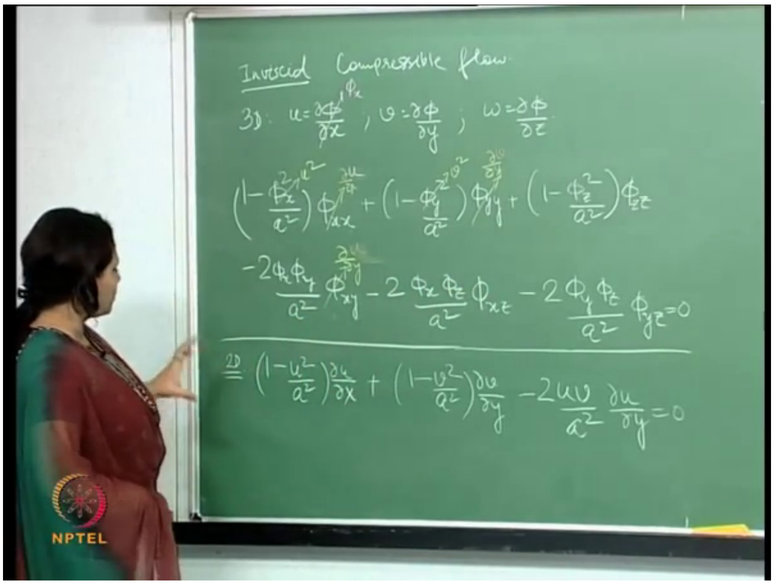
Advanced Gas Dynamics
Dr. Rinku Mukherjee
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture - 20
The Method of Characteristics

So, we introduced the method of characteristics or the characteristics in the last lecture. So, this is kind of the beginning of introducing numerical techniques to solve you know equations and solve for compressible flows etcetera. So, let us sort of um trying to understand the concept behind the characteristics or what exactly the method of characteristics means.

So, let me just sort of begin you know very simply trying to under trying to explain or trying to understand, what the concept of this characteristics is right. And to do that what we will do is we will basically consider the velocity potential equation. If you kind of do not remember what the velocity potential equation is what the potential means ok I think you should just go and look up you know just a brief reminder should help; so I just to remind ourselves; so for we will consider basically solution for an inviscid compressible flow.

(Refer Slide Time: 01:45)



Now if I say if I consider say ok. So, you know the 3 D irrotational flow. So, in 3 D case velocity potential basically. So, if the potential is phi and I will just sort of write out the entire 3 D equation right to kind of just remind ourselves that the complexity of it all you know what each term basically means. So, I have this right. So, this you can see it is pretty engaging right.

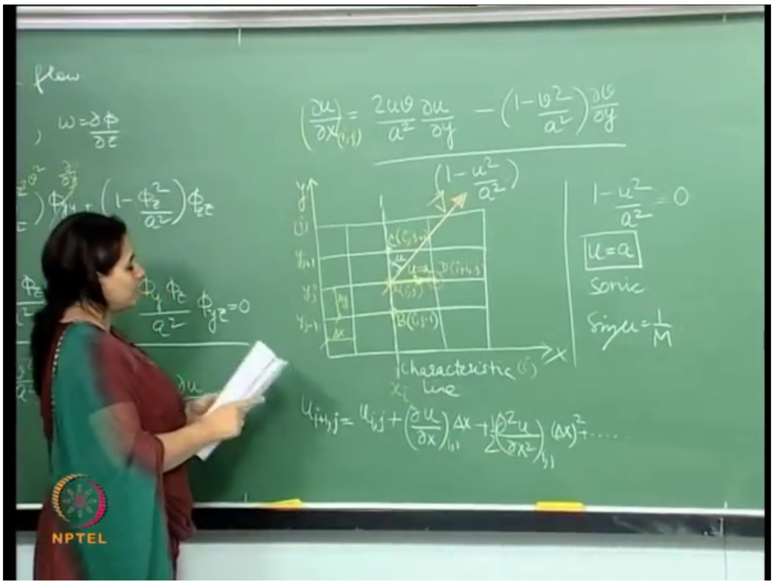
But there are certain pattern to this. So, this is essentially my entire 3 D equation this is my entire 3 D equation and what is we will start with like I said to understand you know the concept of characteristics is that we will start with the 2 D case and let us see if you understand that. So, for the 2 D case what we will do is from here. So, this is the entire 3 D say equation right I leave that here. So, if you see if I take the 2 D case, what we get from here. So, this phi x is nothing but this, this is essentially phi x and I hope you remember that. So, this is essentially phi x and phi y and phi z etcetera. Therefore, if I write it in; so phi x is here. So, phi x is nothing but u in here.

So, essentially this is nothing but u square this is nothing but u square and this is nothing but v square and in here. So, uv. So, then if I write this out what I get is in 2 D is u square by a square. So, phi x x is del u del x is not it? This term this term out here is del del x of phi x which is u I will say let us start it like this del del x of u. So, then what we get here is del u del x plus let us get here, which is 1 minus v square and again here. So, this is nothing but again this is nothing but del del y of phi y which is v.

So, if I do that. So, this is del v del y and this is 2 D. So, we will not have this, this term here. So, we will come back and have this term. So, phi x and phi y are u and v respectively. So, it is minus 2 u v by a square and this is basically if you look at this, it is del del y of phi x right. So, this is del del y of phi x which is u del del y of phi x del del y of phi x which is u. So, therefore, what we have here is del u del y is equal to this. So, therefore, this is the equation that we get from here right.

So, we have del u del x you have del v del y and we have del u del y. So, now, let us write this in let us sort of write an expression for del u del x from here.

(Refer Slide Time: 07:23)



So, what we get is this. So, what I get is del u del x is 2 u v, a square del del y; so 1 minus a square. So, we have an expression for this. So, let us just sort of take a step back and look what we did. So, what we are trying to do over here is that we want to numerically solve the equations which govern our compressible flows. So, now, this is a that we have taken this equation which is the 3 D velocity potential equation.

So, this is the science or nature of this equation, if you look at this now if you. So, basically we are dealing with in this particular case Inviscid compressible flow right. So, Inviscid compressible flow right. So, this is the. So, the velocity potential equations, velocity potential are phi. So, then this is the 3 D equation. So, we said we will sort of start with a 2 D equation and then see how we will be able to solve this. So, we were able to get a 2 D equation which is like this and what I see here is that I can write say del u del x in terms of this. So, what I essentially need here is del u del y and del v del y u v u and v. So, if I have these values then I should be able to get of a get an expression for del u del x.

Now, how do we go ahead and sort of solve this is what is the best way to do that? And this is where you know CFD comes in I am assuming that you have done at least a basic course in cfd. So, you kind of have a little bit of idea about that. So, if you do not I think you have to go back and look at it just appraise yourself you know just the basics of cfd. Now how do we sort of go about this numerically? Now for example, you know let us take a domain we have a fluid flow and we are trying to calculate say del u del x there, now let us take as in a domain like this. So, let us say that and al.

This is say the origin and what we will do is say look at this at a particular x location. So, let us call this as i. So, I think guess you know that you know the indicator and the x direction is i and in this direction is j. So, let us look at this x i point and let us look at this point out here. So, this point let us call that as say a we will call this as A and this point is say i j is that some j and some i. So, say this is y say j over here. So, that is the point a say i j then we will look at say these 2 points. So, we have a point here. So, what this is say we will call this as say. So, let us say we will come here ok.

We will call this as B and this is at the same i, but j minus 1 if you look at this. So, you have the same i it is in the same x, but it is at the lower j which is a lower j this is why j minus 1 right this is y j minus 1. So, this will be say c and this is i and is 1 j up j plus 1. So, say this is basically y j plus 1. Now we have these say we have a domain we discretize that in a certain way. So, this is essentially say delta x and this is essentially delta y. So, say we have these 3 points over here. So, now, if I say need to calculate for example.

Now, say we need to calculate this, say we want to calculate u i plus 1 j. So, which is this point? So, this point is say i plus 1 j. So, we have got like at 1 we have advanced one x or 1 delta x in the x direction. So, if I had to say calculate that, then let us write this out and say the Taylor series. If I were to write this what I will write this as is u i j this is I am writing this in Taylor series u i j plus del u del x at i j into delta x plus the.

So, half del 2 u del x 2 at i j del x square and so on and so forth. So, this is what we get from Taylor series. So, what we see over here is this is essentially are Taylor series right and what we see over here is that if I need to get the derivative d as say if I need to get say u at i plus 1 j, then I can write that in terms of the value at a and the derivative of u here a right. So, in that case; therefore, you know in this case I basically need to calculate del u by del x at say this point. Now if I am able to do that, then I should be able to get you know some value over here in here.

Now, let us do something over here which I guess you have done in your like I said you know your CFD classes in the beginning of CFD classes, now if I can if I will ignore the higher order terms over here. If I ignore the higher order terms here then basically what I will get is my fire difference equations here. So, based on that, what I can do is I can calculate essentially del u del x from here and del u del y as well as del v del y at this particular point a right. Now there are various like forward difference or backward difference or central difference and so on and so forth.

So, you can use those things from those expressions for and calculate the derivatives at basically this point. So, once you do that therefore, I can calculate this I can get an expression for here. So, then once I get you know these values. So, if it is known then I shall be able to calculate then you del x at this particular point. So, this is your basic sort of CFD and there is a catch here though there is a catch. Now we have chosen you know we have chosen this grid right in such a way that is I am able to see use my basic CFD knowledge and you know and then able to calculate this there is; however, one catch which I hope you can see from here, now you can do this as long as this does not go to 0 right.

If the denominator goes to 0 then the derivative is also going to blow right. If this goes to 0 then the derivative will not will be indeterminate and it can also be discontinuous so. So, therefore, there is a condition now what does this derivative this denominator being 0 mean? Let us look at that. So, this catch here is and this is possible only as long as is not equal to 0 you know. So, what happens if it is equal to 0 what you see is that u is equal to a, which means this is sonic. Now, if now if there is such a case, let us use this say grid itself and let us take this grid itself and look at this, now say this condition happens you know you get your condition when this is 0 it; that means, that; so in here in this grid.

So, say we have a velocity component. So, say we have a velocity; this is the velocity of the fluid. Now it so happens that here the x component or you write here is sonic, which means that this component out here, I have a u component over here u is equal to a. So, in that case the derivatives of u over this particular line. So, if I have this line say over here and say this is making some angle say this line out here and say this is making some angle ok.

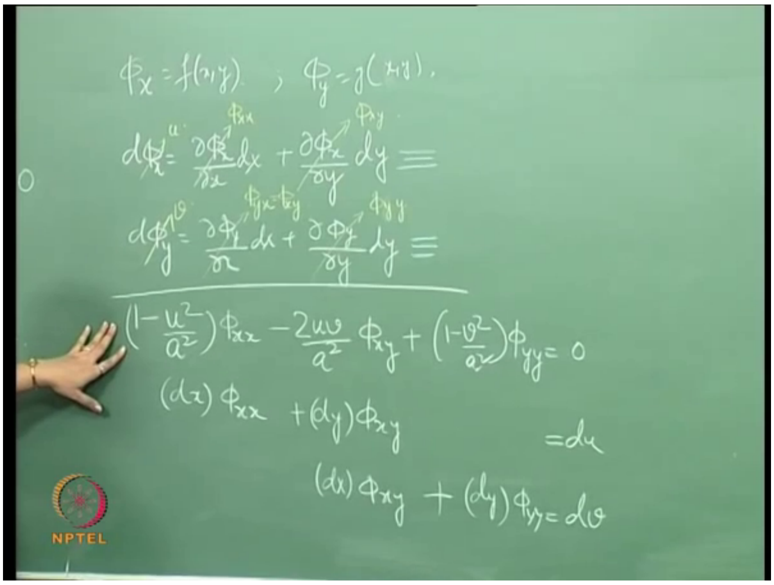
Let me call that as mu let me just call that as mu. So, in here now on this line the derivatives are going to blow. So, where is this del u del x or del v del y etcetera it is going to blow on those on that particular line. So, now, therefore, in this case, now if you sort of recall this. So, you have basically a line here right which makes a certain angle say in this case which is mu right with the flow direction right, across which the derivatives of the flow are may be discontinuous and indeterminate and the u component is here the u component is sonic ok.

So, if you remember this that we have done we kind of did this a couple of lectures back that such a line is nothing but a Mach line right and so therefore, now in this case, this line is also called a characteristic line. So, this is sort of introducing the master of characteristics. So, this is essentially a. Now as you can see here now this orientation this x or y now this does not matter, because the thing is this is at the Mach line here Mach line orientation is basically is orientation is with respect to the flow direction right. So, it makes a certain angle with a mu in such a way that this u out here ok.

The u component is sonic and therefore, the derivatives across this line because if this u is making a now this is important here. Now we have this sonic condition only when this happens only when we have this condition. So, only when we have such a condition when u is equal to a and we have a corresponding line here which is making mu. So, in that case the derivatives across this line are going to blow and therefore, this is called a characteristic line and is in physically nothing but a Mach line. So, yes and I think if you remember of course, that right.

So, this is something that we did the earlier on. So, having said that, let us sort of go back and try and see how you know how we can use this concept. So, like we had say this equations here. So, again for the 2 D case right. So, let me sort of write this again. So, we had this 2 D case. So, we had you know this equation out here. So, should I write this again? So, let me sort of.

(Refer Slide Time: 24:24)



So, we have the 2 D equation and you know we have phi x right this is also a function of x and y right phi x is a function of x and y as well as phi y is a function of x and y right.

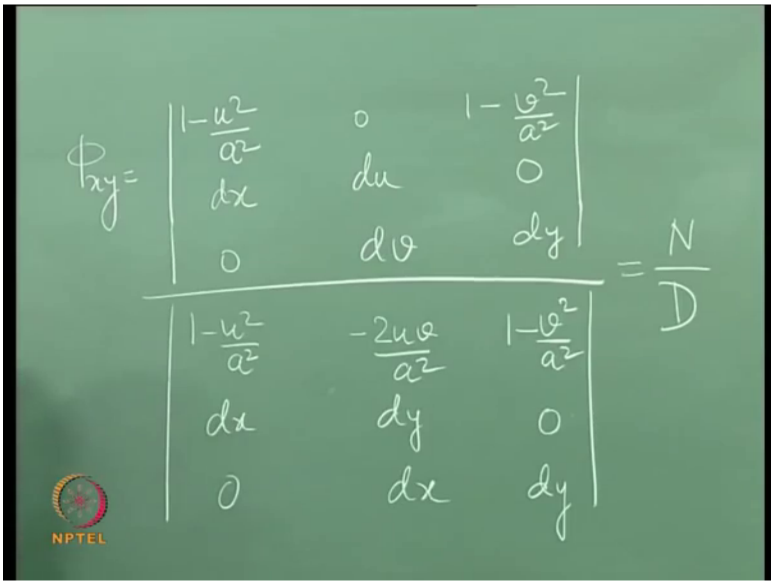
So, therefore, I can write say d phi x, d phi x is right. So, if I can write this then again. So, as we know as we know phi x is nothing but u right. So, then I can write this as. So, I am do this here and I can also write this as phi x x and this also I can write as phi x y. So, again similarly we can write d phi y right. So, this d phi y is nothing but d phi y del x dx. So, then what we have here is phi y x you know this is phi y x which is equal to phi x y.

Now, that is the potential condition. So, just sort of remind yourselves about that. So, then this is phi y y. So, now, what we have do is. So, just to bring everything into place, essentially we say have this equation over here, we going to take this put it there and we will write this out. So, what we have is this is what we have got from earlier on. So, what we are going to look at is this expression this one, this one and the 2 D equation that we have derived before. So, let us just sort of write that down. So, what we get is 1 minus u square by a square del u del x is fee x x k minus 2 u v by a square is phi xy plus 1 minus we get this.

Now, let us look at this equation here. So, what we get here is that let us look here. So, what we get is dx into phi x x, then we get plus dy phi xy, this is equal to d u. Then we look at this equation here and what we get here is that we get dx phi xy plus d y all right. So, this is basically the 2 D velocity potential equation right and this is what we get from here. So, this is what we get from here; so dx, phi x x, d y, phi x y is equal to d u and this here is again dx, phi x y and d y, phi yy is equal to dv.

So, what you can see is basically we have got a set of the system of simultaneous equations, now let us solve for phi x y from here.

(Refer Slide Time: 29:29)



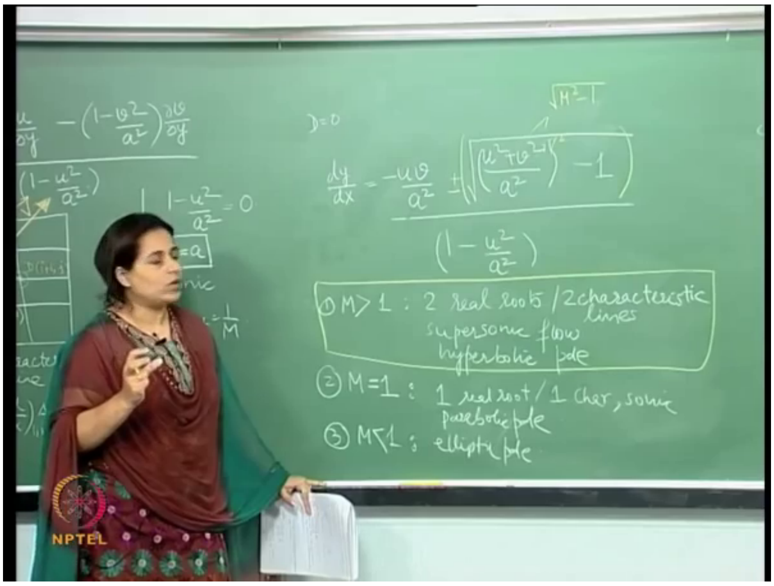
So, we can use Cramer’s rule etcetera, now if I solve for this what I get is this, that and then we do this and what we get here is. So, now, this is what we get now. So, all we did here we took the 2 D velocity potential equation, combine that with this and what we are able to get here is this. So, basically for a given d x and d y; so d x and d y is nothing but say in this case the small you know incremental change in the x direction and y direction respectively.

So, what we are basically doing here therefore, a given d x and d y we are trying to calculate phi x y. So, now, once we have this. So, then we use a Cramer’s rule to get this. Now the point is that how are we going to use this. Now this thing like we just did just now, now phi x y now this will have this will not blow as long as the denominator is finite. However, if the denominator goes to 0 right now if we choose d x and d y in such a way that this denominator goes to 0 then of course, we are not going to have a determinant value we no we are not going to be able to determine phi x y, we are not going to be able to determine u and a u v etcetera; if we choose d x and d y in such a way so that the denominator goes to 0.

Therefore, here what we have seen is that if we have chosen it in such a way that the a in the this particular case right we have chosen and you agreed in such a way that in this particular case, the u component becomes sonic, then we have a line which makes mu with it right such that u is making you become sonic, in that case the derivatives blow on this particular line right. Now, in here d cannot be 0 it is physically inconsistent right we cannot have a d which is equal to 0. So, for this to exist d has to be finite.

So therefore, now if d. However, is 0 what we get essentially is the direction of the characteristic lines, because if d is 0 out here then my derivatives are going to blow. So, which is the same as what we had discussed out here right. So, if here it becomes a sonic and then we get these characteristics lines. So, let us now look what d is equal to 0 means.

(Refer Slide Time: 34:05)



Now, D is equal to 0 what is the. So, basically you know we will you know this will go to 0 the determinant is going to go to 0 and what we get here is this.

So, I am not going through the total math of this what we get is this. So, if d is 0 then what we get is this right. So, what you are trying to say over here is that if the determinant here if this goes 0 then the. So, so for a particular chosen dx and dy right if when a nominator goes to 0 then my derivatives blow across. So, this happens. So, these are basically this will happen across. So, for a particular value of d x and d y; those lines are going to be characteristic lines so on the characteristic lines; therefore, d y by d x. So, what this is essentially giving us is in this, but what this is giving us is basically the orientation or how we are going to locate these characteristic lines ok.

So, this is what it is giving us now, let us see what else information this is giving us if you just take a step back and look at this. So, this essentially is nothing but this is say total velocity is not it this is say v square. So, what this essentially is equal to. So, I can say this is actually equal to m square minus 1 is not it Mach number now if we look at this. So, mathematically when you look at this. So, now, if you look at this now just mathematically, what we have here is minus u v going to write this. So, since we have this now let us look at these conditions here, what we have is that m is greater than 1 and these are the case studies let us do this ok.

So, in this particular case if m is e equal is greater than 1, now let us just look at the math of it. If I had m greater than 1 right what happens? I get 2 real roots right I get 2 real roots; real roots in this case means that I will get d y by d x with what this is essentially giving us is the slope of the characteristics lines. So, I will have 2 characteristic lines. So, here which means 2 real roots which means 2 characteristic lines right. Now and this physically means that this is a supersonic flow right this is a supersonic flow and what we get in this particular case is a hyperbolic pde ok.

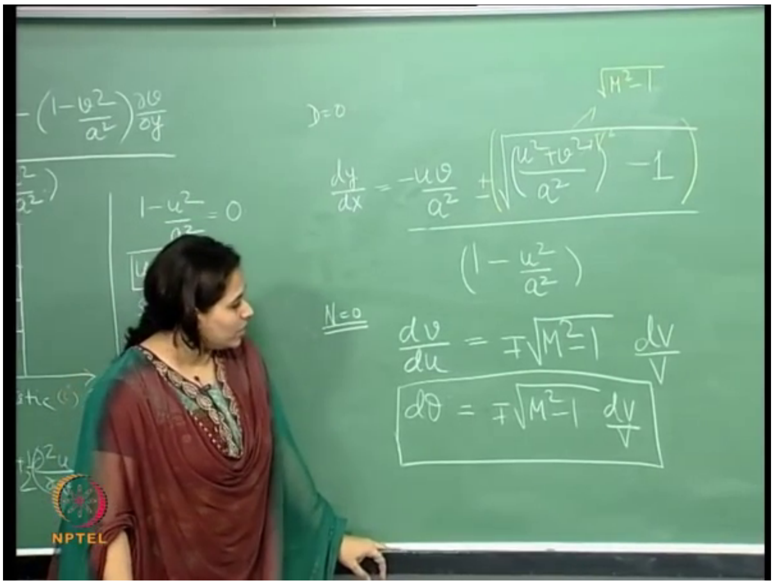
What we get here is a hyperbolic pde. So, mathematically what we get is these 2 real roots which means this is a hyperbolic pde, now physically what this means is that this corresponds to a supersonic flow and we are basically looking at 2 characteristic lines here. Now let us look at the second case if say M is equal to 1 if we have equal to one. So, we have just one real roots out here right. So, if we have is equal to one. So, it is basically it is a. So, we have essentially one real root mathematically which means 1 characteristic line and this is basically a sonic flow, this is essentially a sonic case and this corresponds also to mathematically to an elliptic pde.

Now if we have the third case right if it is subsonic. So, if a m is less than 1 you will have imaginary roots. So, you will have essentially what this is sorry sorry about that. So, this is essentially parabolic right this is parabolic and then for subsonic case we will have imaginary roots right imaginary roots and then this becomes essentially an elliptic pde right an elliptic pde and we shall have. So, this is basically for subsonic case. So, mathematically that is what we are able to derive from here. So, essentially, the basically the orientation or.

So, we are now essentially concerned with this right. So, we are essentially concerned with this. So, when we go to solve for supersonic flows basically we are going to look at hyperbolic pdes right and for supersonic cases and in this case we are essentially using the characteristics right and these characteristics is the orientation of that or how these are going to be located these lines, how these are going to be oriented rather in the domain or flows you know flow area basically. So, is we will be given by this. Now in here in order to calculate the compatibility equations, which is the now we now that we have the um pattern, now that we have how the characteristics are going to be located oriented in the flow field.

Then how do we now all we have here is the velocity is the velocity definition with respect to the domain. For a given d y by d x how are the u and v locate etcetera and what about the properties how are going to be the how are the property is going to you know vary along those characteristics or in the flow field how I are we going to collect to that, that is going to be given by the compatibility equations and we are going to set basically here n is equal to 0 for that. Now let us do that and see what we get ok.

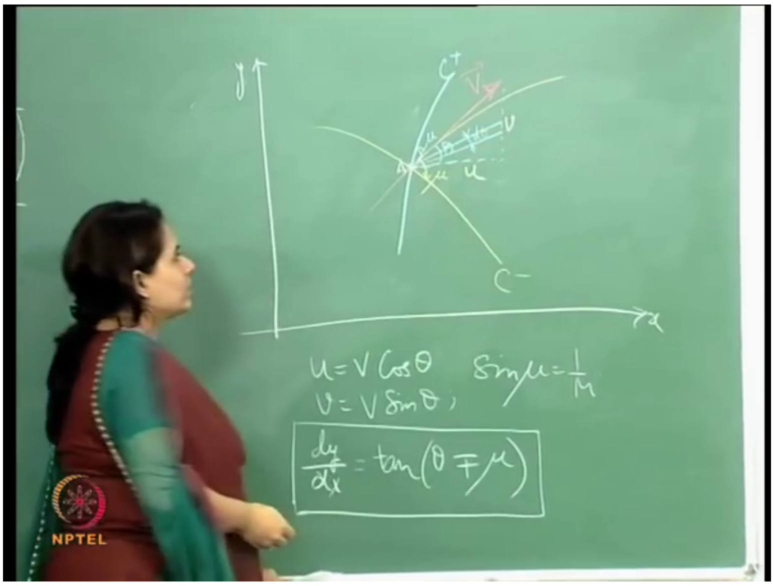
(Refer Slide Time: 42:25)



So, I am going to sort of erase this over here, now we need to discuss this a little more as to we have set essentially n is equal to 0. So, now, the in this particular case, let us first find out the compatibility equations.

Then we will discuss this a little further right. So, let us just put n is equal to 0 if I do that then what I get is this. So, what I get is this actually. So, now this basically now let us to look at this here. So, since I have written that, let just look at the physical picture of this. So, if I were to look at the phys we can we could sort of go back it is become a little congested there ok.

(Refer Slide Time: 43:39)



So, say at a particular point. So, we have this x and y right. So, we have this certain domain. So, say at this point right A. So, this is our A and here I have a certain say velocity here right. So, this is the point here. So, save this at this point right at this point we have this velocity, this we have this velocity component right.

So, what we found that we are basically going to do d y by d x is essentially gives us 2 characteristic lines 1. So, it gives us the direction of that. So, one we have say like that. So, say this we are going to call this as a right running characteristic and let us say look at another one and say we are going to look at. So, this is c minus right and as we said this characteristic line is essentially Mach lines right. So, therefore. So, if you say look at this. So, this is making say positive u and this is making say this is also making u, the arrows indicate the directions of is around I did not write minus u.

So, this is essentially the orientation of this now. So, now, if you let us look at this plot here and let us look at this. So, now, this is say theta, this is the velocity vector there. So, say it is making a theta with the horizontal. So, this is my now this is kind of looking like this is mu let us pull that tail at down a little bit. So, this tail is comes out here. So, what I have over here though say this is d v and this is d u right d v d u or you know u or v. So, let us just see since I have a v over here. So, we are going to just say as u and v. So, essentially what I have over here is a v and we have a u and this is may this velocity vector is making an angle theta with the horizontal.

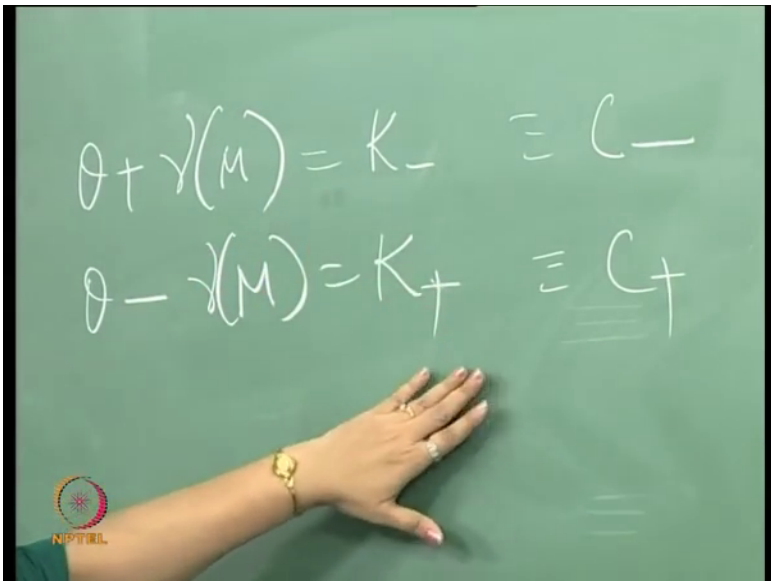
If I do that then what I can write over here if you that look at this picture over here is that u is equal to v cos theta and v is equal to v sine theta right and as we have of course, then something that we can see from here as well. So, if I do that then coming from this particular expression over here, if I look at this right and if I can write this as v cos theta and v sine theta and in here of course, we have the sine mu is 1 by m if I do that. So, from this d y by d x can be written d y by and d d ex right.

This I can write actually. So, therefore, d y by d x this can be written as tan of theta minus plus mu. So, this is the Mach angle. So, now, a question we will we will sort of discuss this a little bit tomorrow. So, the question is are these lines curved or are these lines straight what should they be at least we should know. So, this characteristics is essentially. So, are these curved I have sort of drawn them up curved nothing tell you this has to be a straight line. So, I drew this curve. So, we will just discuss that a little bit or you can kind of see what is going to what is happening over here. So, this is the essentially that and. So, here now if you see d v by d mu d u sorry d v by d u if you if you see from here, if you look at this here in this case. So, what do you with respect to the you know diagram that we have done here we can write this also as d theta.

So, what essentially is that if I take say this is d v right. So, if I do that. So, this is going to be my. So, if I take just a small element out there. So, therefore, this d theta, this is also a. So, this is equal to minus plus and the of course, the negative sign the negative here it corresponds to the c minus characteristics the left running wave and plus correspond to c plus correspondingly. So, what you see over here do you even recognize this equation at all, but this is if you remember this is exactly the same as the Prandtl Meyer equation right this was this is something that we derived for the Prandtl Meyer expansion wave this is on this is exactly equal to that. So, this is exactly similar to that.

So, therefore, what we can write this as if I now integrate this? Therefore, the Prandtl Meyer compatibility equations; so in this case the algebraic compatibility equations we can write them as like we have done before I guess ok.

(Refer Slide Time: 50:49)



So, I am just looking for a little bit of space here. So, let me say if I do that. So, it will be.

This is say k minus or something as theta minus oh sorry that that is correct. So, we have a minus plus. So, this is on the c minus characteristics. So, essentially this corresponds to the left running wave and this corresponds to this. Therefore, these are the compatibility equations for this particular case for a left running wave and a right running wave and this is similar, this is similar to the Raman invariance which we did in the previous lecture this is the we did j plus and j minus. So, this is analogous to that.

So, essentially you know what we will do is we will kind of discuss this n by d thing a little more tomorrow. Now that we have done the math we will go over and understand that little more in the next lecture.

Thanks.