Advanced Gas Dynamics
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Lecture - 21
Application of The Method of Characteristics: Design of a minimum length nozzle

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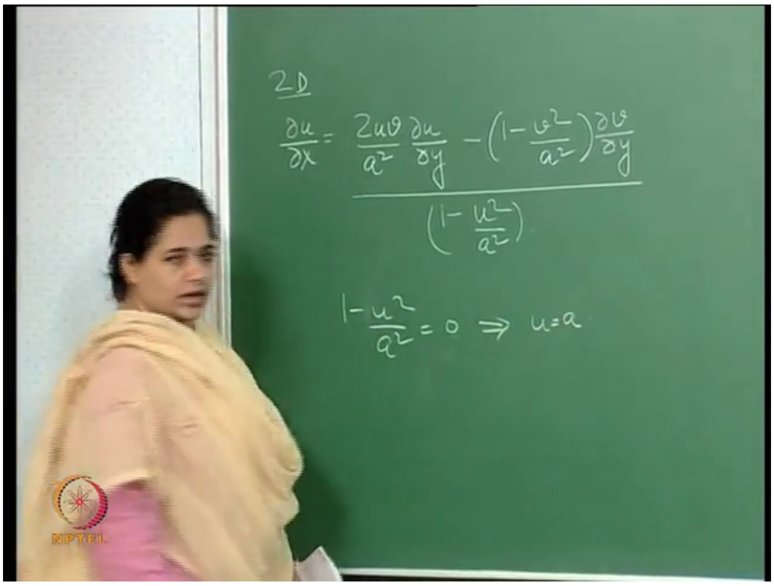

So, we sort of did little more of method of characteristics right. So, we were derive some equations got some you know expressions to play with. Let us sort of go back and revive there are little bit let us do the discussion that we can have left out yesterday. So, let us say so we took we governing equation for say 2D irrotational flow, for the 2D irrotational flow and we got something like we said we could write it like this. And we said we could solve this on a on a computational grid it right that is the basically the beginning of compression fluid dynamics, and we could solve this on a grid knowing the derivatives at you know see location.

So, this is say at a particular location in the flow field. Having divided that domain having divided you know consider a domain in that flow field which we are discretizing you know in certain way. However, there is a cache that this will exist only as long as this is not zero. So, when this becomes indeterminate, when this becomes zero then del u del x becomes indeterminate. So, what we saw is that which means that u is equal to a.

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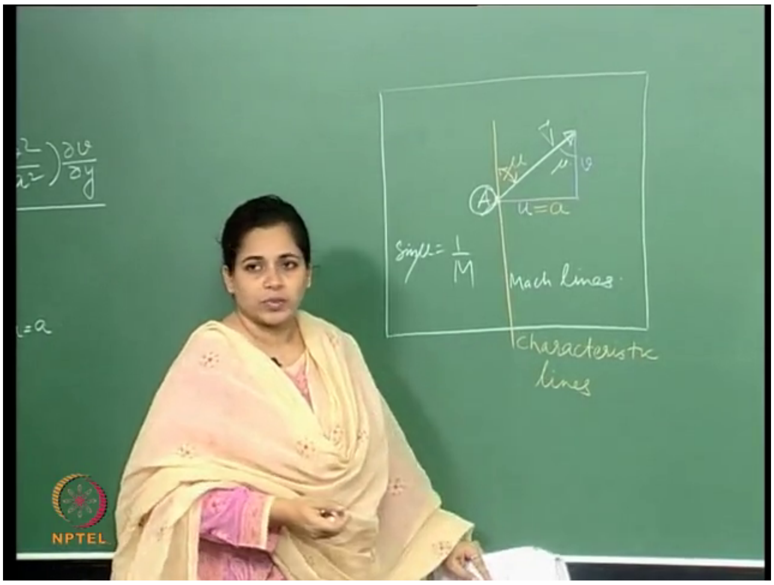

So, if in my computational grid, so therefore, if at a particular so if I say have a grid or say at any point in my flow field, so somewhere at my flow field over here right. So, this is the stream line direction, this is the velocity effecter. Then it so happens that this is the u component and this is the vertical component of this and it so happens that this u is equal to the speed of sound u become sonic. So, therefore, when then if we have a line right this is this line is part of the grid, so if we have this line, which is making an angle mu with the streamline direction.

So, this is a line which makes an angle mu with the streamline direction. And the velocity component perpendicular to this line is sonic. In that case we were able to see that the derivatives of u and other properties will become indeterminate and also become discontinuous may become discontinuous on this particular line, and such lines are basically the characteristic lines.

So, if I am it do that repetition one more time many if you can so basically this is a line right which is making an angle mu with the streamline direction with the velocity vector at say this particular point which is a. And this is basically any arbitrary point in the flow field; and the velocity component perpendicular to this line is sonic, then the derivatives of the velocity component here u a as going to be indeterminate and may be discontinuous on this line. And therefore, this line is essentially a characteristic line. And what we can also see from here is that if you see from here, this is mu. So, sin mu is 1 by m which essentially means that these are also Mach lines, these are also Mach lines.

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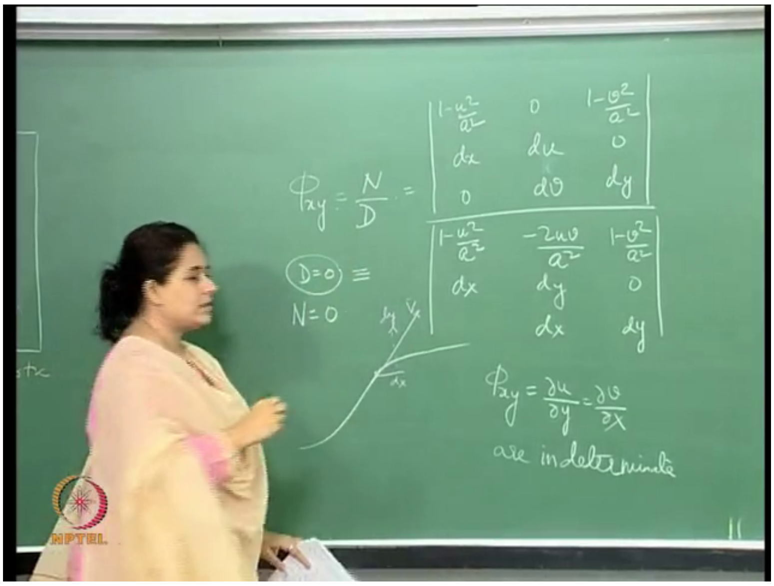

Now, we then we went onto so this is something with a, and then we went onto we worked with this we forgot an expression to find out say velocity potential. Using the velocity potential equation and we found out to be say some numerator by denominator now here to we have to come up with a similar kind of analysis like we did here. Now, this phi xy, this is the phi xy becomes indeterminate if D is equal to 0. Now, basically in a phi xy will always exist only, if D is not equal to 0, but say we put D is equal to 0 then phi xy becomes indeterminate. And what we have seen over here is that the derivatives become indeterminate on these lines which are characteristic lines.

So, therefore, here if we force this D to be 0, so then the derivatives will basically, so for these derivatives to become indeterminate we can set D to be 0. And D is 0 means what we get from here is that the way we choose our dx and dy. Let me sort of write this out this will make a little more sense than. So, as you can see here so therefore, if I choose dx and dy in such a way so a particular flow field. So, you know that this is where do you say for some streamline like that.

So, if I have say a particular at say this particular point say right this particular point this is my streamline direction, this is my v. Now, I choose a dx dy here. So, I choose a dx and a dy, I choose a dx and dy in such a way right that this D goes to 0 then the derivatives of my flow field are going to disappear I go to become indeterminate and may also be discontinuous.

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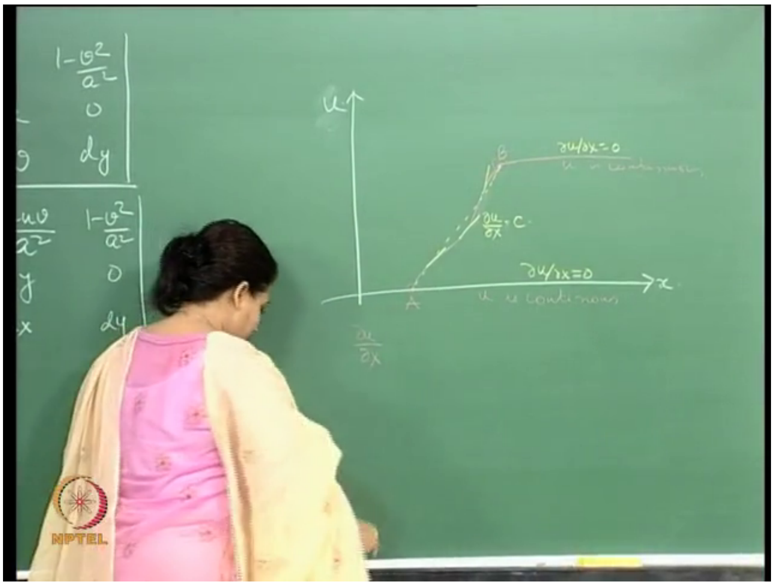

Now, let us look at this now for example, let us say this is a this velocity u. Let us say this is varying something like this say it you know this is the following. So, I have just taken an arbitrary fashion something. So, let us say this point is A and this point is B. Now, if you see from here, so del u del x is 0 right on the x-axis right; on the x-axis this is also pal to the x-axis, so del u del x is 0 over there as well. And somewhere anywhere on this curve at some point, so you can see here so somewhere so del u del x is you know some sort of a constant. So, it will vary you know accorded in this direction.

Now if we have something like this. So, say this is you know at the most it can be a constant. So, in that case basically then you know we can have say a straight line. So, say something like this. Now, as you can see as you can see say for example, at this point A and B; at A and B, the derivative is not uniquely determined right. So, in here u is continuous, here also u is continuous. At points A and B also u is continuous, but del u del x is not independently determined is not uniquely determined. So, it is discontinuous at A and B. However, right the value should lie somewhere between this 0 and c, it the value should be somewhere between 0 and c.

So, therefore, for this to be at least you know for my derivatives to be at least finite, it has to be at least 0, it has to be at least a 0. And to enforce that I have to make n is equal to zero you know somewhere I am doing here. So, what this what essentially I am trying to say is that now I am choosing directions which here are the characteristic lines on which the derivatives become indeterminate it may be discontinuous. So, for which I set D is equal to 0, but to the for these derivatives we indeterminate, but at least finite the n also has to be 0 which means that phi x. y which is del u del y is indeterminate are indeterminate. So, therefore, we found out use of setting D is equal to 0, what this gives us is how the characteristic lines are located with respect to the flow field; they do not necessarily have to be straight lines like we had done for the grid here.

Now, so what D is equal to 0 does is essentially tells this how these characteristic lines are located what is their orientation with the local velocity vector. What N does is essentially gives us the relationship of the properties the way they are related to each other or how the change over these characteristic lines. So, therefore, now what we also did is that we were able to find write a the equations of the characteristic lines setting D is equal to 0.

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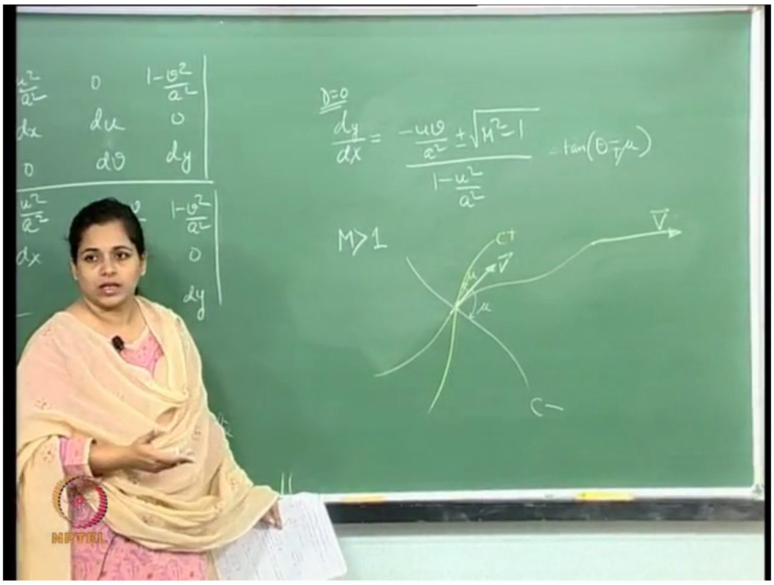

And we found this out, we were able to find dy by dx is equal to right, and we also said now this is also equal to equal to this. So, if we have something like this, this is dx and dy. So, essentially the orientation of the characteristics is given by that and says, this is u and v. So, we found out this and for. So, this was the relationship we got by setting D is equal to 0. So, what this is essentially telling us that we get two lines right two characteristic lines provided M is greater than 1, so which is for supersonic case and what we get here are hyperbolic PDs.

So, for compressible; so for our cases, so our study here what we going to do is essentially consider the case where we have, so m is greater than 1. So, we have basically two a characteristics. So, at a particular point say at a particular point like we said. So, say this is say this is some sort of a streamline some sort of a streamline. So, therefore, my velocity vector here is this, this is my velocity vector. So, what I have over here, it could be any line you know like that. So, let us to say this is C plus this is the right running characteristic and we can have another one, so this is C minus. And this is making an angle say mu and this is and making an angle mu, note the directions by the arrows. So, these are basically C plus and C minus.

Now the question to ask here is are these straight lines ask candies ah characteristics be straight lines. Now, I have just look at this, now for example this is a streamline. Now for example, at say this particular point now I have say at this particular point this is the direction of v. So, what we see is direction of v is changing right on this particular stream line. So, the direction of C plus and C minus will also change at this particular point. So, they do not necessarily have to be.

So, they will be differently oriented at say this particular point. So, C plus and C minus do not necessarily have to be straight lines, they can be a curved line and we will see that. Now, having then these that is so much as what we done with this now. And we finally, came up with the comp compatibility equations what we came up with the compatibility equations by setting n is equal to 0.

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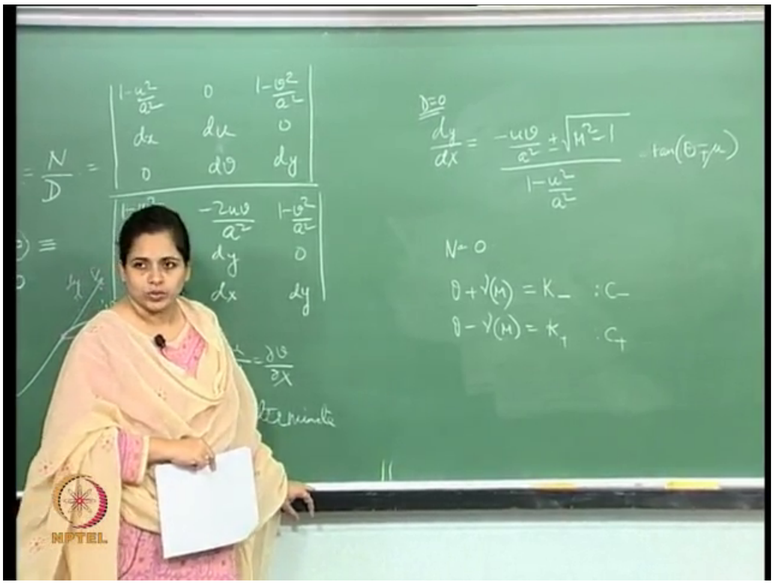

So, let us see what that looks like and let us see what we will do with that. Now, so we set n is equal to 0 and what we found out was that we were able to these are constants. So, this is the Prandtl-Mayer function and theta is theta is basically the streamline geometry definition here. So, to tan theta is v by u, so that is what we get by theta. So, now, here this is similar to the Riemann invariance that we did previously. And this is valid for the see electronic characteristic and this is valid for the right running characteristics. So, this is what we have found in terms of the compatibility relationships. Now, let us see what I mean what sort of a problem we can solve with this, what we will do with all these information that we have it.

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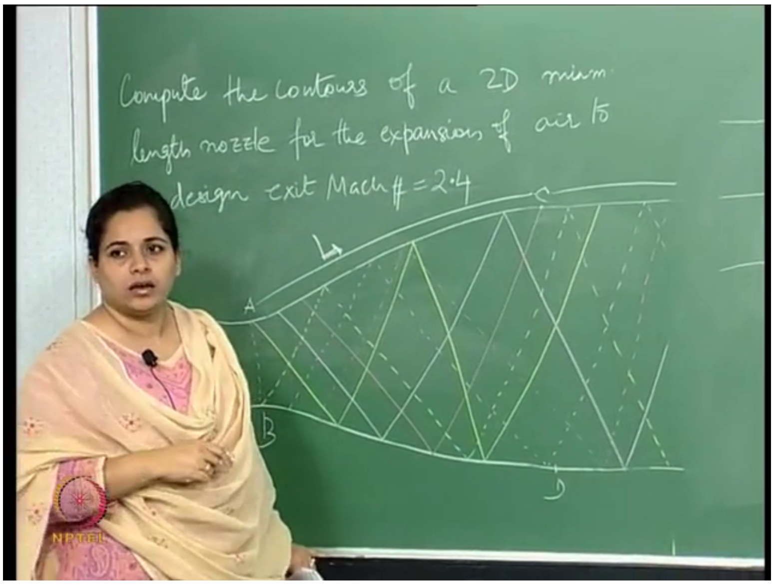

Now, let us say we have this sort of a problem to do. So, this is the problem that compute the contours of a 2D for the expansion of air to a design exit Mach number. So, now it seems that we are kind of at the cross rules. When we started with this course we talked about leverage experiment and how to get supersonic flows we need convergent divergent nozzles to get flows which are at such high speeds, we need to sort of design you know the ducts through which it was flowing. So, we seem to come to a point where we are going to try and design such a duct, depending on what sort of Mach number we needed the exit.

So, depending on how fast we need our flow to be, we want to design the designer duct what that is we are going to do. So, do a converging diverging nozzle, so that is what this is about, but there is something here, which I guess I need to explain. What is this 2D minimum length nozzle, what does this mean? So, let us see what this means. Now, for example, what we could basically do is that you know you have flow this is what we have done right this is what we have done, this is this is basically the converging diverging nozzle. Now, you have basically subsonic flow coming in here, this is the throat region right, this is a throat region the flow is allowed to expand, and then basically we should get uniform flow at the exit basically we will get uniform flow at the exit.

So, now what happens here is that now you allow this, you allow this thing to expand over a certain length. So, what we are going to basically have here is that if the flow is going to expand. So, in this particular case, so say we going to have this. So, we have go to have say several. So, this is an expansion fan. So, I am going to represent this expansion fan like this. So, what happens is now it hits the wall over here, hits the wall over there, reflects from there. And when it reaches say this particular point over here, so let us I am going to call this say it does not reflect any more, it stay stays parallel and it is you know parallel and what we get is uniform flow in this region.

Now, similarly what happens here is that it goes over there, and that is it does not reflect anymore then we have this does, not reflect anymore. Now, we can do the same thing over here. So, there you have this flow expanding over here, we could have the same flow generating from here as well. Let me do it this way, let me do this by dotted lines. Now, for example, this now that comes out here. So, this goes, so this goes then does not reflect anymore. So, similarly we have for this, and then again we will have say.

So, what essentially is happening is that there is a lot of reflection. So, we have this flow which is sort of expanding it this region right it expands in this region. So, the flows expanding from here it gets reflected in its expands from here gets reflected from this wall it hits here and finally, it reaches a section where it does not reflect anymore and it moves it is just sort of becomes parallel. And we get uniform parallel flow and the from this duct.

So, similarly you have flow which is expanding from here, it reflects from here, it goes here reflects from there, and finally it does not reflect anymore and it goes past this. How are we representing this, we are representing this essentially when I have these lines, if you remember the shock tube right, we represented this as a as an expansion wave. The expansion wave right that is what we are doing over here as well these lines essentially mean so now these are basically characteristic lines or Mach lines right this is series of Mach waves traveling.

So, now, we are doing this. So, you can see that this will need sort of long sort of a you know it will need a long duct when you allow this to sort of I know expand over the sort of this region. And finally, you have and you can see this you can see the difference in the region my artwork is not very good it is not at all symmetric, it should be actually, but it is not very symmetric. But you try to get you know with the point here. Now, you can see that this area out here is all intersection of various characteristics; you have right running characteristics and left running characteristics. So, these are all sort of interacting with each other. So, therefore, these lines when they interact with each other, now these are going to be curved except now once you come to here, so if you come to say this region you have only say in the right running characteristics and only one type of characteristic. So, this becomes a simple region. Whereas, something like this where you have an intermingling of lot of both types of characteristics this is a non-simple region and here the lines are curved. You know here, however, it becomes a straight lines and beyond this point of course, we have we have flow which is parallel and that is at the exit Mach number.

Now, therefore, you have supersonic wind tunnels which are going to be very long when you know allow for this gradual expansion, but you want to decrease the you know if you want to decrease the length of width the size of with you want to use it in rocket a fast moving aircraft then what you do. Then what we can do is essentially decrease sort of this expansion region, we want to decrease this expansion region, basically the non-simple region. So, in this non simple region we can decrease that to a certain point. So, that these become not just a spread out expansion fan like that, but a centered expansion fan it will expand only at a particular point. So, when you have that then let us see what it looks like. Let us do that.

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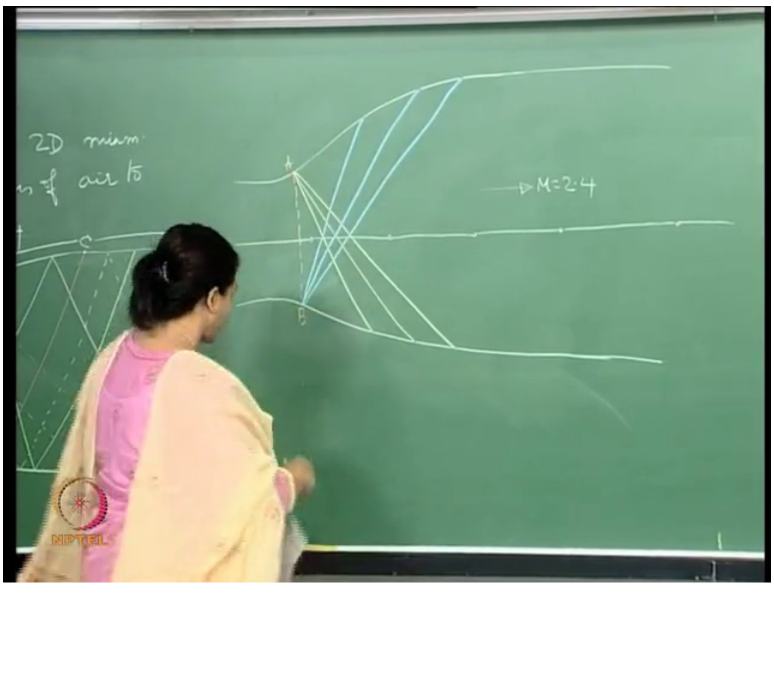

So, let us do this here. So, for example, for other thing is so essentially ok. Now, see the artwork is not good, because it just becomes straight after a point; I am sort of curving it in, so that is it and let us put a center line here. So, if we have this, now this is the throat region this is the throat region. Now, as you can see that you know we do not have like we have essentially a multi-dimensional flow coming in over here. So, it for all practical purposes, this is not going to be a straight line, this sonic region because it is not expected that it becomes sonic exactly along this straight line, it is going to be a curved line, but for our practical purposes we will just take there as a straight line.

So, let us call this as say A, B, and this is essentially the sonic throughout region. So, then unlike in this particular case, when we allowed it to expand over a region, we are going to reduce that region. So, but this particular diagram basically, so ac this length is your expansion length and after this it because this is a straightening portion is straightens out, so that sort of deals beyond say C, D is the simple region. So, here what we will do is that this is you know we will decrease that l to completely we will reduce that l to completely 0, the expansion region to completely we will get just get rid of that, so that we do not give it any region to expand, it has to expand at a particular point.

So, to do that, so let us say we will so essentially this is the wall right. So, let us do this. So, let us say you know write it like this. So, I have basically expansion for now what is going to happen here is that this is not going to reflect it is once it reaches this wall here it will directly just flow out parallel that is the difference when we reduce the expansion region. So, similarly we have so that is it that is pretty much what you can see, now as you can see over here that this region out here becomes non simple is not it. Now, these are all like interacting with each other this region becomes non-simple. So, and rest you can consider there us straight lines. So, therefore, this is what we do when we have a minimum length nozzle. So, you can see that now we have basically centered expansion wave it is a centered expansion wave and the middle portion here, it also likes it is like a surface or a dividing line and you can see that see from here the symmetry of this.

So, during our calculations as long as so what we are asked do here is that if I have a flow, then my exit Mach number here. So, now, when the flow comes here, I will get a uniform flow like this and this Mach number is 2.4. So, what I need to do now is to design this wall basically design this wall. So, design this minimum lake nozzle. So, what we will just sort of do that here now as you can see now this one lets sort of say. So, let me define some geometry. So, what we are going you do here is essentially use the symmetry. So, what now we will basically design just the top part of it. So, we going to get not look at this part. So, let us just say erase that, let us just say get rid of that we are going to deal only with this.

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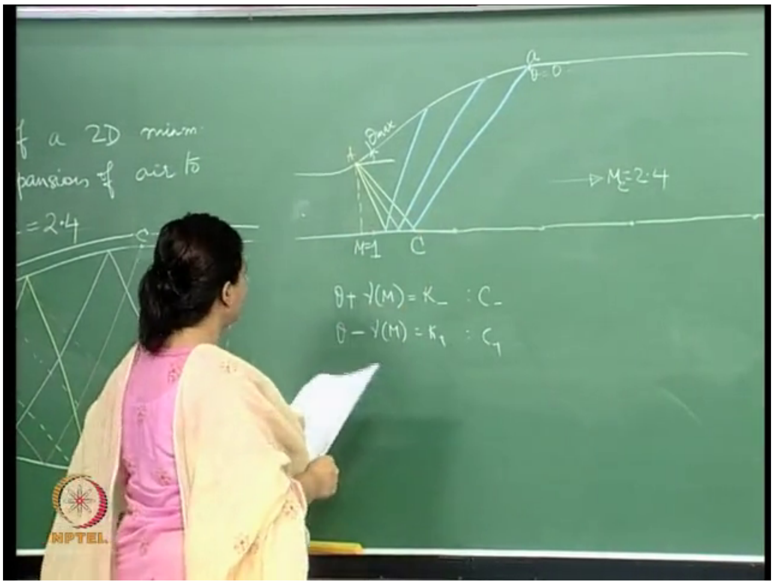

Now, and let us see what happens. So, this is what we have. So, this is what we have and let us see how we going to go ahead and do this. So, let me call this as some number. So, M Mach number is, so let me call this as Mach number at the exit, which is M e. And here Mach number, so here actually Mach number is one this is the sonic region. So, we have this. So, this is the minimum length nozzle and then let us call this as say this point is a, small a. So, now, if we have this and let us say call this point as c, we will call this as c. I get rid of that.

So, now, if we sort of do this over here, so you can see that these are the right running characteristics and these are the left running characteristics. Now, here design wise say design wise, so this is essentially this is say theta max and you can clearly see here that here at this point of course, theta is equal to 0, so that is our geometry. Now, let us use c compatibility equations. Now, what we had was that theta is equal to a constant; this is on the left running characteristics like it on the left running characteristics and theta on the right running characteristics. So, based on that what we can say here is say along this characteristic. So, along this characteristic, so what we would this is a right running characteristic.

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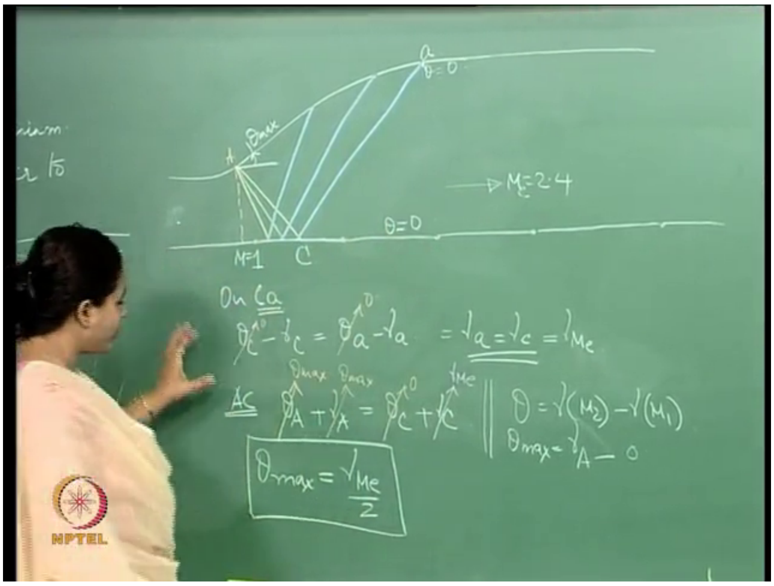

So, let us write it here. So, say let us see. So, on ca this is the right running characteristics. So, we can write say at theta c. So, theta c you think that is ok. So, essentially what we what we are looking at is this characteristic, we looking at ca which is a right running characteristic, and therefore we using the compatibility equation which is which is this right. So, therefore, theta c minus and nu c is equal to theta a minus nu a. Now, here now what is theta over here, theta over here is also 0. So, theta a is also 0. So, theta a is 0 and theta c is also 0. So, what we get from here is the theta, so that what we get from here is essentially mu a is equal to mu c, we get that.

So, we get this and also now we also know that this mu a right this mu a which is equal to mu c is also equal to the mu which corresponds to the exit Mach number is not it. So, therefore, this also we can write as this is corresponds to the exit Mach number. So, this is as much as we know from here. Now, let us look at for example, now let us look at say AC, which is a left running characteristic.

If I look at AC, so and again apply the comparability equation here. So, what we get is theta a plus nu a is equal to theta c right plus nu c. So, now what we are looking at you see when we took see A when I wrote this equation I am using this C here, I am also using c here. What I am doing here is that I am considering the right running characteristic hence we are getting this from there and here we are using a left running characteristic between A and C. And therefore, this is what we are getting now what we did just now is that theta C is 0, theta C is 0 and theta A is of course, theta max right is theta max.

Now, theta C let us sort of look at this over here. Now, this mu A now mu A is essentially for a Mach number, which is M is equal to 1. And if you remember the Prandtl-Meyer equation is right. So, I can write this as theta max is equal to say mu A minus mu A and this is minus mu 1, which is 0 right. So, mu A is also equal to theta max. So, mu A is also equal to theta max, which means and now this mu C is of course, equal to the Prandtl-Meyer function corresponding to the exit Mach number. So, this again is mu exit Mach number.

So, therefore, what we get from here is theta max is equal to of the exit Mach number by 2. Now, this is very interesting. So, now, you can see that if I have to design my, so at just let us take it step back and look at this. So, what we have found out here using the compatibility equations that given the exit Mach number, this is all that we are given that is all we are given. So, how do we start out to design you know this nozzle. So, what we found out is that theta max which is here right is equal to half the Prandtl-Meyer function based on the exit Mach number.

Now, so essentially what we saying here is now if you have a length you need a minimum length of this. So, you can start from theta max and gradually go up and straighten it up. So, you know you need this minimum theta max; you need this minimum theta max. And if you decrease the length below this then it will not be shock free, then you will have shocks. So, to have shock free parallel flow here based on this Mach number, this is the minimum length you need. So, this is the length that we shall be needing. So, now, let us sort of do this do this problem and let us see how we will sort of go about this. it is the same thing.

So, our specific problem here, so let me sort of you know draw this save one more time or shall we do it over here. So, let us sort of do this over here itself because this is what we are looking at. So, having found this out how do we go about designing for our particular case. Now, let us say in this particular case, where I have done is said taken and say 7 - 1, 2, 3, 4, 6, 7, 8 lines.

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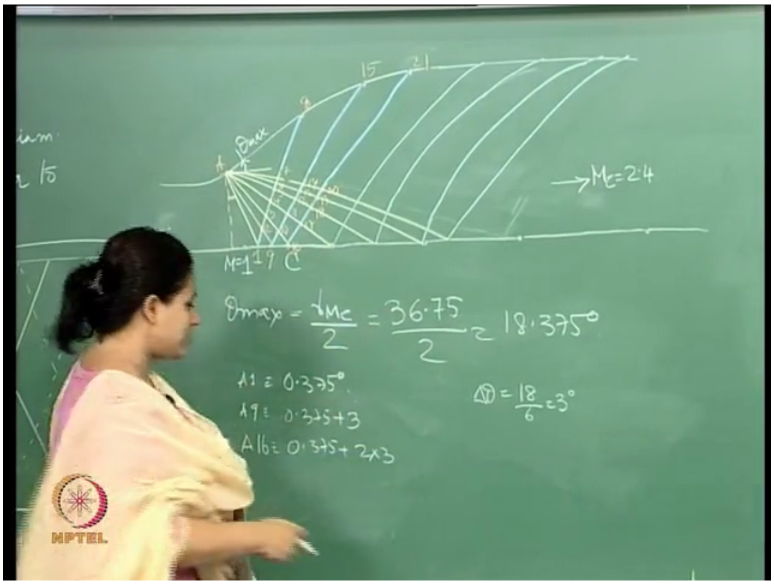

So, say 1, 2, 3, 4, 5, 6, 7, 8. So, now we have this and yeah. So, what is happening here what I have done here is and you can see I am just representing the expansion wave, you know now this is a numerical method right. So, now the more number of characteristics that you take you will get you know better estimation of the properties within that region. So, in this case so obviously, when you write a computer program you can actually do it with you know more accuracy when you have more and more curves. So, I am sort of this giving an example out here.

So, similarly we will have say this will go out and we go out, basically how we will go about solving this problem. So, we will have things like, this is my essentially my region. So, what we will do here is, so this is my theta max. So, how do we number this? So, what we are going to do is we are going to number this whole thing. So, we are call that as A. So, we will call this point as 1. So, this you can see here this is we have actually created a grid, we have created a grid in a very different way, our grid looks different because it consists of Mach waves, these are characteristic lines.

So, this is 1, this is 2, this is 3, this is 4, 5, 6, 7, 8, 9 I am sorry it looks it looks a little congested, but you know I am afraid you know you really cannot do too much about this here. So, this is how I am numbering it. So, what I do is on each first line from here, so I said 1 and 2, 3, 4, I go along to the wall. Again I come over here or did I ok, so I guess I took one more extra line does matter. So, I can sort of this to maybe decrease one more say if I say do that it will give me some more space actually if I do that. So, let us say I decrease that.

So, therefore, you can you can really take as many lines you know as if you want and that is basically going to give us a better estimation of things. So, if you have that. So, then this is 6, and this is 7, and say this point is 8. So, then we come here this becomes 9, this is 10, this is 11, this is 12, 13, 14 yeah and so on and so forth. So, then you come here this becomes 16, 17, 18 and so on and so forth.

So, this is what we get now. So, and I note my exit Mach number my exit Mach number is and that is all I am given. So, my exit Mach number 2.4. So, therefore, I can find out my theta max. So, my theta max right is corresponding is right and what I get from here is essentially. So, what I get from here is at this, so having found this out, so theta max is essentially 18.35. Now what we are doing? So, essentially this is my theta max and you can see that these characteristics the way I have you know inclined them. So, this is really up to the user as to how that person does that.

So, now let us say that this angle let us take that you know that A 1, so that makes an angle 0.375 degrees. So, then therefore, each of these are then looking at then basically that angle is divided into remaining 18 degrees divided into 6 – 6 divisions right. So, therefore, A 1 is this, then say A 9 the theta is 0.375 plus yeah so therefore, my delta theta becomes 18 by 6 which is 3 degrees. So, it is three. Then A say 16 right A 16 that becomes 0.375 plus 2 into 3 and so on and so forth, and so on and so forth.

So, what we shall do here is again use the compatibility equations and properties, and we will complete this you know problem. So, basically using the and see if how were to locate, so what you have to understand here is that we do not have this profile at this point of time, we do not have that. All we have is we are asked to design this. So, we need to locate where 8, 15, 21 is with respect to A, we do not know that here; all we know is this based on that we were able to find theta max.

Now, we been able to you know orient these Mach lines, which are in the expansion wave. Now, let us see how we will design where these points are located and only then we will be able to you know design this point design this. So, we will complete this in the next lecture.

Thank you.