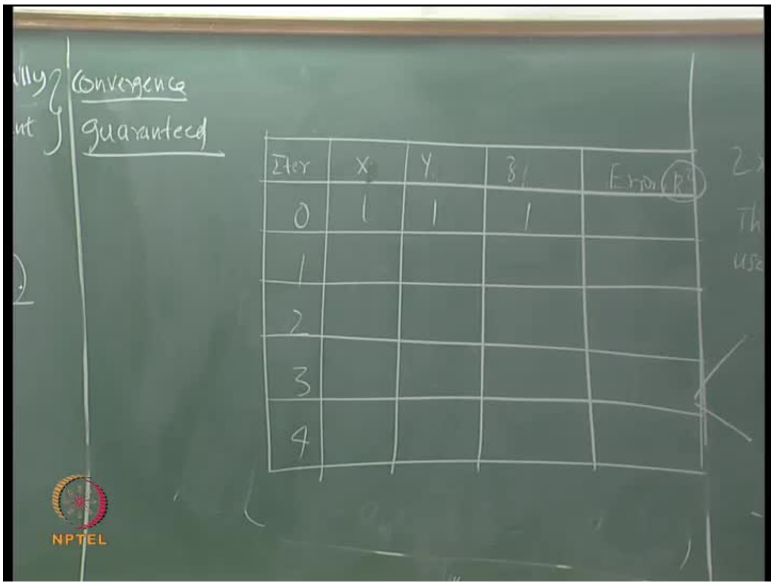

If you are sure, please you can use omega which is not 1, if you want. I cannot put x i because I am going to update on the same table. So, it is difficult for me. Error is that r square. We are understanding the r square sigma of x I, x i plus 1 minus x i whole square plus 0 is 1 1 1. So, using these values of y and z, you will get the latest value of x; from this value of x, you will get y; using x and y and old value, you will get z. Then you have to proceed like that.

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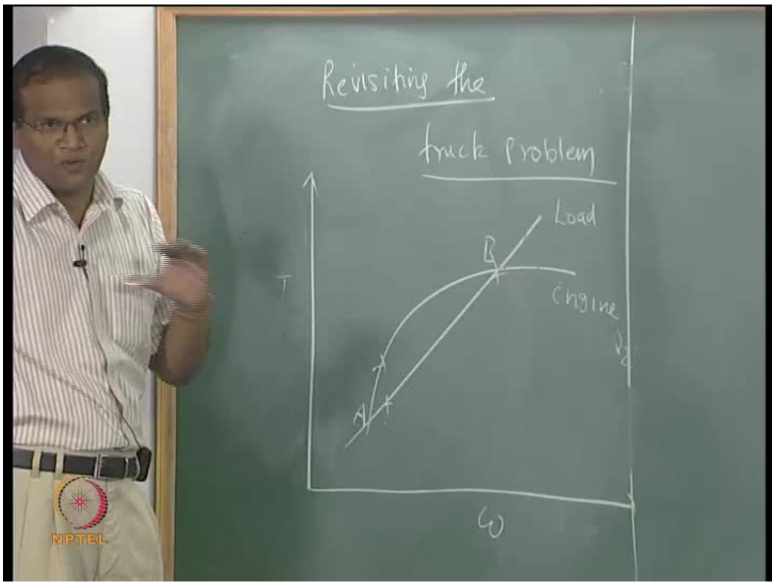

I will just quickly discuss as a truck problem; just a minute. I do not know whether all of you finish this examining the stability of the operating points of the… So you got a load curve like this. This is he engine. So, this was B; this was A. Suppose you try to raise the engine; you try to increase the speed beyond this. When you increase a speed beyond this what happens is the load keeps increasing. Therefore, it brings the engine back to this; it brings it, brings the engine back to the speed. So, this will be a stable operating point but I guess, if you if a actual rate here, what happens? The engine is got more torque but the load is less. Therefore, it should move to this, but numerically it we get this? This was what Abhishek was telling me the other day. But numerically are we getting this? This is the physical stability of the system.

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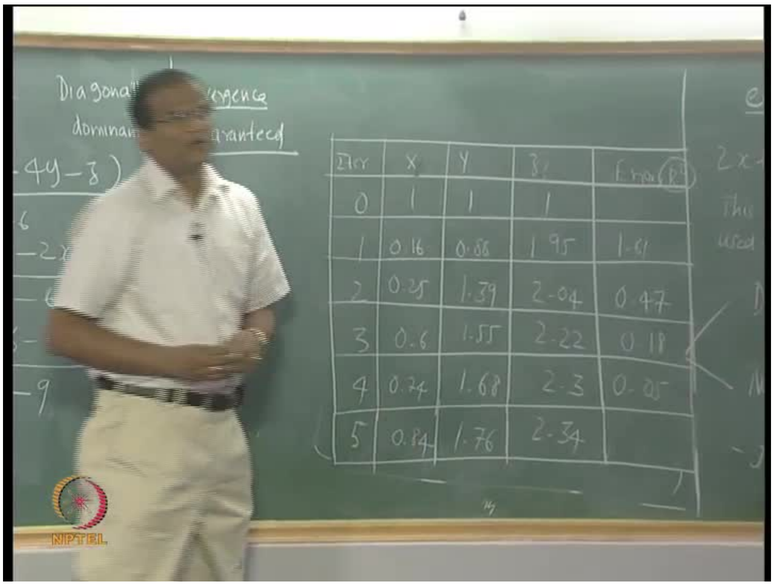

The numerical stability of the successive substitution I do not know. So, physically this is what will happen. So, when truck is climbing up hill what is a competition between the engine torque and the torque demanded by the load?

What you get here? 0.16

Student: 1 by 6

1 point Error? We will… What is error?

Student: 1.6

Next one?

Student: 0.25

Student: 1.39

2 point

Why is this fellow doing it so slowly? Third one?

Good. 0.6 then 1 point… ok

What will be the error here? What is the error Vikram?

Student: 0.47

Iteration 4. We can just work for 3, 4 minutes and then will disperse; 3, 4 minutes. I think you have the 10 o' clock class; 9. 48.

Point? Is not good enough. What Surekha? What is your problem? Next one.

Very slow

Point… so, only converging. I think we will get it in the fifth iteration.

No? What are you getting? 0.84. Then? But it is not unstable right? It is converging slowly.

0.84 then 1 point…

Why don't you use omega 1.8 or something? Why did you update the value? Use omega 1.8; actually it will be the convergence.

Where is it heading?

Point…How did you get the correct answer?

What is the correct answer? x is equal to 1

Then we call 1. 1 right?

Student: 1.875

Student: 2.425

Let see minus 6 plus 7.6; is it okay? 1.6 is not all right. 1 point?

What is z 2.4 minus 6; this what? 7.6.

1.6 plus 2.4; alright, fine, good. 4 so (( )) 4 here. 2 minus 10.4 11.4 Minus 9.4 plus 7 point

Student: That is 7.2

You have got some slight error here; slight error. That is the here 45.9. This is too much.

More or less. So, let us stick to the other one; 0.98 is 1 point

So, this will be the mass balance. From the mass balance, you get this x, y, z. You can see that is lot simpler compared to the Newton Raphson method for unknowns. But it is not diagonally dominant, it can extremely painful or it can go very slowly; it can go very sluggishly. Just like you remember x to the power of 8 minus 1. We tried solve to using Newton Raphson method; 0.00.

We will stop here.