Design and Optimization of Energy Systems

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Lecture No. # 10

Convergence Characteristics of Newton-Raphson Method

We will continue with our discussion on the Newton-Raphson method. The first part of the lecture, we will look at some convergence characteristics of the Newton-Raphson method. Then, we will apply the Newton-Raphson method to one variable problem in heat transfer. And then, we will go on to the two variable problem namely, the truck problem. With that, our discussion on this successive substitution Newton-Raphson method will end. In the next class, I will teach you Gauss-Seidel method, the system of simultaneous linear equations, and very much useful in chemical industry, petrochemical industry, and so on. And then, that will round off our discussion on simulation.

And then, as you can see, whether we use the fan and duct problem; whether we are solving the fan and duct problem or the truck problem; you are getting some characteristics. These characteristics are invariably obtained basically when we do some measurements, which the manufacturer does. So, he has got discharge versus pressure and all that. So, this will be available in the form of discrete point. You have to generate curves from that. In the next 2-3 weeks, we will look at regression curve fitting, exact fits, approximate fits and so on, so that you are able to generate these curves and use it for a simulation. And, simulation has to be logically followed by optimization. So, modeling, simulation, optimization; in between somewhere regression is also there. So, that completes the whole thing. Then, this can be applied to any field you want – electrical engineering, chemical, mechanical this thing; of course, all our problems, our focus is on thermal systems and energy systems. So, this will give you a broad base with which at least you can start off; I mean this gives you a starting point; you can pick up other additional techniques and advanced techniques for multivariable non-linear problems and proceed. That is the intent of this course.

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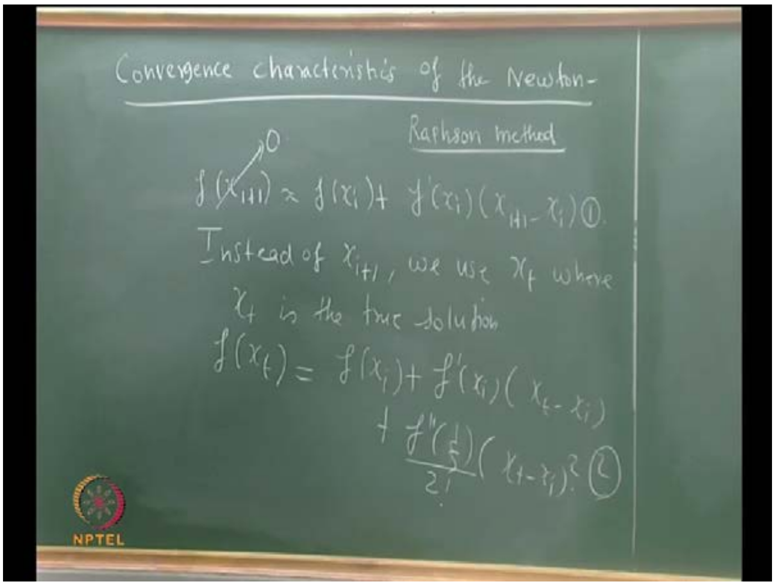

Now, what about the convergence characteristics of the Newton-Raphson method? What you mean by the term convergence characteristics? Any answers? What do you understand by the term convergence characteristics of a numerical scheme?

Student: (( ))

No; that is ok. That is very qualitatively say how does it converge. Quantitatively, what are the measures?

Student: The error is the two successive…

How is the… Why the relationship between the error in one iterate and the successive or the next iterate? How does it go? How does the error in one iteration or the error of a particular iterate; what is its relation to the error in the previous iteration? If it is linear or quadratic or cubic and so on, then whether the error… So, the number of decimal places. If it is accurate, one decimal places; if it is quadratic, the next iterate will be accurate to two decimal places, four decimal places and so on. So, whether Newton-Raphson method has a linear convergence or quadratic convergence, we will have to examine. How do we examine that?

Student: You take an example.

You take an example, that is… That is what the easy way of doing it; that is a way of doing. But, mathematics professor would not be impressed. What he says is you take an example; solve the problem – Newton-Raphson method. Do not take that example – x to power of 8 minus 1 or something; I mean take a decent example of f of x equal to x minus 2 sin x or whatever. Then, look at… You can see that the number of decimal places… Look at the previous example. It jumps, no – the number of decimal places accuracy? So, you know that it has a terrific convergence rate. You can plot it and find out whether it is linear or quadratic. If you fit a function linear or quadratic; approximately, you will have an idea. That is how an engineer will solve the problem; that is all right.

But, from a mathematical stand point, how do you analyze this? You have to start writing the Taylor series, Taylor series expansion; you have to write the Taylor series. But, now, you have x i; you have x i plus 1. There is a hypothetical x true or x real; the ultimate true rule. Then, you expand the solution at i plus 1; you expand the solution around x of true value. And, Taylor series expansion – you subtract one from the another and just try to see whether you are able to get it. That is the whole story. Can you start? I will have a delayed start. You just start; I will start after few minutes. You understand what I am saying? One you already know; one equation you already know; that is, f of x i plus 1 is equal to f of x i plus f dash of x i into x i plus 1 minus x i. This is one equation. The left-hand side, what is the… What about the left-hand side? It is forced to 0. I put approximately equal.

Now, if I say that… Now, instead of x i plus 1, I use x true; instead of… There are several ways of doing it. I can have the quad… For example, I can have one more term – the quadratic term; I can have one more term – quadratic term. And, this is approximately equal. Can be replaced by exactly equal. Are you getting the point? For example, f of x t is equal to f of x i plus f dash of x i into x t minus x i plus f double dash of xi by 2 factorial into x t minus x i the whole square. Now, this way it may lead to some trouble; I have expanded all right; but, I made a small mistake on the board. What is that? From i, I am going to i plus 1. Next, the logical steps should be…

Student: i plus 2.

i plus 2; that i plus 2 I am considering as true value for example. Therefore, what I should do is… Left-hand side is all right. Right-hand side – I think… What is the correction you want to make? Where do you want to put i plus 1? The point is very simple. Is this all right? This step is all right. This step – we are doing some expansion. So, if I use i, i plus 1, i plus 2, I will not get anywhere. So, instead of i plus 2, somewhere I am saying, declaring that, it is true value. Therefore, I will get an error between two successive iterates. But, here I am not getting that, because x t minus x i – I do not know whether it is successive iterate or not; what should I do?

Student: Minus 1.

Then, I will get f dash of x i plus 1, no?

Student: You have x t minus…

Or, we leave it like this. We leave it like this and let us see what happens.

Now, maybe when we can subtract 2 from 1 or 1 from 2 and see what happens. This expansion is correct? Taylor series expansion is correct?

Student: It should be above or around the point…

All are close to each other; do not worry about that, Vinay. Is there an error in this or it is ok?

Student: It is fine.

It is fine. Now, this is the true value. Therefore, f of x t must equal to 0. Now, there is no confusion.

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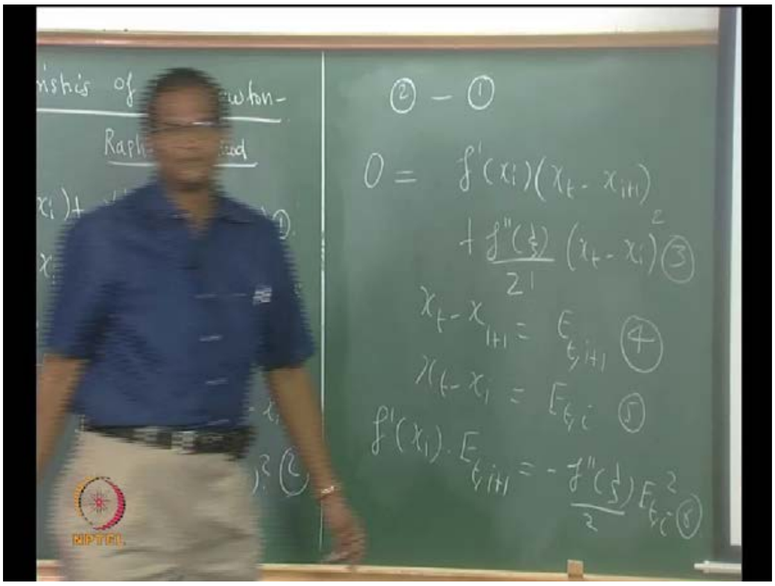

Now, all of you please do 2 minus 1. Left side is 0.

Student: (( ))

What what?

Student: (( ))

I am forcing; x i plus 1 is always 0.

Student: (( ))

No; in the Newton-Raphson method, f of x plus I – I am forcing it; I am forcing it to 0 numerically. So, that is why I have put this (Refer Slide Time: 10:05). This is approximately equal. This is exactly equal, but I want to retain the second order term in order to ensure that you do not object. Now, if I do this, what happens? This is slightly shady proof, but it is ok. f of x i gets cancelled. So, 0 is equal to f dash of x i into.

Student: x t minus (( ))

x t minus?

Student: x i plus 1.

Plus?

Why minus?

Student: Plus.

Plus f double dash x i divided by 2 factorial into x t minus x i the whole square. This fellow is there. Now, what is the big deal?

Student: x t minus x i plus 1 can be replaced by the (( ))

Good. So, we are progressing. Go ahead; this is somewhat this thing. I did some cheating here by putting only the second order terms. So, it is all right. Now, what he is saying is, this is actually the error term, true value and this thing. So, I can call it as error in the true value and i plus 1-th iteration. Correct? There is no problem about this nomenclature, error in the true value at the i plus 1-th iteration; true value minus x i plus 1; no problem. We can call that as 4. By the same token, x t minus x i will be error in the true value at the?

Student: i-th iteration.

Good; error in the true value at the i-th iteration.

And then, you can take one of these terms to the left-hand side, so that f dash of x i into E t i plus 1 equals minus f double dash x i divided by 2. Where is whole square? Not yet. Right side is E t i square. Please correct me if I made any mistake. Vikram, any problem; you are frowning?

Student: (( ))

I have already ensured that is quadratically converging. That should be edited. Anyway we are seeking… Anyway we are seeking a solution; we are expanding f of x and all that very close to the solution. It is not already we have reached. We are looking at the convergence, when it is closed and all that. Therefore, I will make some additional cheating now; I will say that…

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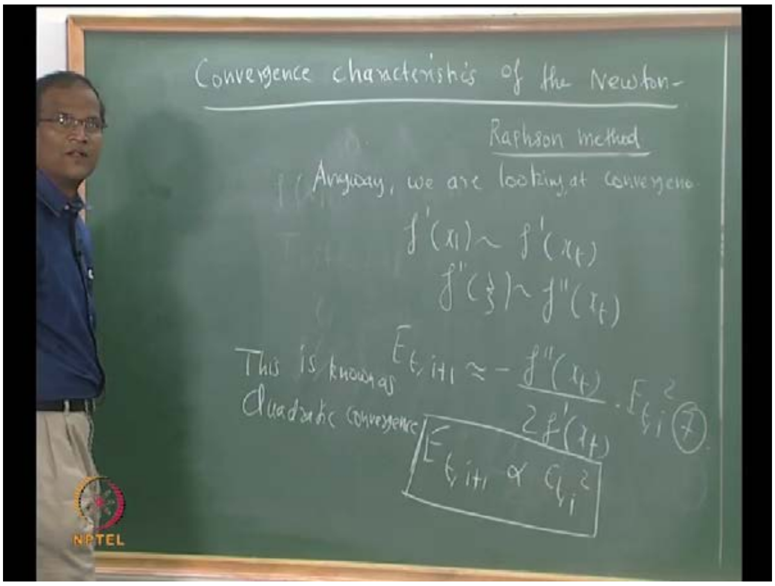

Therefore, I will say f dash of x i can be put as f dash of x t. All within limits; and, everything at the true value. Now… Around the true value, these two are stationary. How they are stationary? Because I made them stationary. Now, I am just going to tell you that, E t i plus 1 is proportional to E t i whole square; that is it. So, the error in the i plus 1-th step is proportional to the square of the error in the i-th step, because the other two are constant, because we are looking at the values around the true value. You may say, sir, these are not true values; one is at xi and one is at x i; but, I am saying x i, x t, x i plus 1 – they are all very close. Therefore, that is not a serious matter. I want to know how E t of i or E t of i plus 1 behaves with respect to E t of I; that means if this is…

Suppose you get an error of 0.1; next, it will 0.01; then, it will be 0.01 square; point… So, 0.01 to the power of 4. Therefore, the Newton-Raphson method exhibits quadratic convergence. So, this is known as quadratic convergence. People who are interested, can do a similar performance, similar exercise; can perform a similar exercise for the method of successive substitution. And, I guarantee that you will have only a linear convergence. We have already seen it live. We did one problem with the third iteration, we got the root. So, if we apply the same technique to the method of successive substitution, it will result in a linear convergence. So, this is known as quadratic convergence. Akshay Gulati; you surface; what happened – 2 o clock? Putting sleep or 1 o clock? But, you are sleeping. So, this is called the quadratic convergence behavior. As Vikram said, the other possibility is; take a numerical example and see how actually it will work out. You can see that it is quadratically converging.

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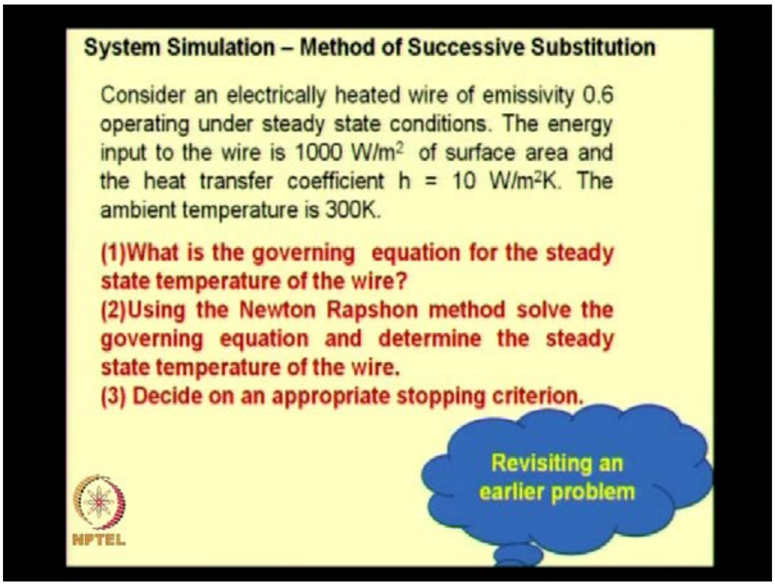

Now, we will go to a problem, which we already worked out using the method of successive substitution. So, problem number 11 or what? Problem number 11. Today is the 11-th or 12-th lecture. So, we are in 11 – problem number 11. At the end of the course, we will have done at least 40 problems. So, class notes is very important for this course. So, problem number 11 – consider an… As a simple way of writing, it would be revisit problem number? Revisit problem number 5 or 6 or whatever. What was it? Revisit problem number 6. But, if you had been absent for that class, you have to write it down.

Again, revisit problem number 6. I will say it all over again. Consider an electrically heated wire of emissivity 0.6 operating under steady state conditions. The energy input to the wire is 1000 watts per meter square of surface area and the heat transfer coefficient h is 10 watts per meter square per Kelvin. The ambient temperature is 300 kelvin. So, we are looking at steady state. There is an input energy to the wire and it is dissipating heat by both natural convection and surface radiation. The ambient for the radiation is same as the ambient for convection. In fact, we had a small discussion about this in one of the earlier classes. It is not always the case that the ambient for the convection should be equal to the ambient for the radiation. Generally, it is a case. What is a governing equation for the steady state temperature of the wire? You just copy, paste the governing equation from problem number 6 using the Newton-Raphson method, rather than the method of successive substitution. Solve the governing equation and determine the steady state temperature of the wire. Decide on an appropriate stopping criterion. The appropriate stopping criterion would be t i plus 1 minus t i whole square. So, I have worked it out; it takes about four iterations. So, maybe 10 to 15 minutes.

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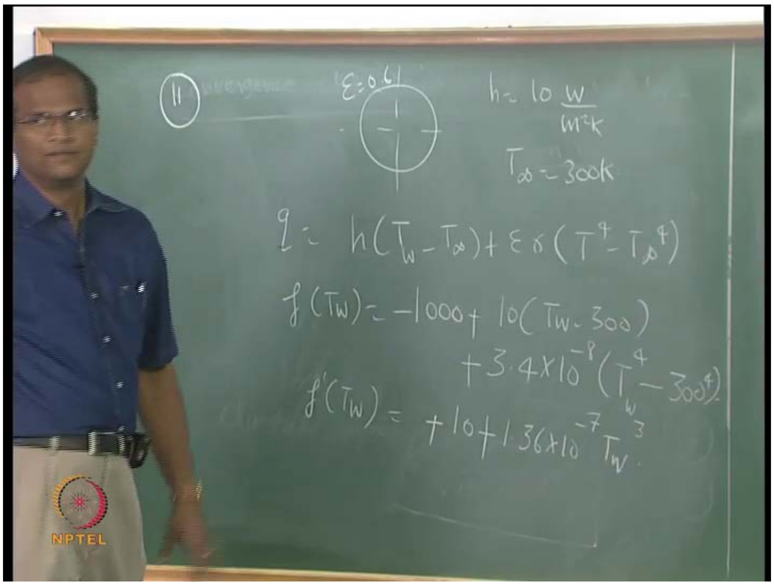

Is it T s or T w?

Student: T w.

1000? This one is 1000. 0.6 into 5.67 is how much?

3.?

Student: 3.4

3.4?

Student: 02

3?

Student: 3.4024

That is ok. Then, 3.4 10 to the?

Student: Minus 8.

Minus 8. Now, we need to get the f dash. 4 is how much? 12, 13.6. Correct? It is always negative is it? Always negative?

Student: (( ))

This not good; I do not like it.

Student: (( ))

This will work right?

Student: That will work.

Then ok. But, how are we work. How have we rewritten f of…

Student: Minus (( ))

So, what you have done is…

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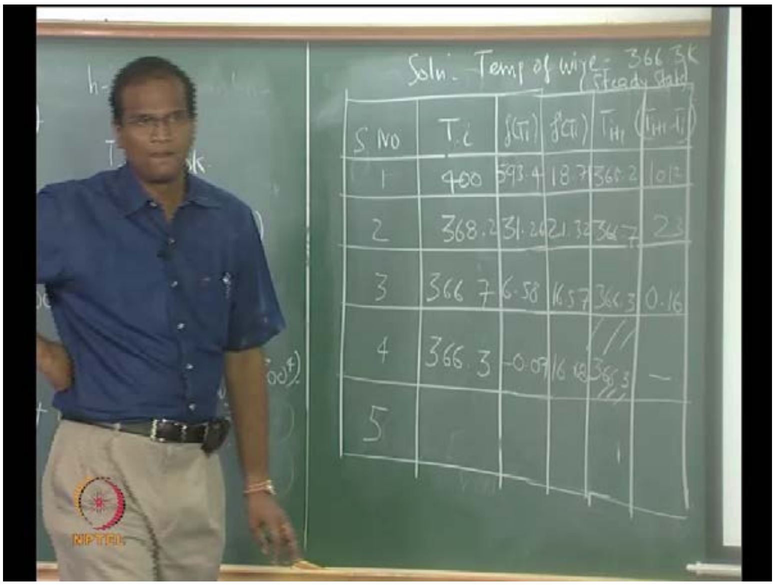

This minus sign can be very daisy; it may lead to the downfall. We should try to avoid this minus sign.

I will just put it as T i instead of T w i. What is the good guess?

Student: 400.

No 360

Student: 400

400; we will start with 400. What is the answer?

Student: (( ))

400. Now? You have to write the algorithm. Anyway, you can get rid of the w. This is not good man; this is not good enough. So, f of T i. Give me some values; 400.

Student: T i plus 1 is equal to 363.7.

No, before? Give me the split…

Student: Put it as a formula here.

Student: 595.34

Then?

Student: 18.701

18?

Student: 368.17

So, that is about 32… 32 square is how much?

Student: 1012

No; I want the steps.

Student: (( ))

It is also being shown to others. No, I want to know whether you have… Is there a quick way of doing it in the calculator? What do you do? Recursive relationship?

Student: No, sir. (( ))

You put it as a function?

Student: Yes sir.

No; see… but, you would not have an intuitive feel; how? f ultimately should become? f of T i must become 0. That you can see when you do this. He is very sincere. 31.26. Then? 366.8. 1.4; this is about 2. 3… This is 366? 7

68.?

Student: 57.

Student: 366.30.

So, 0.4 into 0.4; how much is it? 0.16?

Student: 0.16.

We can go; one more. Can you see the quadratic convergence?

Student: Yes, sir.

We will stop with this. Anand, tell me the value. You are getting something different?

Student: Sir, I started with 350 (( ))

But, it is self correcting algorithm. If you go wrong in one step, it will correct; but, you will take three iterations more than your friends; but, it is all right. But, eventually, it will converge. It will not go astray. We will fill this column out and then… Sureka, you got it? Third… Second step you got.

Student: (( ))

Two steps? Anybody else?

f should…

Student: Minus 0.07.

See that means it is already there. Minus? Derivative will stay close to 10, 15, 20. Derivative is stable. That is very important. If that fellow starts misbehaving; if he becomes very close to 0, then we are in trouble. See this is very… 0.09 by 16. So, I will put 366.3. So, this is our answer; solution. So, we started with 400. Fourth iteration – we got successive substitution. How long did it take? It took 16 iterations. So, this is quadratic convergence; that has to be linear convergence. So, I can argue like that also. But, it is very powerful. If the problem is well-posed, then there is no… You would not experience any trouble. So, this is a powerful root extraction technique. Even in two variable problem, you can put down into one variable problem and solve it. But, I do not expect you to do that. But, for some cases, you want to get quick solution. If we treat a problem as a one variable problem; using the Newton-Raphson method, it is very powerful to get.

The other will be a… There are other techniques like the bisectional algorithm; the bisectional algorithm, where you find out the two places, where it crosses the 0 mark. And then, you start bisecting. And then, you bisect, keep on bisecting and then do it. That is one thing. Then, there is a Regular-Falsi method; then, there is a sequent method; so many methods, which are available. Then, Newton-Raphson method is one of the more important ones; and, its convergence is quadratic. Now, how do we extend the Newton-Raphson method for multiple unknowns? Because it is of practical relevance for a course like this; because invariably in thermal system design, you have more than one component. How do we handle Newton-Raphson method in multiple unknowns?

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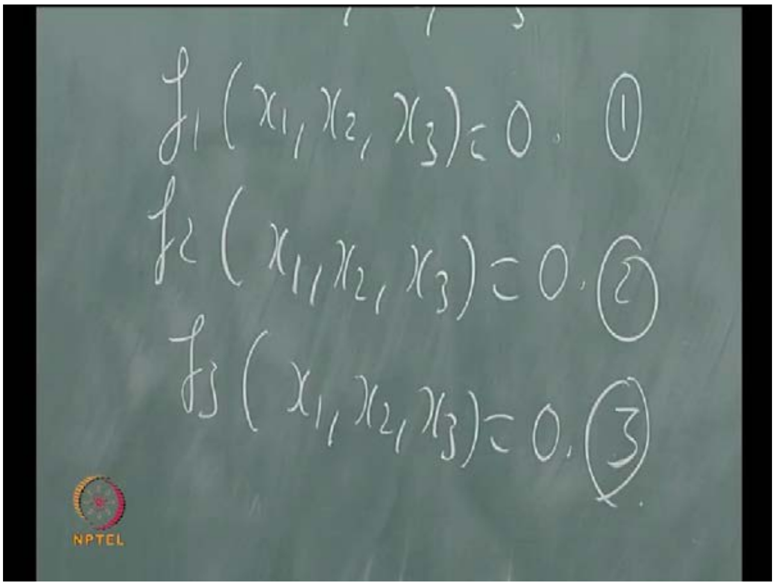

Let us consider a three variable problem; where, the variables are x 1, x 2, x 3. No; for example, it will be pressure, temperature, density; whatever. Why do I write three variable pressure in three variable problem? So, x 1, x 2, x 3. How many equations you need to close the problem mathematically?

Student: 3

3. We got three equations. So, you can start with x 1 of i, x 2 of i and x 3 of i. You want to proceed to x 1 of i plus 1, x 2 of i plus 1 and x 3 of i plus 1; which means you have to write f as the Taylor series in a Taylor series expansion. How do you do that? Can you write the Taylor series expansion for a function of several variables?

Student: (( ))

Use partial differential equation and stop with linear terms.

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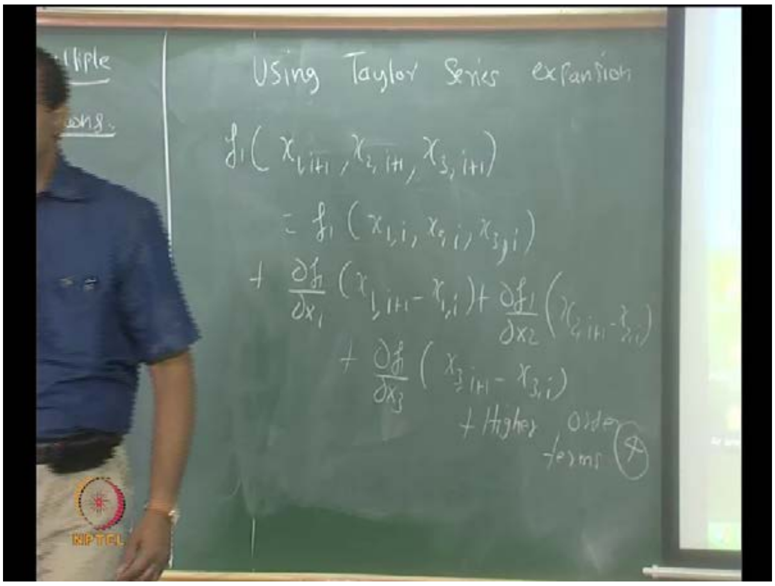

Using Taylor series expansion… I am expanding around i plus 1 from i; i is close… i plus 1 is close enough to i, so that I can neglect higher order terms. If the delta x is small, then I am allowed to… Delta x 1, delta x 2, delta x 3 are small; I am allowed to do that. So, this can be equation 4. Should we also get equations 5 and 6? Sure, we will get equations 5 and 6 for f 2 and f 3 respectively. Now, the goal is to make the left-hand side equal to 0, because we are seeking roots to the equations f 1, f 2, f 3. Therefore, we force f 1 equal to 0, f 2 equal to 0, and f 3 equal to 0. So, if we do that…

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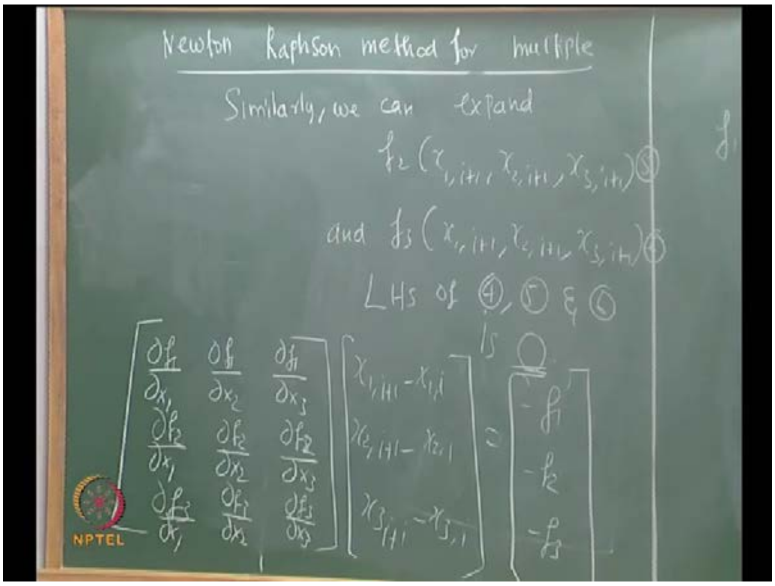

I will call this expansion as 5; this expansion as 6. LHS of 4, 5 and 6 is?

Student: 0

Correct. Then, how do you write this algorithm? The left-hand side is something equivalent to what is called a Jacobian matrix. It is a sensitivity matrix. The sensitivity of… The partial derivatives of the various functions with respect to respective variables – we frequently refer it to as sensitivity or the Jacobian. So, what you have to do is, unfortunately, for us, the Jacobian matrix is not fixed, because all the elements of the Jacobian matrix keep changing with iteration. There are certain derivatives, which will get fixed.

For example, in the engine… In the truck problem, if t is equal to 11 omega; we will put f 2 is equal to t minus 11 omega. Then, dou f by dou t will be 1. That fellow will remain 1 throughout the iteration. Dou f by dou omega will be equal to minus 11. He will remain the same. But, if you have got 3 omega plus 4 omega square or something, it will be 3 plus 8 omega. Then, when the omega changes, that guy will change. So, some of the elements will be fixed; some of the elements will change. So, you start working out like this. Take an initial value of x 1, x 2, x 3 i. Watch very carefully. Take an initial value of x 1 i, x 2 i and x 3 i. Substitute in the respective equation for f 1, f 2, f 3. It is possible for you to calculate the values of f 1, f 2, f 3 at the initial starting point. So, the forcing vector… The column vector on the right hand side is known. Then, once you known x 1 i, x 2 i and x 3 i, evaluate all the partial derivatives at the point x 1 i, x 2 i and x 3 i. So, the Jacobian or the sensitivity matrix – all the elements are known.

You also know this. Solve this system of equations using Gauss-Seidel or invert or whatever; and, get this value. You can write this as delta x 1, delta x 2, delta x 3. So, you solve the system of equations for delta x. Delta x is x 1 i plus 1 minus x 1 i for delta x 1. You can do the same thing for delta x 2 and delta x 3. Now, you will get the new values of x. You will get x 1 i plus 1, x 2 i plus 1, x 3 i plus 1. Substitute; on the right-hand side, you will get the new values of f 1, f 2, f 3. With the new values, rework the sensitivity matrix. Now, you feel bamboozled, because it is very tiring and all that. But, it is eminently programmable. This is the (( )) You can write a matlab script; it will do in no time. But, the moment it exceeds 10 or 12 variables, handling the matrix sometimes, some issues are associated with that. Matrix inversion will not work properly for certain number of variables. Then, you have do Gauss-Seidel or Gauss elimination or things like that. But, if you have a practical problem; if you are really talking about system simulation of a power plant and all that; then, these are the kind of things, which will be involved.

Now, I have just given a general case, where for a three variable problem, this can be extended to any number of variables. If it is a two variable problem, you will have only two elements. So, in the fan and duct problem or in our truck problem, we will have to solve only a 2 by 2. You will have to invert a 2 by 2 matrix each time. But, you can see compared to successive substitution, this will be extremely fast. And, the Jacobian matrix, the partial derivative matrix also gives you additional information of the sensitivity of a solution to the sensitivity of f 1 and f 2 to the operating variables. Is it time to close or you want to start working out the problem?

Anyway, in the next class, we will use this… We will use this… In the next class, we will use this technique to solve the truck problem. And, with that, we end our discussion on Newton-Raphson method. I will teach you the Gauss-Seidel method subsequently. Any doubts?

Student: Sir; in our derivation, when we force that error in the i plus 1-th iteration in proportion to error i square; we tend to exactly normalize that. And, this will only be valid… This will only make sense if the error values are normalized, so that my error values are less than 1.

No, error values… In fact, I told you, why the successive substitution does not work; g dash of alpha must be modulus less than 1 and all.

Student: Yes sir.

See we are already… We already very close to the solution. Therefore, we are looking at the decimal points and all that. So, it would not… When it is wildly swinging, what you are saying is… What you are saying is; if it is 40; next time it will be given 1600; what is the point? That is what you are saying?

Student: Yes sir.

No.

As I said, we are seeing the convergence (( )) error values are larger. So, is the derivation is such that there is some normalization?

No, is there any normalization?

Student: No.

Student: f dash of zeta.

No, that is that. There is one factor called f double dash. That is one normalizing factor.

Student: No, like the…

No; but, you can… Suppose that confuses you; you will say that, already you have come to level, where it is less than 1. If it is less than 1, 0.9 will become 0.8 and then 0.64. It will go quadratically low. Is that thing understood? He says if it is 40, it becomes 1600; 1600 will become… But, there is 1 minus f double dash xi divided by f dash something, no? That will take… That will cool things down. That will ensure that these fellows do not misbehave. So, that is the normalizing factor.

Thank you.