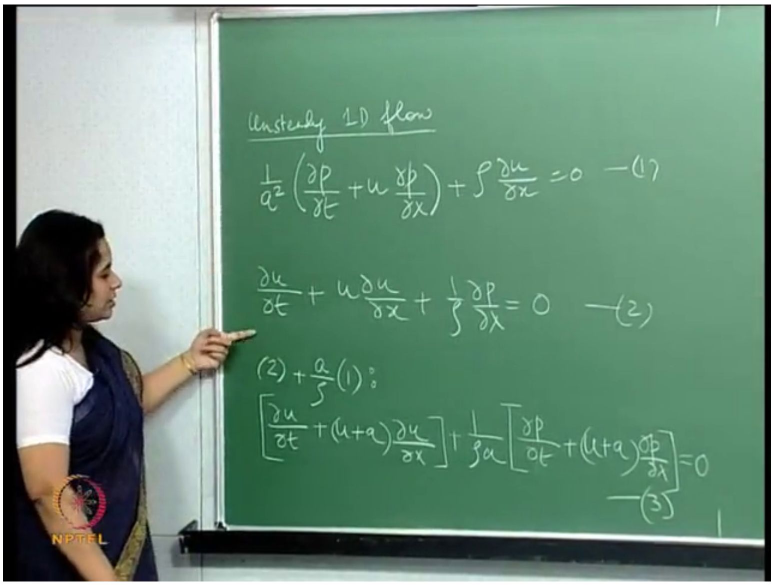
Advanced Gas Dynamics
Dr. Rinku Mukherjee
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture – 18
Finite Wave Theory: An introduction to the Method of Characteristics

So, we were on finite waves, right. So, we said that for the unsteady 1D flow the continuity equation and momentum equations. So, let us start from there. So, what we had was the continuity equation for unsteady right, for unsteady 1D flow we got this the last lectures. So, we wrote it out to be. So, let us call this as say 1.

(Refer Slide Time: 00:29)

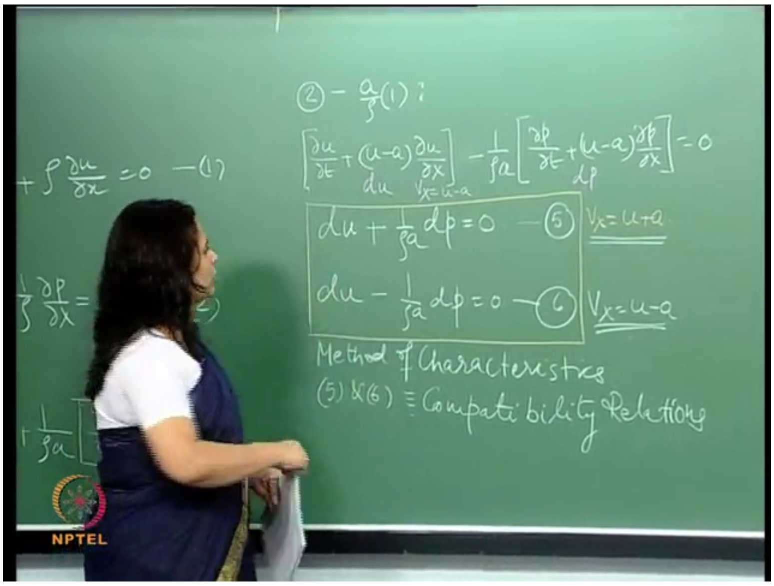


And this is from the continuity and from the momentum conservation what we got was this. So, this is the other the momentum conservation equation. Now what we going to do here is basically what we tend to do always which is play around with these equations right and see what if we can get any more information from this. So, our main task out here is basically to solve right, to solve for the pressure the velocity etcetera right and in order to do that we sort of going to play around with these equations.

So, now let us do this. So, we will do two things over here. So, let us take the second equation and add to it. So, what will we get in this case if we do that? So, what we are doing is multiplying this equation first equation by a by rho, a by rho and adding it to the second equation. So, what we get is this you can sort of do it yourself once to see whether this is correct or not. So, then what we get is this right this is what we get let us call this equation as 3.

So, this is what I did. So, I multiplied the first equation with a by rho and added that to this. So, I end up over this. So, now, the next thing is let us do this let us take, the second equation and subtract from it this, a by rho into the first equation right and if I do that then I get this. So, this is what we get.

(Refer Slide Time: 03:33)



Now let us look at this equation over here rather this, this part of the equation over here. So, if you see that this is essentially. So, del u del t plus u plus a del u del x right, if you remember the concept of the derivative right or the total derivative, so this is essentially the instantaneous velocity the instantaneous acceleration right and this is the convective term right. So, this is happening because of the change at each location over time and this is because of the movement of the particle from one place to the other therefore, that is the convective term.

So, therefore, this term I can actually write it as du is not it. So, this is du when of course, the velocity with which it is moving let us call that as v x is v plus a and the it is similar here. So, this is the change in the, this is the local change in the pressure and this is the convective change in the pressure. So, we can also write this as dp right. So, therefore, we can write equation 3 in this form, we can write equation 3 in this form right.

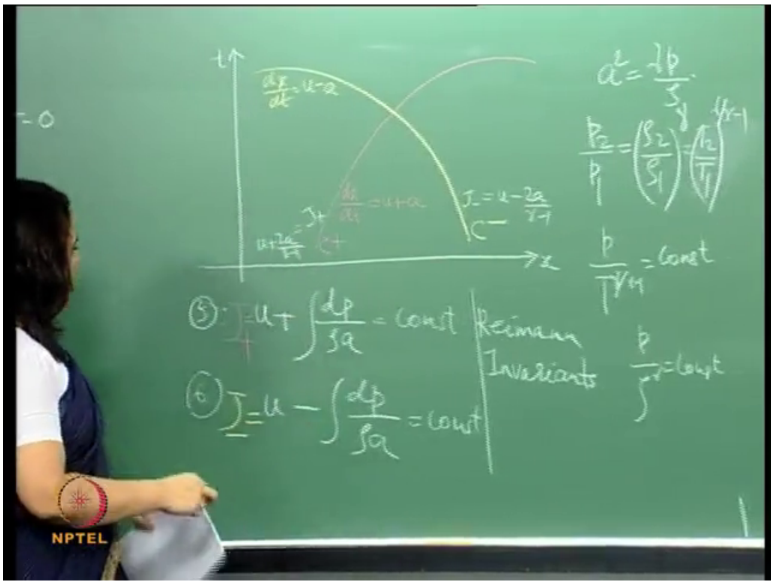
So, if I do that, if I sort of do that let me call this as equation say 5, now similarly if you look at this equation here now let us just note here in this case v x is u plus a. So, now, if you look at this equation similarly, so again this also we can say this is the local velocity change and this is due to the a convection, and this also a we can say as du there is; however, a difference and this is dp difference; however, here is that the velocity with which the particle is moving is u minus a right.

So, again for that case we can write du minus one by rho a dp is equal to 0 let us call that a 6, in this case we access u minus a right. So, we can write that. So, therefore, what is this exactly mean, what is this exactly mean? And you have a Mach wave right if you have a Mach wave then if the particle is moving in the direction of the Mach wave then you have this equation right and if it is moving in the direction which is opposite to the Mach wave then you have this.

So, that is what this equation essentially means now if I now these 2 equations actually, these 2 equations 5 and 6. Now these 2 equations actually they form the basis of a very powerful method and especially compressible flow which is the method of characteristics, we will sort of begin that. So, this is these two equations of the basis of the method of characteristics and these two relations are called the compatibility relations and these 5 and 6 are the compatibility relations.

Now we will see why we call it method of characteristics. Now what we will see over here is that the solution to this, solution to these equations essentially lie on 2 lines, lines. So, if the solution to this lies say u plus a. So, essentially if I draw the dx dt plot if I plot this.

(Refer Slide Time: 09:05)



Now if you kind of recall what we did last time. So, this is the solution to this lies on a certain line and in that line the particle velocity is what we are trying to plot in that xt xt diagram now in this case we know that the particle is moving in the direction of the Mach wave right. So, dx dt is positive.

So, therefore, if I have say this is an arbitrary curve whether it is going be a straight line or a curve we do not know. So, just for conjecture said at this point of time we are going to draw a curve. So, here, this is my dx dt right the dx dt for this curve is u plus a this is the line. And for this one this lies on another line which is moving opposite to the Mach wave. So, in that case we will have something like this say this line and for this case we have, now these are the lines which is which are call as the characteristics these lines are call characteristics and this.

So, therefore, this is going to be called a C plus characteristic for a right running wave and this one is going to be call C minus, well C minus for the characteristic wave. So, essentially you know this is the basis of the method of characteristics. So, what we will. So, now, that we have sort of got an ideas to what that is, let us go ahead and do some a Mach and see how we were going to actually use these characteristics if at all we are going to use these to solve you know some practical problems namely the shock tube expansion wave.

So, now, if you look at this what we going to do and here is integrate these equations 5 and 6 and let us see what we get. So, if I integrate this if I say integrate equation 5. So, what we going to get is right. So, this is what we going to get and let us call this as. So, this is what we get from 5 and from 6 what we get is again this is a constant ok.

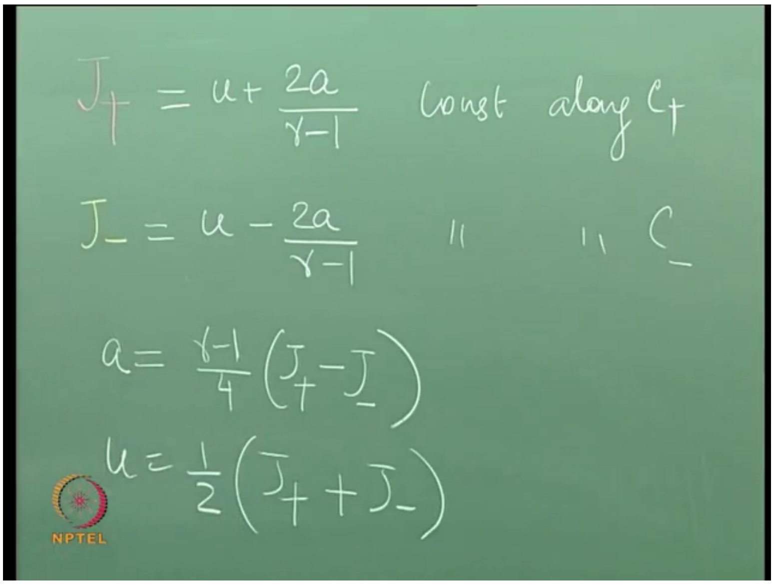
Now, let us call this as J plus let us call this as J plus and let us call this as J minus. We are going to call it this. So, essentially you can see what we are doing. So, this integral this J plus is actually constant over this C plus characteristic is not it and the J minus which we got from integrating equation 6 is this is the integral which is constant over the C minus characteristics. Now let us go and let us try and integrate this part how are we going to do that if can we integrate this.

Now, without going into the, I am not going to do right do all this detailed integration here, but I usually tend to give out my notes to students if they ask for it. So, I have it all sort of done in detail over here. So, if you want to look at it you can you can ask me for it I will scan you and send it or you can do it yourself, I think it is not too I will just give you brief ideas to what are the relations I used to integrate this right. You can do it yourself, but if you cannot then you can you know take a look at my notes ok.

So, what we will do here is use these relationships before I do that now this J plus and J minus have another name they are called basically Riemann, Riemann invariants. So, these are called, so said these are basically the Riemann invariance J plus and J minus. So, now, say now for a calorically perfect gas we know this is true and this is isentropic right. So, using the isentropic relationship which is p 2 by p 1 and so forth, so what we can write is. So, let me write this. So, p 2 by p 1 is rho 2 by rho 1 to the power gamma is equal to this is equal to t 2 by t 1 gamma by gamma minus 1 right. So, gamma by gamma minus 1 right this is what you get. So, therefore, from here what we can also get is p by t to the power gamma by gamma minus 1 is a constant and p by rho to the power is equal to constant ok.

So, now if you look here, if you look here, a you can represent from a is rho p by gamma then rho you can you can get dp and you can get rho. So, then you should be able to integrate this. Now if I do that if I go ahead using these relations if I integrate this, what I get is this what I get is this. So, basically then what happens is that, so, then J plus now J plus becomes right this and J minus right.

(Refer Slide Time: 15:59)



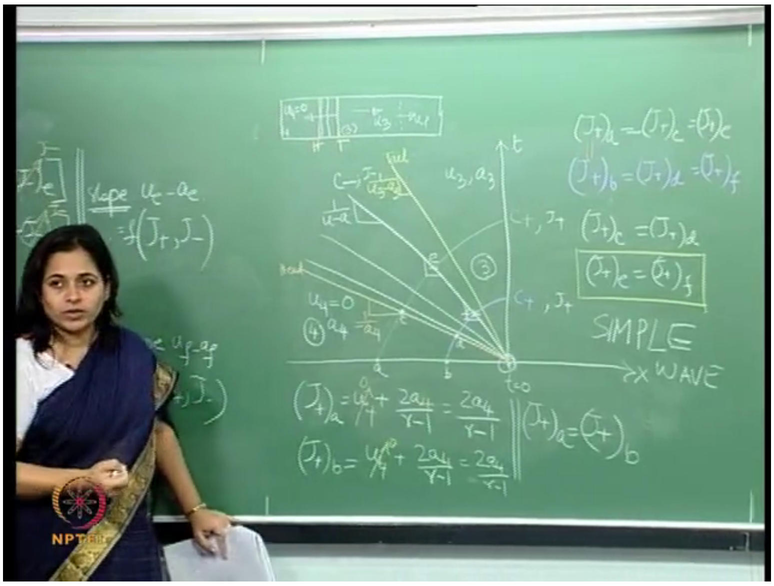
So, needless to say that this is constant, so this is a constant along C plus right and this is a constant along C minus. So, what we have essentially been able to do here just sort of take a step back. So, we got these compatibility relationships, we got these compatibility relationships combining our governing equations for a finite wave now when we do that and then we see that on integrating these two compatibility equations we get these two Riemann invariants and we see that this can be further simplified into this and what this is telling us that on a right running way when I write running characteristic C plus.

So, we have J plus in here we have J plus and on a left running characteristics we have a J minus and J minus is nothing, but, and this is constant on a particular C minus this term a combination of the velocity and the speed of sound is constant on a C plus C minus characteristic and similarly here this is u plus 2 a by gamma minus 1. So, essentially what we see is that the solution right to the compatibility equations lie on certain lines which are known as the characteristics and this is the new information that we are getting now.

Now, we have to see that how we are actually going to use this, how this is going to be put to use. So, therefore, now if you look at this now if I come back and again look at this I can write here that a from these two equations is essentially, so a is gamma minus 1 by 4 J plus minus J minus if you if you sort of you can you know cross check this. So, and u is J plus plus J minus. So, I am basically using the writing the velocity sound and the local velocity in terms of the Riemann invariance all right.

So, now let us do this. So, now, that we have sort of found out a way in which we can you know evaluate a finite wave let us go and see how we going to apply this for example, to the expansion fan in the shock tube all right. So, to do that let us, let us go back here. So, this was our shock tube.

(Refer Slide Time: 20:05)



This was our shock tube now we have done then we have when we break that, shock tube is essentially what its basically two chambers in which you can have two different gases or the same gas which are under different conditions. They are separated by a diaphragm which we break by a mechanical means or electrical mean at some point of time and what that does it sends a shock wave a travelling to the right of it which is moving into christen flow and we were able to calculate the properties of the a fluid once a shock wave moves behind and in front of it and that we have done. What was left to do is when we break the diaphragm there is a an expansion wave which travels to the left of it which travels basically in the direction opposite to the shock wave and what we left to do is find out the properties in front and behind of this expansion wave.

So, this is what we have. So, if you this was the contact surface which is moving again to the a right and we had an expansion fan right say something like that and that was moving into this region. So, this region was say 4 and this region is 3. So, this is a and this is and sort of breaking that vision out. So, therefore in here this is say at time is there a nonzero time at some instant of time when we break it then this is what happens. And what essentially happens is that you have the shock wave which moves to the right and that induces motion in the fluid which is behind it.

So, we will have a velocity in this as well let us call that as a u 3, now this is a contact surface which moves you know to the right basically following the shockwave. So, now, that we have this let us draw an xt diagram for this, let us draw an xt diagram for this. Now what you can see over here is that this expansion of f right all of it originates from one point and then it proceeds is not it because at the time when there was the diaphragm was present there was no show there is no expansion fan and then as we break the diaphragm right then all these waves they basically originate at one particular point and then they move forward from it.

So, therefore, at the time 0 at time t is equal to 0 that the wave is essentially not there, so just about at time t is equal to 0 yes the wave is situated here all the waves are essentially situated over here and then they move from x is equal to 0. So, that is what this origin here means. This is essentially the location at where the waves originate all right. Now if I do that. So, let us call this.

So, this we can call say as the head, let us call this as the head and this was the tail of the this is the head and tail of the expansion fan. So, in an xt diagram what are we going to do here basically the velocity is not, is not it. So, xt diagram. So, here say this is or say let us let us do this. So, say this is the head this is the head of it and this is the tail of it. So, if you look here. So, say this is the tail and this is the head. So, this is the head and that is a tail and then in between of course, we have several waves so on and so forth. So, what we see over here if you look at this picture. So, essentially we have several waves moving into the region 4 here ok.

Now, u 4 of course, is quiescent is not it u 4 of course, is quiescent. So, therefore, now we have several ways moving into this. Now at t is equal to 0 all the ways originate from here and then they move right and in this case we are taking this you can see I am taking x 2 in the negative direction. So, that is the reference frame here it is moving to my left and so they originate from here and then they move into this region 4. So, therefore, this is this is the expansion fan and this corresponds to this over here.

So, to the left of it is moving into region 4. So, this is actually my region 4 and this is my region 3 is not that right, so this is it. So, therefore, in here of course, now let us write down. So, in u 3 region which is corresponding to this on the xt diagram we have essentially velocity is u 3 and speed of sound is say is a 3 right. Now in here what is u, so we have velocity u 4 this is quiescent. So, this is 0 and a 4 is, a 4 is yeah so that is the velocity of speed of sound.

So, this is essentially my region. So, therefore, if you remember now if you what we did just now right what we did just now here if you look at this, this is the xt diagram right. So, we said on a left running characteristic. So, now, on a left xt diagram on a left running characteristic we have the slope is dx by dt is u minus a is not it, which is that that the particle is moving away from the wave and it is moving with the velocity basically u minus a right, so dx by dt. So, now, these say are the characteristics. So, if you consider these are C minus characteristics right because the particle for a left running for left running wave right.

So, let us say these are say C minus characteristics, these are all C minus characteristics on which this is the slope is not it. So, the slope is u minus a, so this is u minus a. So, this is essentially dx by dt there is a slope of this characteristic right. So, if I look at that then in that case what is the slope of the head of the expansion wave, what is the slope of that because u 4 is 0 right. Similarly what is the slope out here, so this is, is not that right. So, that is what we have from here.

Now, what should be asking at this point? So, I say that for this is my expansion fan and I am drawing the characteristics because why am I doing this, what am I doing this here basically what I need is the properties across the expansion fan right and how am I going to do that. Well I say that these are my governing equations right these are my governing equations of the of an expansion fan and what I have found out that the governing equation solution to that especially solution to that lie on lines, lines which are known as characteristics and these characteristics for a left running for a left running characteristic. So, I am basically plotting that on x by t, I am doing an x by xt plot for that and on these lines which are essentially the C minus characteristics. So, these are the slopes these are the slopes.

So, in this case what is these are left running because the particle is moving in the direction opposite to the actual wave now if this is a. So, then now the question to ask is what is the difference between this plot and this plot. When I did this and I said the solution to the governing equations or to the compatibility equations basically lie on lines and we are not specify whether this has to be a curve or it has to be a straight line these are just lines. So, therefore, we did we do these arbitrary curves. Here how I have drawn straight lines I have drawn straight lines and I am calling them as C minus characteristics. So, I think first thing we need to do is proof, let us prove that it is indeed a straight line it is indeed a straight line. So, in order to do that say I will draw for in this case I will draw to the C plus characteristics, if I have to draw C plus characteristics so say is there anything else here that that is it.

Now, let us do this let us draw to say C plus characteristics. So, these are C plus characteristics. So, I have that and let me draw one more, these are also C plus characteristics. Now let we going to call these you know we call these location something let us call this as a b. So, then here on the head on this characteristics this is c and this is d and anywhere in the middle say this here anywhere in the middle this is e and this is f. So, what essentially we saying is that we drawing this C plus characteristics.

So, what we going to try and prove is that these lines are indeed these characteristics that indeed straight a straight for this particular kind of expansion fan. So, what I have done here is drawn the C plus characteristics and what I am doing here is essentially you know these are in intersecting the C minus characteristics. So, therefore, I am calling these I am specifically locating those points and calling these as you know giving them some names.

Now, let us see now in this case. So, what we know is that yeah these are things that we know right now. So, if I go to say in region 4 if I go to in region 4. So, let us say how will we calculate say now J plus is constant on a C plus C plus characteristic, is not it. So, these are C plus which are associated with J plus and this is J plus and J minus is on a corresponding to, so the C minus characteristics and J minus is constant on C minus characteristics.

Now, let us calculate J plus for a. So, this a point is in is in the region 4 is not it. So, in here this is this is essentially what. So, now, this is the J plus, in the a region so this becomes is not it and this u 4 is 0. So, what we end up getting here is that fine. Now let us calculate J plus of b if I calculate J plus of b what I get again is u plus 2 a by gamma minus 1. Again this is a and this is a, this goes to 0 right, so this is again to a 4 gamma minus what. So, you can clearly see that J plus a is equal to J plus b.

And let us just stop here for a moment and think about this, this may not be true for every possible case what we are doing is we are taking the characteristics and looking at it for this particular case. Now what you see is that this is a region right now this wave is moving into this wave is moving into this quiescent flow this u 4 out here and they are all originating at the same point ok.

Now, think about this now if this did not originate at the same point, if I had another characteristic this head or this head that I have drawn over here if this did not originate at this point t then a and b would most probably be different. So, this is so, when I say J plus a is equal to J plus b this is not for every possible application it is just for this particular case. So, we are applying this you know concept of methodology only for this particular case just want to emphasize that.

So, therefore, this is my u 4 region. I independently drew to C plus characteristics and I am then I and I basically for t x is equal to you know on this x line where t is equal to 0 I am trying to calculate J plus and J plus a and b and when I do that the a and b are both in this region in this quiescent reason where u 4 is 0 and what I see is that J plus a is equal to J plus b. So, J plus a is equal to J plus b right.

Now, J and now let us look at thus this, this C plus characteristic. So, this green the green C plus characteristics and we know that J plus is a constant over a C plus characteristic. So, based on that what we can write here is that J plus a right now this is constant. So, J plus a either along the same lines. So, C lines or C lies this point C lies on this green characteristic. So, therefore, this is also equal to J plus C this is also equal to J plus e is not that true. So, essentially what you are saying is on each C plus characteristic has the same is associated with the same, with a particular value of J plus. In this particular case I have found out J plus at this point a which lies on C plus characteristics right now though therefore, J plus a is equal to J plus C J plus e which is equal to this value which we have found out.

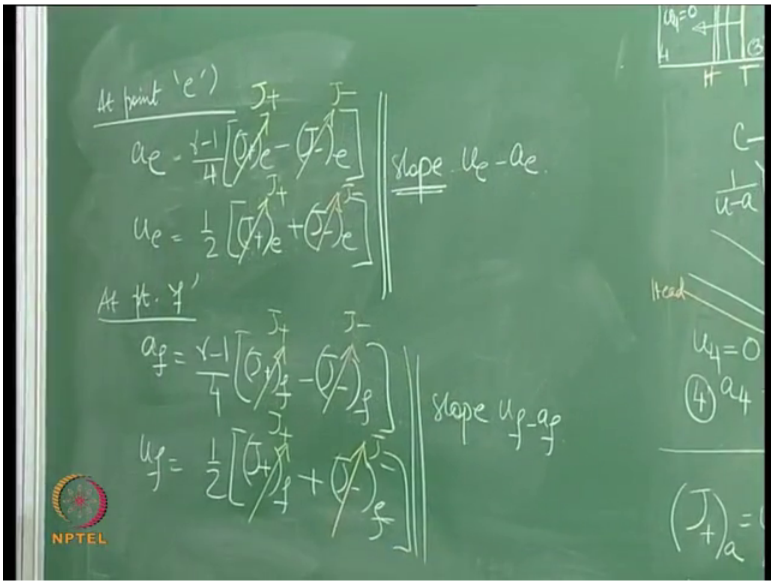
So, similarly when I look at this purpose C plus characteristic what I get let me write that. So, what I get is J plus b is J plus d f. So, what I am doing is just going at different points of this C plus and I am I have on all these points b d and f they all lie on the same C plus characteristic and hence they all have the same J plus which incidentally we were able to calculate which is equal to J plus of b and which is equal to this, is not it.

Now again since we know that these two are actually equal to each other is not it if that is true then what we get is if that is true therefore, these 2 are also equal is not it. So, if a is equal to b, therefore, from here we can also write the J plus C is equal to J plus d and J plus e is equal to J plus f is not that true. So, now, these two are equal to each other then we move out here now the reason we kind of are interested by when we say J plus C is equal to C J plus d because we see that this the these two points also lie on a C minus characteristic they also lie in the C minus characteristic and therefore, that is important

So, therefore, what we are saying is these two points, these two locations C and d then they are also they also lie on a C minus characteristic they lie on the same C minus characteristics, but they lie on 2 different C plus characteristics right and what we were able to see here is that therefore, this particular case J plus of C is equal to J plus of d right and similarly J plus of e is equal to J plus of f.

So, now let us look at e and then let us look at e and f because that is like for any a particular characteristic within the wave. If I look at e and a now what we will do is in this particular case if you look at those two points, now as we did we shall be able to calculate the velocity of sound and the local speed using J plus and J minus is not it. So, now, if I look at say at the point e if I look at the point e which is lying on this C minus characteristic, so then say at the point e, so I can write the velocity is not it. So, now, at the point e I know J plus e and I know J minus e is not it.

(Refer Slide Time: 42:03)



So, therefore I can write this and u e is half of J plus e plus J minus e which we have sort of a written earlier yeah I think this should be a minus yes this is a minus. So, I got that wrong here. And at the point f similarly, what we have is a f is gamma minus 1 by 4 again J plus of f minus J minus of f and u f is half J plus of f plus J minus of f.

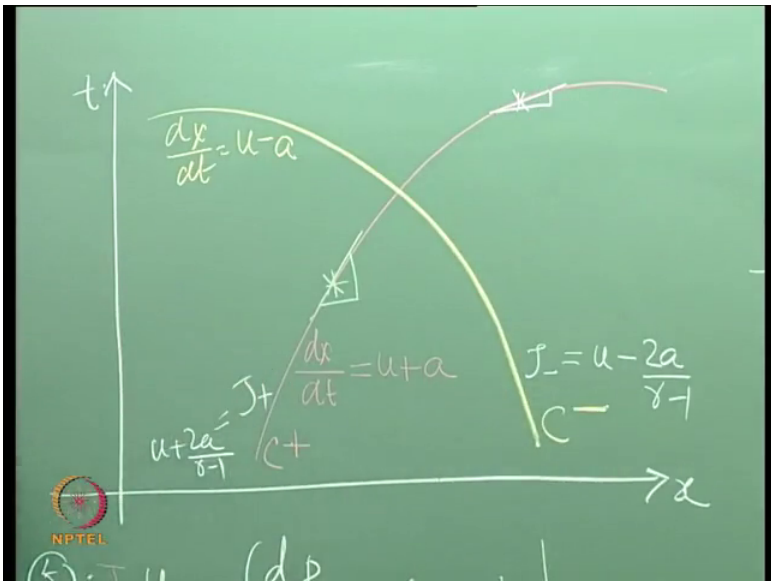
So, now what is the slope, what is the slope of the characteristic at the point e? So, what is the slope here, what is the slope here at the point e on C minus characteristics? It is this right, u minus a. So, I can write here the slope here is u e minus a e right. If I do this like u basically I have u e minus a e and similarly here you see what I am doing. So, then slope is u f minus a f.

So, what I have done essentially is taken two points I have taken two arbitrary points on this C minus characteristics right C minus characteristic which lies anywhere in between this which represent any part of the expansion fan and what I am doing here is calculating the slope here and calculating the slope here ok. So using the information that we can get from the characteristic, when I write this here slope is u e minus a e here and slope here is u f minus a f now what we have seen so far if we go back is that J plus e J plus e is equal to J plus f. So, in here, let us say in here, this J plus e is equal to J plus f. So, why do not we write this as say let us just write this as say J plus. So, we will just write this as, we will just write this as J plus again we will write this as J plus we will write this as J plus ok.

Now, this e and f they lie on the same C minus characteristics. So, this will have the same J minus is not it. So, therefore, this we can write as say J minus this is also J minus this is J minus and this is J minus. So, therefore, what happens to u e minus a e if you look at this. So, this is essentially so I am going to just write it like this function of J plus and J minus similarly here this is also f function of essentially these are same.

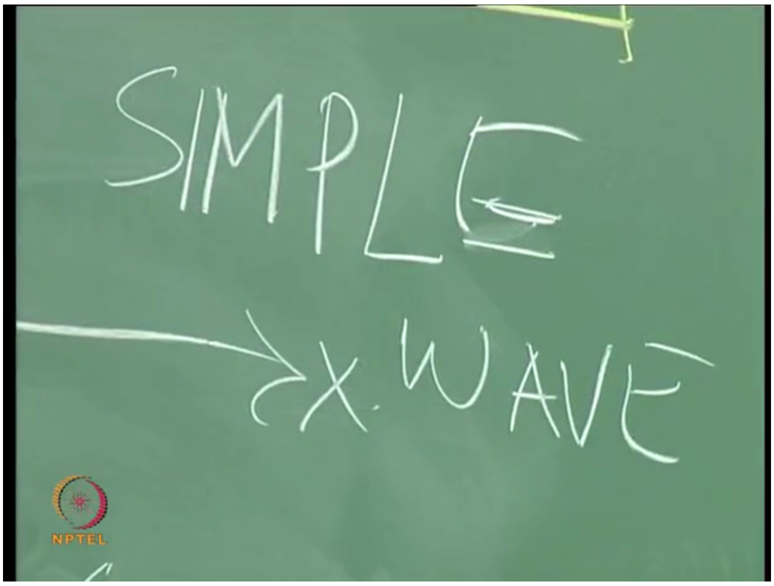
Now, so therefore, the slope at two points, at the slope at two arbitrary points right of this C minus characteristic is same and therefore, we can say that this is actually a straight line. Now I could take this e and f or more points or I can take more than just two points because these are arbitrary points. So, what we were able to prove is that this slope here is the same as the slope here and therefore, for example, if I go back here say now for example, look at this point let us say I have a point over here in that case the slope here will be something like this is not it and say I go to a point here in this case the slope will be something like this and you can clearly see that the slopes are not same right.

(Refer Slide Time: 47:39)



However if you look here if the slopes are same which is what we get hence this is a straight line. So, this is a reason, so therefore, this C minus characteristics therefore, or characteristics usually they become a straight lines when the wave here is moving into a quiescent flow and therefore, this is usually called, it is called a simple wave all right.

(Refer Slide Time: 48:30)



So, simple wave and also it is also called a centered wave because it is a simple wave because it is moving into quiescent flow as a result of which all the characteristics are straight lines and also it is a centered wave because the waves are generating and a single point and then they move away from there all right.

So, now that we figured that out, so we have been in the sort of we have in the process of trying to figure out the properties you know across say an expansion fan in the shock tube. So, what we have you know we have still have not done is the actual relation between the properties which we will do right after this. Now that we have proved that we basically have a simple wave which is centered at t is equal to 0. So, and these characteristics are a straight lines. So, we will continue with this and get a relation between the properties of the waves and then we will see how the you know properties vary when you know such a wave starts traveling in the shock tube.

Thank you.