Advanced Gas Dynamics
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Lecture - 28
Quasi 2D Flow - II

So let us begin with the Taylor Maccoll equation which we said that we will solve by Runge-Kutta method right. So, let me write this down.

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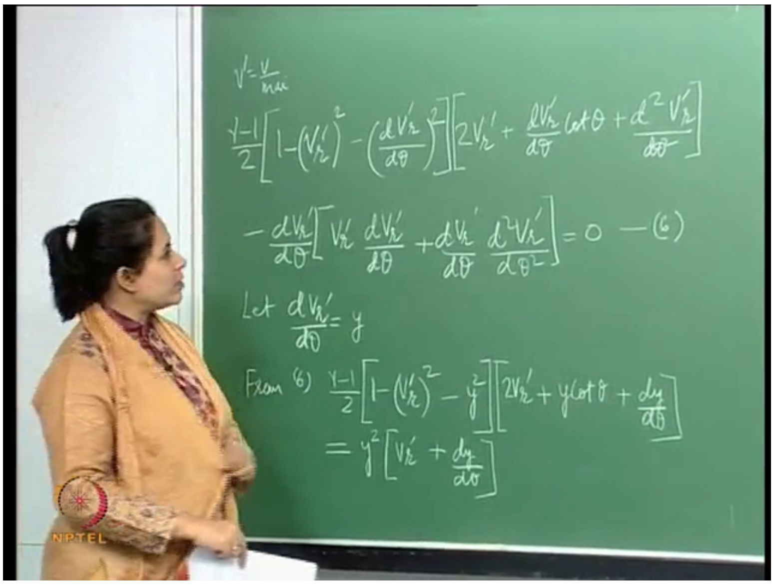

So this was primarily the main task, when we try to solve the flow past the cone right. So, this is essentially the equation 6, we and if you remember right; so right. So, this was the unknown, this was non dimensionalized form which is v dash here right ok.

So, this was our Taylor Maccoll equation which we needed to solve numerically, we cannot solve the closed form in order to get the, solve for the flow when you know past a cone right. So, we said that you know this is an equation which has basically just if you if you look here, just one unknown which is V r V r dash right. So, if you are able to find V r dash from here then we should be able to get the rest of the properties you know which is v theta dash and then correspondingly the thermodynamic properties ok.

So, let us begin. So, let us say here now this y out here is just a parameter is just a parameter, it has is it dummy variables. So, to speak; let us say d V r dash d theta is equal to I if we do that then what happens in this equation 6. So, let us this was equation 6 as we derived it. So, therefore, what we get you can see from here itself this and then let us take this on the right side and write that as.

So, we have this out here now if I you know what could this out here, then I will write from here.

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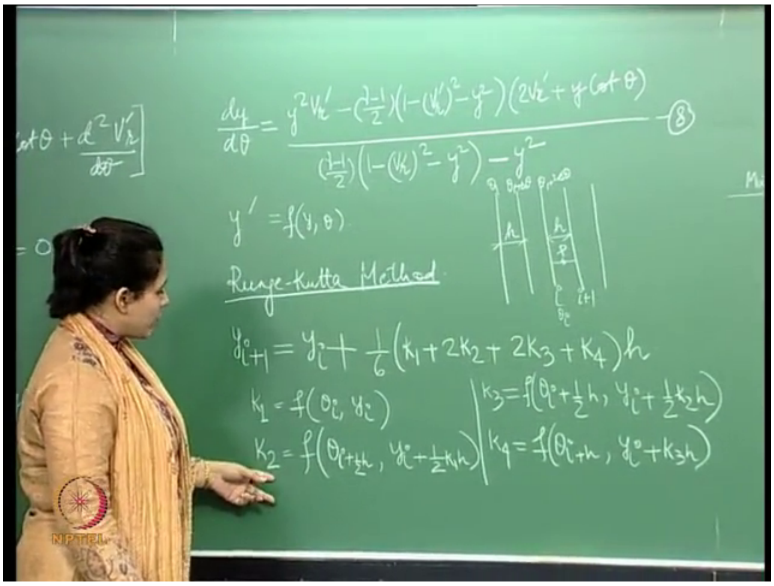

So, from this equation then I will write it like this just rearranging the variables right from here from here. So, I am just writing this out as dy by d theta as in this form. So, what you can see basically is that this is this essentially a function of if you see over here right.

So, this is the function of y and theta or what we can say is basically is you know I can write this that y dash in if I write this in here, this is a first derivative. So, y dash is basically a function of y and theta now. So, now, what we gone a do out here is solve for V r dash right. So, now, what is the Runge-Kutta method? So, if you are not familiar with it and sort of just walk you if you do not have any idea of what you know Runge-Kutta method is. So, let me just walk you through the procedure and I think I will suggest that you know consult any standard text book you know for what this method is all about and you should be able to figure it out pretty straight forward ok.

So, essentially what we saying here let me just write out what this Runge-Kutta method is right. So, what you saying is that say in you know and in some space out here. So, if I discretize my space at various points. So, at a particular point say I plus 1 right then this value is actually given using the Runge-Kutta method as so, right which is the value at the point i and in that to that this ok.

So, now, essentially let me say let me just do this over here. So, I have say some domain like this let us call that this is i and these are all equal divisions. So, say here and this is my point i plus 1 this is also h. So, what I am saying here is that the function value at i plus 1 out here is equal to the function value at i plus this out here this and h is the space length discretized space length. Now where in this particular case, since we have the y y dash is the function of y and theta and so, you know here. So, it is a function of theta i and y i. So, this K 1 is essentially the function of theta i and the function at the point i at the point i out here ok.

So, in this case of course, as you can see we are differentiating over theta right; so this h. So, essentially this means that theta 1 and theta 1 plus delta theta, theta 1 plus 2 delta theta and so on and so forth. So, this is essentially we are differentiating over theta right. So, if I go from say. So, when I am calculating the value of my function here at i plus one. So, K 1 is the function at i at the location i. Now K 2 is the function value at theta i plus half h and y i plus half k 1, K 1 which I found out here in to h ok.

So, when I say theta i plus half h what do I mean. So, theta i is here this is my theta i right. So, half h is over here. So, what I need to do here is calculate the function at the point not theta i, but here, but this point and correspondingly the y function comes out to be y i plus half of K 1 into the into h this is K 2 K 3 is equal to. So, similarly K 3 is again theta i plus half h which is here, but the y i s y the y part is y i plus half of K 2 into h and K 4 is of course, the function at, K 4 is the function at theta i plus h which is here this is the theta and the y is y i plus K 3 into h. So, this is what the Runge-Kutta method is all about ok.

So, we are basically moving in step here in the theta direction right so and this is how we gone a go about it alright. So, if we do that now let us see for our particular problem here how does this work now. So, how are we going what is this theta in this particular case? Now if you remember what we did last lecture. So, essentially what we have is that.

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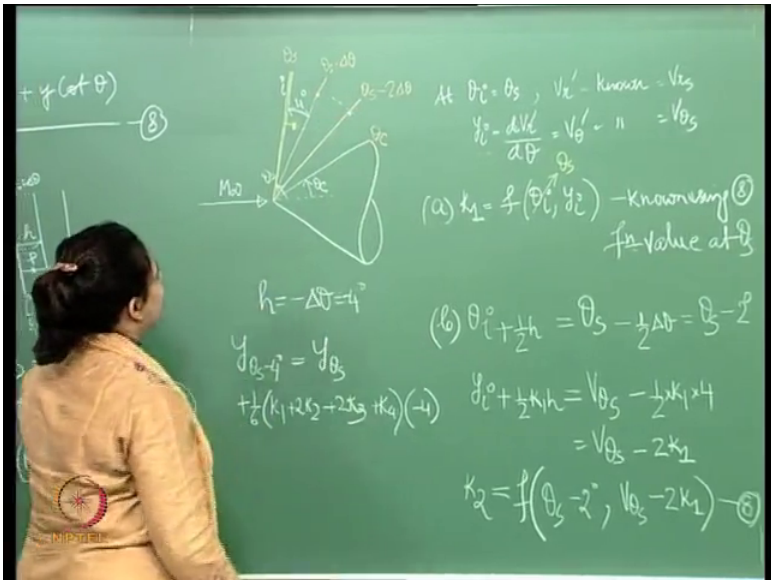

This is our cone right and we have a free stream which is coming in right and so, this is our shock wave. So, this is our shock wave. So, this is theta s and this is our surface which is theta c. So, what we doing here is essentially moving from theta s to theta c right we are moving from here to here.

So, that is. So, our steps that I show over here a basically we are moving from one theta to the next theta to the next theta and so on and so forth right you move away for the shock towards the surface of the cone right. So, let us start. So, at this point and let us call this as. So, this is known here right at when we start here, when we start this is known to us we are r dash. So, then here therefore, right this is also known right and let us call that as ok.

So, let us start form this theta out here. So, when we start from right at the shocks. So, we basically know V r dash V theta dash if we start from there. So therefore now, essentially what we are saying here what we trying to do here is that you know so on and so forth. So, with the on these radial lines, this is my theta s. So, this is say theta s minus delta theta, it is theta s minus 2 delta theta this is theta c.

So; obviously, you know that the more number of thetas you take the you know the more number the more of the properties one will be able to calculate; so if we do this, starting there right. So, basically what we are looking at for example, say y i plus 1 is essentially the function value at this radial location right at theta s minus delta theta; so starting here, if we started over here. So, let us say this is my i. So, this is i. So, let us say this is i out here, if that is I right then K 1 is the function of this right.

So, now in here if this is so, in this particular case this theta i is nothing but theta s right and y i is as we have over here. So, if we do this. So, this is what we do and now. So, let us say over here now it is called this particular equation, let us call this as 8. So, if you look at this particular equation over here now I know the function value at theta s right. So, v theta s and V r dash is known right at theta s at theta s and y again here is v theta dash right which is also known. So, therefore, this is the function value at y i right and theta is equal to theta s ok.

Basically or y at theta s, so this is known; so now, you one can calculate this. So, this is also therefore, known using equation 8 did you get that let me (Refer Time: 19:33) say a one more time. So, what we doing here is starting at this at nearly shock right the angle there is what theta s and correspondingly the y i or d V r dash d theta right d V r dash d theta is let us go back here. So, this is my y you remember. So, this is the y which is d V r dash d theta. So, d V r dash theta which is the y s v theta dash, this is also known this is also known to us. So, if you look at this function here which is y dash right V r dash V r dash V r dash all of this is known right at theta s and y at theta s or I in this particular case and y i, y i and y i this is also known right. So, y i is d V r dash d theta which is v theta dash. So, this is also known. So, you can basically calculate this particular expression write. So, you can calculate the function value which we calling as y dash this is the function this is known this is known at theta s.

So, therefore, and that function value. So, what we doing here is that calculating the function value at theta s k. So, which is known basically using or you know this is the say function value actually, function value at theta s. See if I do that let us go head. So, let us say in this particular case let this be this delta theta be 4 degrees just say it is 4 degrees, this is just for our calculation purposes. So, let that let that be 4 degrees. So, if I do that so, in this case. So, therefore, just know from the mathematics point of view, here what we said is that we will be moving you know forward. So, when we write this when we write this formula writes when we write this formula. So, to speak this theta i plus 1 is ahead of the i right in here; however, when we are doing this our i plus 1 is actually we going behind it we are going back.

So, what only thing is a h here we will let us write as minus delta theta write we are sort of moving back from here to here then if I do that then. Therefore now, again let us see. So, when we come here, so a here i plus half of h, theta i minus half of h. So, what we gone a now do is try to calculate K 2 right for K 2. So, we have i i plus half h rather right. So, this is basically equal to theta s right minus minus half of delta theta is that makes sense ok.

So, which means theta s minus two degrees and. So, y i was v theta s right v theta x minus delta theta s 4. So, we can write this or basically what we can write this as. So, y i plus half K 1 h right y i plus half K 1 h is this right. So, now, what we need is essentially the function value which is k 2. So, K 2 we need the function at theta s minus 2 degrees and the y value is that makes sense. So, let us just go with this one more time, what we did out here is that this theta what you trying to do here is calculate this. So, say let us call this as a let us get down to b ok.

So, we calculate it k 1, now we come to the K 2 for this, what we need to do is locate the theta which is this and also calculate the corresponding y which is this right now. So, theta i plus 2 h is essentially in this particular case i minus half of h which is minus half of delta theta. So, this is the theta s theta i is theta s theta s minus half of delta theta. So, basically what we doing is we are going from here and coming half way we are coming half way right which is what we have done over here minus half of theta which is theta s minus you know a half into a 4. So, we have just taken 4 here just the let this be 4 degrees here for this particular purpose and then y i plus half of K 1 h. So, y i which was known which we which is known actually is V theta s right V theta s minus half of K 1 into 4 and K 1 is what we have calculated in step a right, therefore V theta s minus 2 k 1.

So, therefore, my K 2 now becomes the function at the theta value theta s minus 2 and the y value V theta s minus 2 k 1. A K 1 is something that we already calculated and theta s minus 2. Now, what we do is comeback in here and if you look at this function value. So, y here and y here and y here and y here are these new values which is at this particular value which is v theta s minus 2 K 1 and theta is minus 2 and we put that in here. So, we get the function value at y and theta everything. So, V r dash still remains the same.

So, when we go here. So, therefore, now we are able to calculate again using. So, we calculate again this using 8, we put that in and we calculate this using the equation 8. So, let us go forward again.

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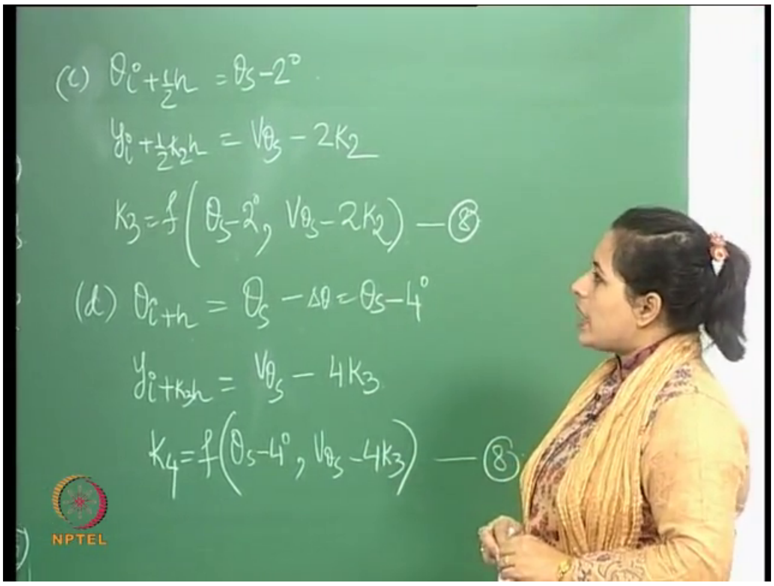

So, K 3 now K 3 all let us to say let us. So, what we need now k 3, K 3 is essentially right. So, this is the same as this value here which is equal to right now y i. So, this is essentially y i was essentially v theta s minus 2 K 2. Now K 2 is known from the previous form this calculation out here.

So, therefore, what we need is that K 3 is the function value at theta s minus 2 and again using equation 8. So, you come back in here. So, using the same theta to change the y change the y values here and what we get is the function value. So, we get the function value at theta is minus 2 for V theta s minus 2 K 2 finally, what we need is for K 4 right. So, then we get theta i plus h. So, theta i plus h is essential theta. So, far we did we calculated two values at the midpoint over here. So, theta minus h is this ok.

So, theta; so this is nothing but theta s minus delta theta which is theta s minus 4 degrees right. This is it and then y i plus K 3 h. So, this is again y of i which is V theta s minus 4 into K 3. So, then we get our next the arguments. So, theta is essentially theta theta s minus delta theta which is this. So, now, basically the K 4 is essentially this radial line this radial line and y out there is v theta s minus 4 of K 3, K 3 which you calculated in c ok.

So, therefore, K 4 then again becomes the function which is theta s minus 4 degrees again we get this from equation 8. Now having done this therefore, now we can get the function value at the radial line theta s minus delta theta right. So, therefore, our function value now becomes. So, well I mean you know this is let us say write it over here ok.

So, therefore, our function value at theta s minus delta theta or say minus 4 degrees here theta s minus 4 degrees here is equal to y of theta s plus one sixth of K 1 plus 2 K 2 into h which is minus 4; so in to minus 4. So, essentially, what we have done here is that. So, now, if you look at this now just take a step back and if you look at this, what we did here the our main you know job was to use a Taylor Maccoll theory solve this and get the properties you know in this space as well as on the surface of this cone. So, what we have done is we have drawn several radial lines, we started from the shock from the shock using this theta s and the corresponding y i and V r dash is known to us over here. We started from there and now what we have been able to do is that this function y here the function y which we defined as this; so d V r dash d theta this y ok.

So, if what we have been able to do is calculate that at this point (Refer Time: 33:44). So, we started from here we calculated that and we get this. So, now, therefore, now we are we know the value of y theta s from here and hence we can calculate other values and now the next step to do to come here would be again to repeat all these steps right all these steps instead of now this becomes the i and this becomes your i plus 1 then you integrate we come from here to here. Therefore, theta i when we started was theta s and we move from here to here. So, now, again we will start and do the same things, but starting from here and moving to here; so moving with 3 theta s minus 4 to theta s minus 8 ok.

So, basically y is therefore, obtained. So, y is obtained here. So, which is d V r dash d theta right. So, this is my v theta essentially y is nothing but my v theta which becomes known right. So, what we need to do is get V r dash and how do we get that. So, when I do this what I get from here.

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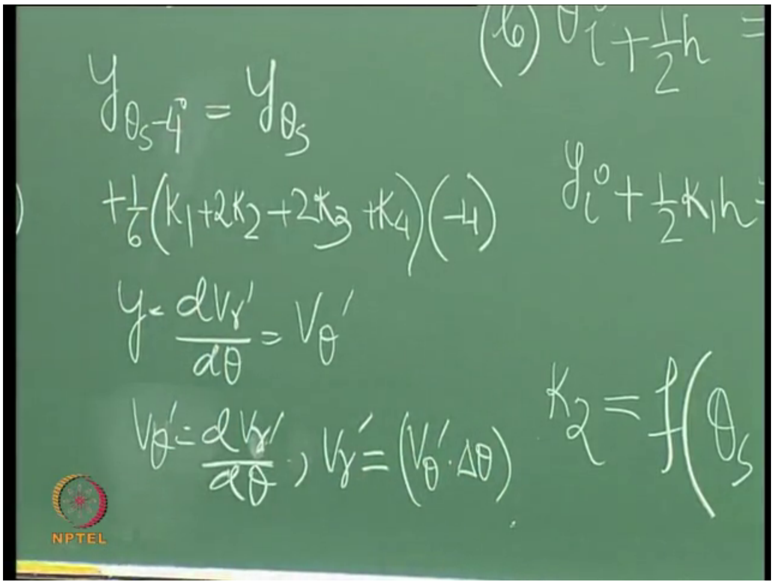

So, essentially this y, so y out here is this. So, when I do this when I get a solution from here what I essentially get is V theta dash. So, once we get V theta dash, we need to get V r dash I will basically just integrate this. So, V r dash is nothing but correct ok.

So, in this case all we need to do is V r dash we will get as V theta dash into delta theta that is all. So, once we do that. So, that is it once we do this. So, what we know is v theta dash and V r dash at this particular radial line. So, once we know that. So, what else is there to do? So, let us see I will come back and need that equation. So, let us say erase this part. So, if I do that.

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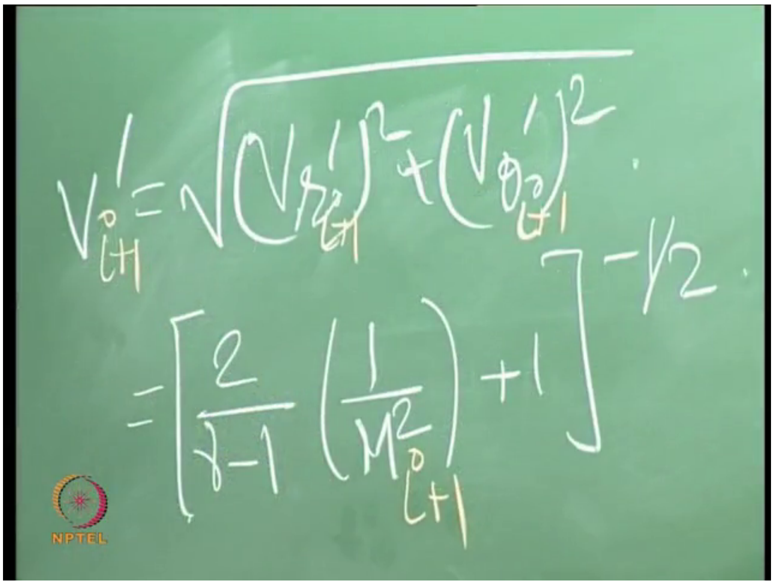

So, therefore, what I get V dash right. So, what? So, once we get V theta dash and V r dash what we get is V dash. So, in this case what we have calculated is V dash on this radial line and if I if you remember let me write this down one more time now this V dash.

Ok. So, therefore, once we get V dash we can get the corresponding Mach number.

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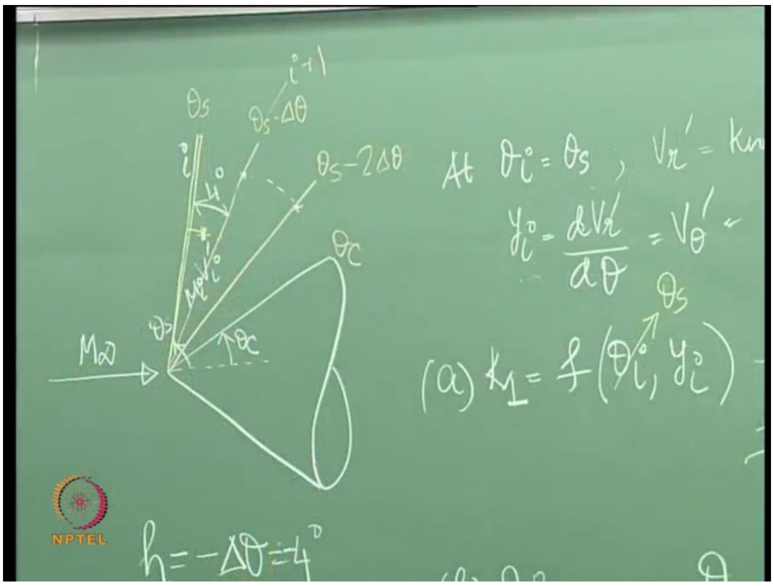

So, what we can now get is the Mach number say. So, say V dash i and Mach number i and once we get the Mach number i. So, you know from here basically if we will get say

V r dash i V theta i let us write this way. So, what you get is I hence and rather in not I actually. So, let us say i plus 1 right. So, this is my i plus 1 ok.

So, now that we have this if we have this no no of course, we can calculate the properties, we can calculate all the properties temperature pressure density etcetera because now we have the Mach number. So, that is it there is basically the solution to this problem. So, this was the. So, as you can see here. So, basically what you can see here is that you know the best way to implement this is using a computer program right. So, of course, I mean you know nobody is gone a do this by hand it is gone a it is exhaustive numerically. So, essentially this has to be implemented by writing a small computer program. So, if you do this then what you know you can take more and more numbers of these radial lines divide this theta into more lines and then that gives a better estimation of properties in this region right.

So, the numerical aspect of Runge-Kutta method is this, it is a pretty straight forward actually and what basically we are doing is using the Runge-Kutta method we are actually solving the Taylor Maccoll you know the Taylor maccollequation at each and every radial line and calculating the property calculating the V r dash and V theta dash and hence V dash at every radial line ok.

So, where Runge-Kutta is the procedure with which we are basically solving the Taylor Maccal equation. So, now, if you implement this using you know a computer program, let us just look at some of the results which you might get this is something that I think you should you know do as an exercise and see if you agree with me. So, let me just set of give you because now again we have done the 2D part of this right 2D wedge. So, we cannot have an (Refer Time: 40:28) we have theta beta m relationship. So, we know what sort of a cone angle would the relationship basically between the cone angle, the shock angel and the Mach number right we have the theta beta m charts.

So, how would that look for a cone right? Now essentially for a cone the shock is weaker for a same theta c and for the same Mach number for a 3D cone the shock is weaker than that of a wedge. So, anyway its let us look at you know just we should be able to get you know properties when from a numerical values then you then you should just sort of cross check now say we have this.

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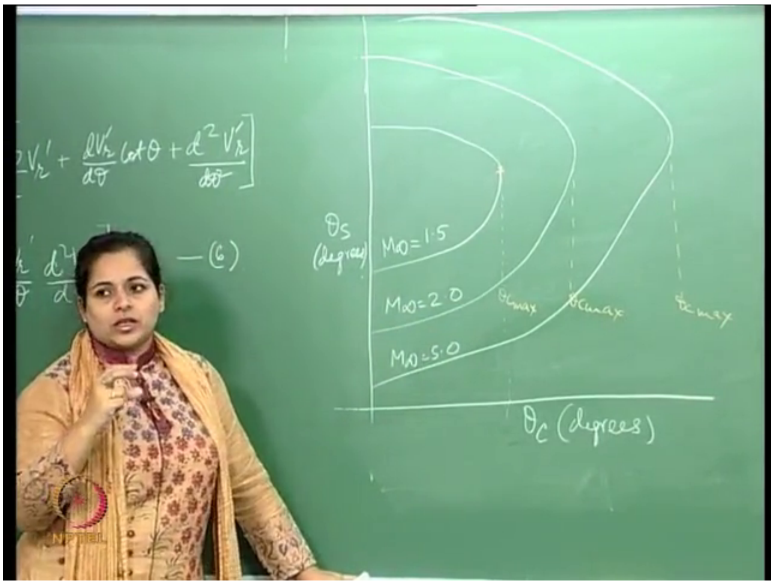

So, essentially say this is theta c this is in degrees, theta c and this is theta s this is also degrees this is not gone a up to not going to be too scale of course, ok.

So, if I do this. So, then what happens say something like that so on and so forth. So, say this is a Mach number say let say something. So, this is similar to the wedge and is just that these numbers are gone be different. So, this is your theta c max. So, now, essentially again you can see that for a particular theta c and for a given Mach number you actually have 2 theta s. So, basically the strong shock and the weak shock the weak shock is more common.

So, this is similar to the wedge, but the numbers are different and again if from the numerical solution if you solve that, now you see that if your theta c or the half cone angle if this is the say; for Mach say 1.5. So, this is the corresponding theta c max right this is the theta c max. So, now, say if you if you have a Mach 1.5 flow right and you have a theta c which is larger than this corresponding theta max, then you know you will not have any solution the Taylor Maccoll equation is not going to give you a solution positive.

So, essentially after this the shock is detached. So, Taylor Maccoll equation is going to give you solutions only when your theta max out here is less than you when the theta is less than the theta max. So, this is also like we did in did for the wedge. Now also the basically the cone the allowance of the cone for the theta c, the cone allows a larger theta c for an attached shock. What I mean is that for a given Mach number that for that for a given Mach number for the cone, the shock is going to detach for higher value of theta c compare to the wedge. For the same Mach number for a smaller value of theta c the flow is the shock is gone to detach when it comes to a wedge right. Whereas, for the cone for a same Mach number that flow this shock is going to be attached for a larger theta c ok.

So, if I were to set off do a bit of a comparison, I can we just know shock its fine. So, basically say. So, essentially if I were to draw this.

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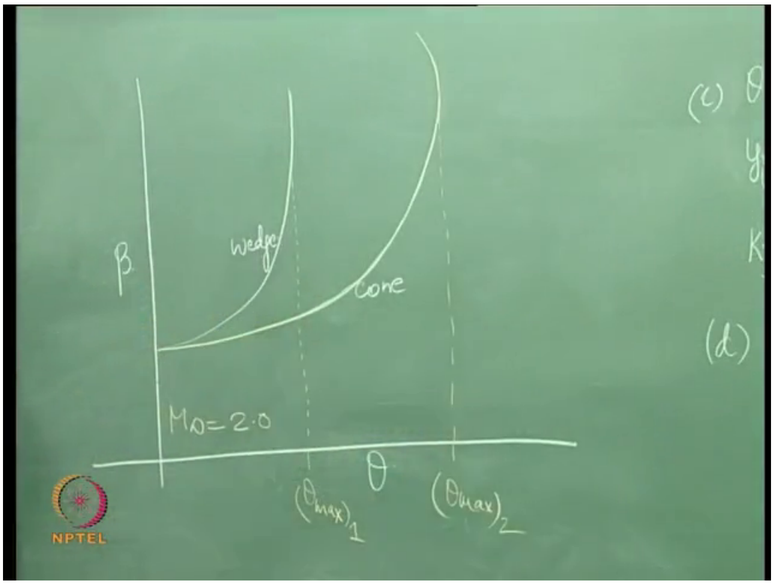

So, this is the shock wave angle and this is the cone angle or wedge angle over right. So, therefore, at the same Mach number right. So, we have say theta max here and let us call this as 1 and we have ok.

So, if I is I just said that if that for the same Mach number. So, say that for the same Mach number say this is for say 2. Now for the same Mach number the cone has a larger theta c max compare to the wedge. So, if you would look at his diagram here, which would be the core and which would be the wedge it is easy to see right. So, what you can see here is that this plot or this I am just drawing the half of it. So, that curves up ok.

So, for in this here this you can see has a larger theta max for the same Mach number compare to this one. So, essentially what we saying here is that this is the cone, this is the cone and this is the 2D wedge. So, therefore, if you have for a you know for the wedge the flow is gone a detach therefore, the wedge at Mach number 2, the flow the shock is going to detach at theta max 1 and for the cone for the same Mach number the shock is going to detach at a larger theta max which is theta max 2 ok.

So, well that would be kind of will the curtains down on you know our quasi 2D flow or 3 flow that we started with, hopefully when we have a little more time may be will go ahead and do further you know not quasi 2 d, but you know we will actually have an angle of attack and 3D flow and we will see the physical differences between you know the 2 cases in this case just 1 bit you know before I close this.

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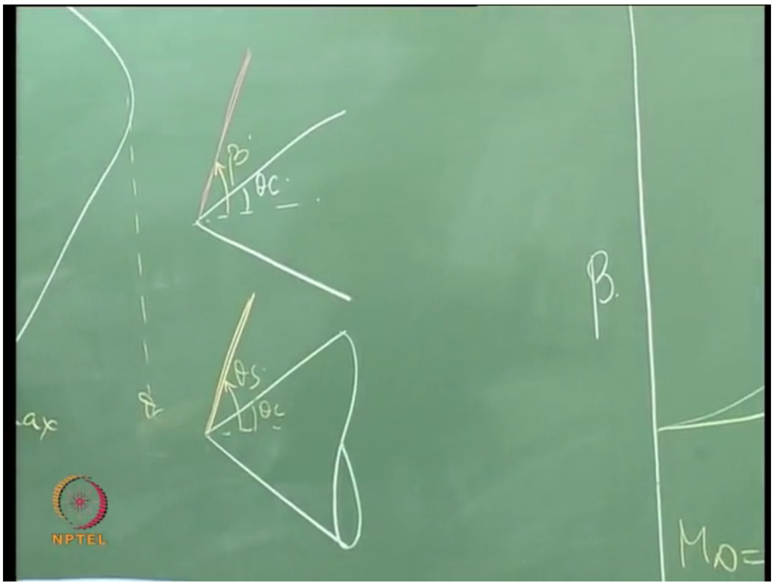

So, essentially you know the (Refer Time: 49:15) just plot this out. So, this was our 2D wedge this is our 2D wedge and so, then we have say a shock here and I am drawing the same thing say I am drawing the same thing over here for the cone and again I have say a shock wave and again this is theta c, this is theta s this is your beta right. See if you were look at these two pictures you know just to visualize this, this is in the plain of the board this here this here is in the plane of the board.

So, this is the 2D picture what you see over here is at this is actually curling away from the board like this, this is the 3D picture of this. So, you actually here what you see is the section of it. If you cut a section through the cone that then this is what you will see. So, in here the difference is that although I am drawing to the board essentially this is curling away from the board like that. Therefore, that makes it 3 d.

So, we will stop here and continue with 3D hopefully with little more time.

Thank you.