Advanced Gas Dynamics
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Lecture – 13
Area - Mach Relationship

So now, we have been talking about shock waves expansion fan etcetera, for some time now right. Now going back to now all the experiment where he basically you know first steam through a certain type of a duct converging diverging now. So, and he was able to get supersonic flow. Now we have an done anything to that respect, right. We have not really said that how if I you know what exactly is a relationship, has to how much there should be an area change so that we can get a certain you know change in Mach number.

So, we will we will do that now and see some things interesting. Now let us consider this you know interesting thing. In the sense we have been talked about this before which is the quasi 1D flow. Now if you have a ducts like this, right.

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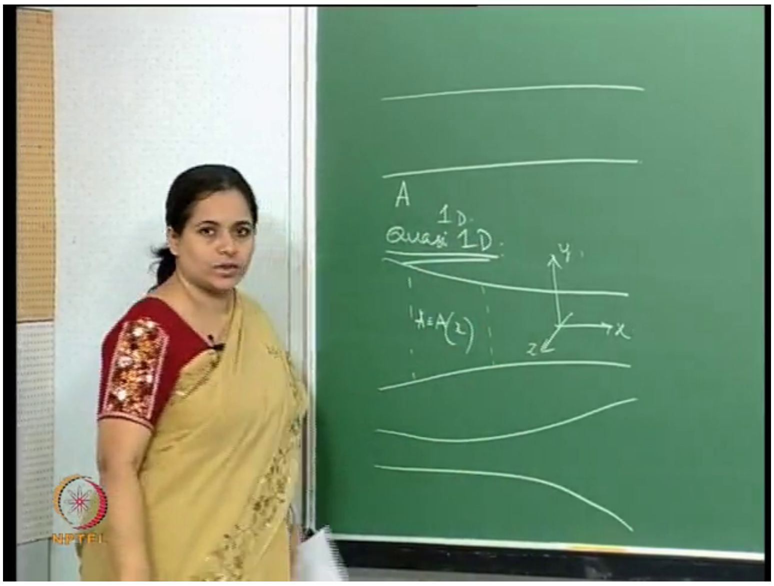

So, here this has an area a you know, and this is a constant area right. So, this is A 1D, you can say this is 1D, but what if you have something like this. You know, or if you have what if you have something like that right.

So, in this case as I think as we had said right. So, if you have a reference frame say like this. So, therefore, basically at every x location you have a different area. But that area it does not change in any other direction at that particular location therefore, this is called a quasi 1D flow. So, let us see what we can do with something like this. So, say we have you know a say, quasi say 1D flow right.

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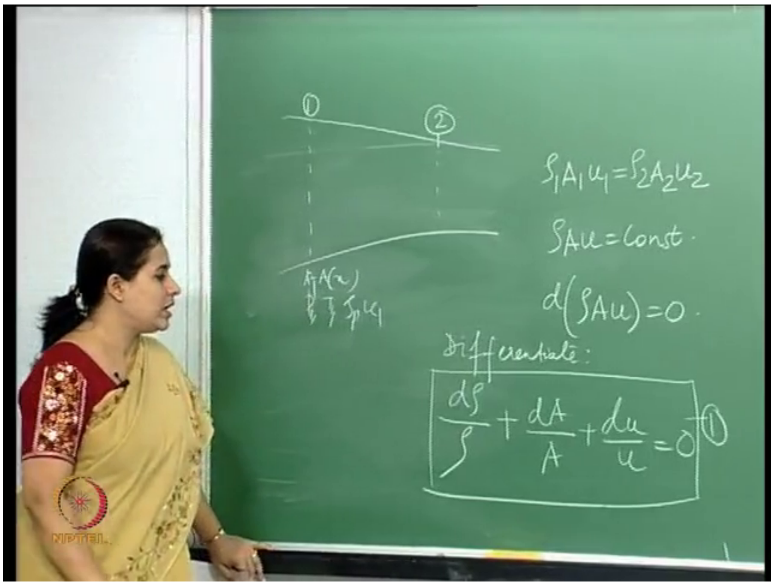

So, we here a is basically A x and you have you know pressure, temperature, density, velocity, etcetera, right. Now let us do is saying let us take a particular location here, and let us say there are other particular location over here right. So, then in that case we have P 1 T 1 rho 1 u 1 A 1 so on and so forth, right. Now we know from the equation of continuity that, right. Now this is from the equation of continuity right. So, therefore, what this means is that by which means that; rho a into use constant or it also means that, right; the receiver.

Now, if I have this now, let us differentiate whether going to going to the details of the let us just say high school map. I will just go ahead and do this. So, I am going to differentiate this. I am going to differentiate this. If I differentiate the above equation what I get is, right. So, we get you know if I differentiate, I will get this and let this let us call this as equation 1. So, we have a certain relationship between in density area and the velocity. So, a relationship between the changes in density area and velocity. So, if you change the area there is a corresponding change in density and velocity.

Now similarly from the momentum conservation equations we get this.

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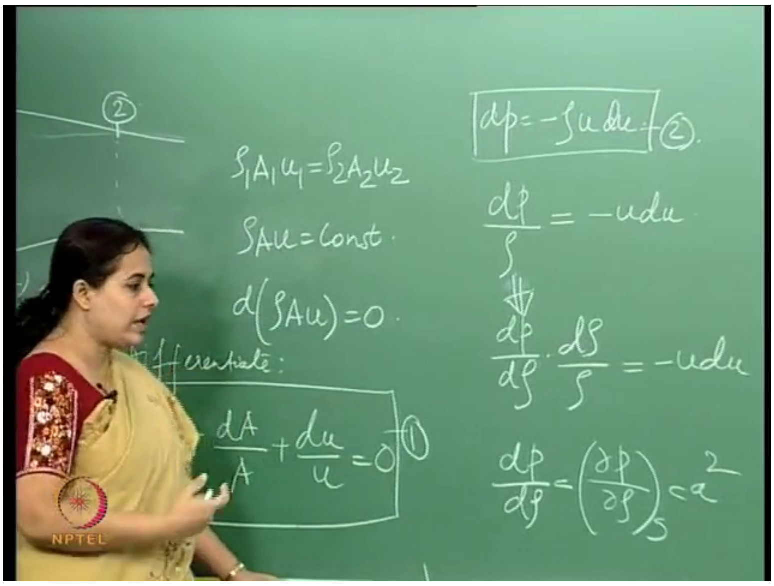

So, from the momentum conservation equation we get this. So, in this case, and let us call this as say the second expression. Now from here let us write it like this, right. This is equal to yeah, or this this term is something that we will write like this. This term say now this term is something that we will write like this, now d P so now, you will know where we are getting at. You see I just 2 is to write it like this. So, we get this from the momentum equation.

So, I just rewrite it is this form, and then this is something I write like this. Now d rho by rho is something that we get in this particular expression over here. Now if you remember d P d rho, right. Where a is the speed of sound, now this is something that we have sort of done before. So, the velocity of sound is related to d P by d rho, the change in density with the corresponding change in pressure, right. If done is before. So, what we can do is a replace this term d P by d rho by this a square right.

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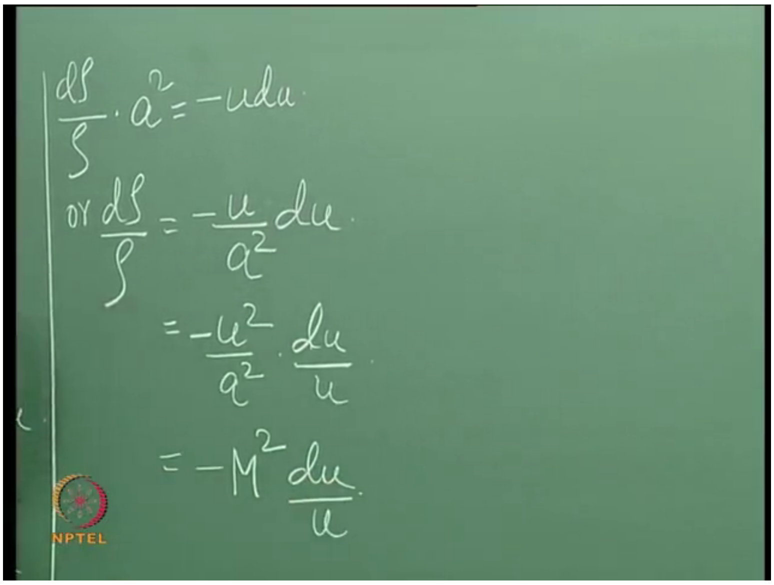

So, what we get is, right into a square is equal to minus mu into d u, or we can write this as d rho by rho is equal to minus u by a square into d u.

Again, this we can write as minus u square by a square into d u by u. So, this you can see we can write as M square d u by u. So, what we can do now is coming back to equation 1 here, we can write this d rho by rho, right in terms of Mach number.

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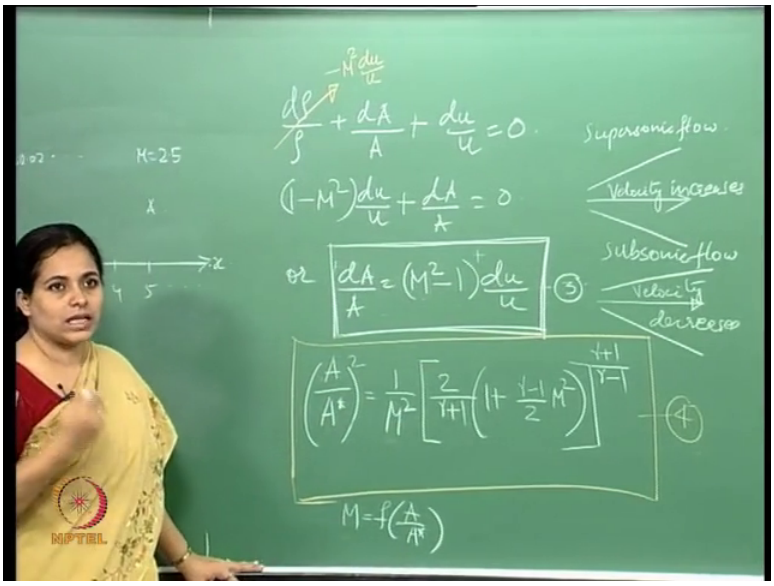

So, what we can do now, let us do this over here. So, let us sort of rewrite this. So, this is the equation we have. So, we have this d rho by rho plus dA by A, plus d u by u is equal to this, right. And we found our an expression for this.

Now, this is something that we can write as minus M square du by u, right. If you write that then, what do we get or we get is 1 minus M square d u by u plus dA by A is equal to 0. Or we can write this as dA by A is equal to M square minus 1 by u. Now let us so, what we have done here is basically using the same all governing equations, right. We use the continuity equation, and the momentum equation, and we related we connected the speed of sound, right. To those equations and what we have able to get, that for a given Mach number if you change if there is a small change in the area there is a corresponding to engine the velocity, right. Let us whole this equation d r and we have going to come back to this, and see what this physically means. So, let us call this this is basically the area velocity relationship.

So, let us call this say equation as 3. So, let us keep this and we have going to revisit this in a bit, all right. Now again let us go back to our continuity equation and do this do this tough. So now, let us say that, let us say so, say this is a location.

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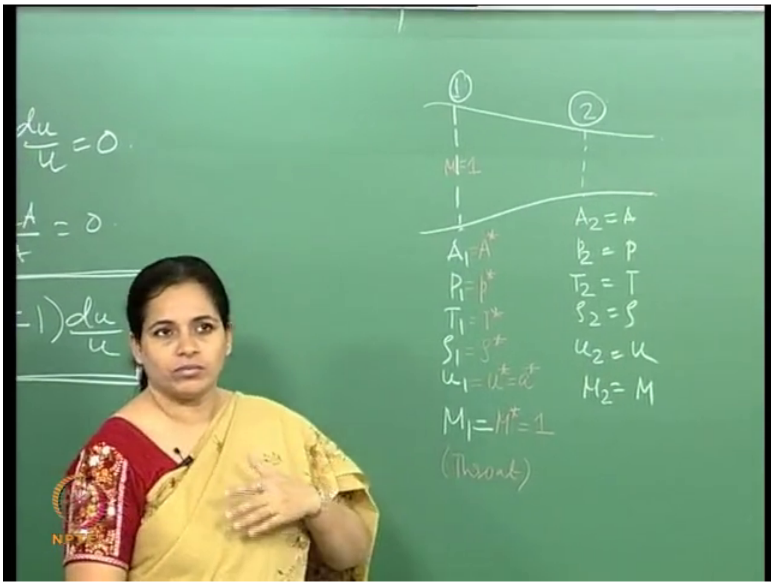

This is a location and this is another. So, this is location 1, and this is location 2. So, this is area, pressure, temperature, density, velocity and this is area. So, say A 1 P 1 T 1 rho 1 u 1, A 2 b 2 T 2 rho 2 u 2.

Now, let us consider this location 1 as a throat. If you remember what that is. So, if this is the location where the Mach number is equal to 1. And we denote this the throat by a subscript start, if you remember right. So, what this becomes here is A star, this becomes say P star, this becomes T star, this becomes rho star, and this is equal to u star which is equal to A star. So, that again M 1, right is equal to M star which is equal to 1. So, this is essentially my throat.

So, this this is my throat, and in here we are going to just call that a say A P T rho u and we say have will have M 2. So, that is equal to M. So, basically, we are looking at a particular location, and we are going to look at that location with respect to the throat. Now let us see if we get any special information from by studying this right. So, again so, we have this we again we were play around with our equations and you know the variables. So, again what we have is rho one. A 1 u 1 is rho 2 A 2 u 2, right. We have this.

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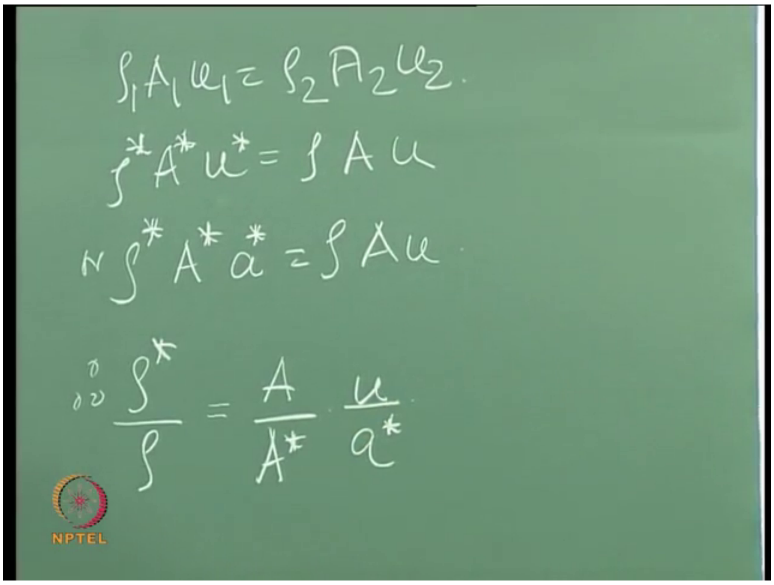

Now, what we are going to say is that one is the throat, isn't it? So, if one is the throat then we write this as rho star is A star u star, right. This is equal to rho a and u, isn't it? Now this u star again this of course, becomes so, then what I write this again or we can write this as rho star A star, that is the area, this is the speed of sound is equal to rho a and u. This is what we get right. So, therefore, we can write rho star by rho, right. This is equal to A by A star and u by so, that is equal to A by A star.

So, or let us write this.

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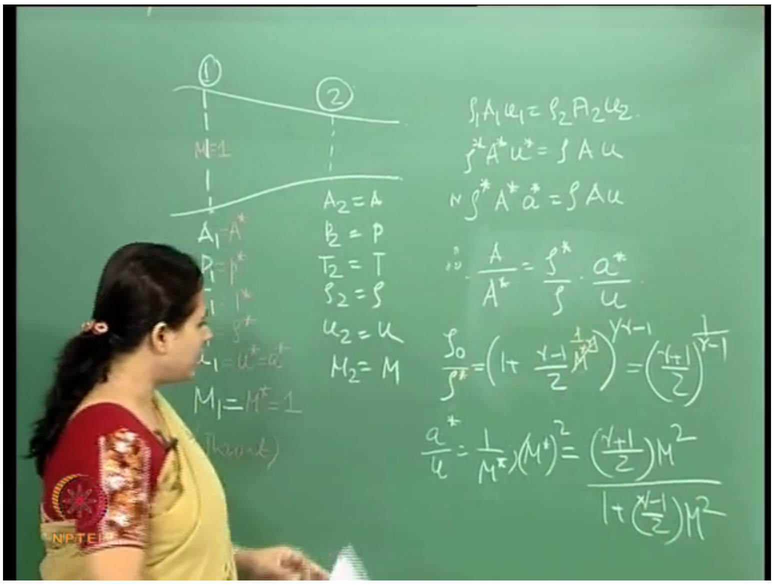

Say A by A star is rho star by rho into A star by u, right. We have this in here. Now again so, basically this is something that we get by applying the continuity equation between the throat and any other location. Now if you remember we have relationships like this. We have a talked about these relationships before, right. So, this is the stagnation density. So, we have similar relationships. So, stagnation density the ratio of stagnation density and density at any location is a function of the Mach number, right. And this gamma is given for a given gamma.

Now, this is something that we have developed before, isn't it? So, and this so, this Mach number corresponds to this density. In the sense the at a particular location, the local density is rho and the local Mach number is M. So, then the relationship of that local density and Mach number with the corresponding stagnation density is this. So, this and. Now, therefore, if I say the local density corresponds to the density at the throat. So, then this becomes star, isn't it? So, this becomes star and I can sort of write this. So, if I may write that, and this becomes M star in that case. And we know that M star is equal to 1, right.

M star is equal to 1. So, therefore, so this basically becomes 1. So, what we are left out with over here is this, right? So, what we get is gamma plus 1 by 2 raise to 1 by gamma minus 1. So, rho star by rho naught by rho star is basically just a finite number for a given gamma. So, therefore, you know so, yeah this is a. And then again here, now here you can clearly see that A star by u is equal to 1 by M star, isn't it? This is 1 by M star, and we also have this M star.

Now, M star we have seen from before that M star square is also related to the local Mach number as so. Right, the local Mach number is related to M star as so. Now we will do is so, therefore, what we will do is so, essentially write this in this form. So, we have an expression for this in terms of the Mach number, and how are you going to do this we will just going a play around with that and we are going to say that this is. So, this is equal to A star by u right.

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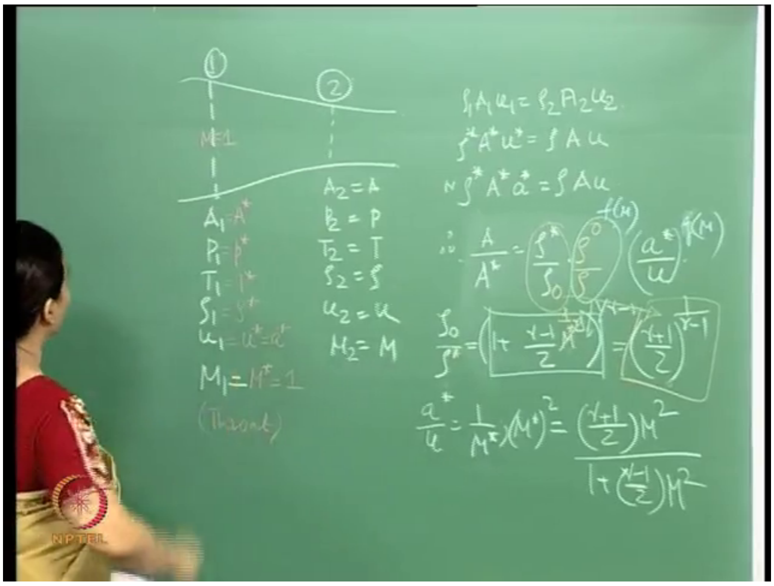

So, what we will do here is we will say, right. We get this so now, for rho naught by rho we have an expression in terms of M, and rho star by rho naught we have an expression which is equal to this.

So, therefore, we have an expression in terms of yes. So, rho naught by rho star rho naught by rho star is this right. So, this is a term which is equal to this. And rho naught by rho is equal to this term which we use originally. So, this is basically a function of the local Mach number, and this also is a function of the so, this is also a function of the local Mach number right.

So, if I write there out if I, in if I write out the equations in that fashion what I will get is this. We come back here and write it. So, then what I get is A by A star, right. This is equal to 1 by M square 2 by gamma plus 1. So, 1 plus gamma minus 1 by 2, M square to the power gamma plus 1 gamma minus 1.

So now let us say this is so, essentially what we did over here, now this is a very important relationship wherever interesting relationship; is that you can also see from here is that the local Mach number here. See the local Mach number the Mach number here is a function, right. Or the ratio is a function of the ratio of the local area to the area at the throat. So, this Mach number here this is the local Mach number, this is the function of the loc of the ratio of the local area and the of the ratio of the local area and the throat area, and what you can see over here is that for you know at the for a say a given Mach number you can get a for a say throat at the throat the Mach number is 1, right.

So, if I go along say a duct, along I go along a certain path, and I need say a certain Mach number. Say I need say a certain Mach number say I need say 2.5 mach. So, given that Mach number, you can relate the change in you can get the area required. So, you need to find out the total area change that will be required to go from say Mach is equal to 1 to Mach is equal to 2.5. So, there is what is given in this relationship. The mathematically speaking what this is doing is basically for a series of Mach numbers you will get series of ratios like this, right.

So, for say a given area the throat, you can get several you can get relations for this area. And you can see here that for a given area basically you have 2 values of Mach numbers. Over here now what I have done basically in here is that I have just plotted this. I have taken a series of let me let me just say what exactly I have done now you show that in a so, I said that I am going to say this is my x direction, and this is my y right.

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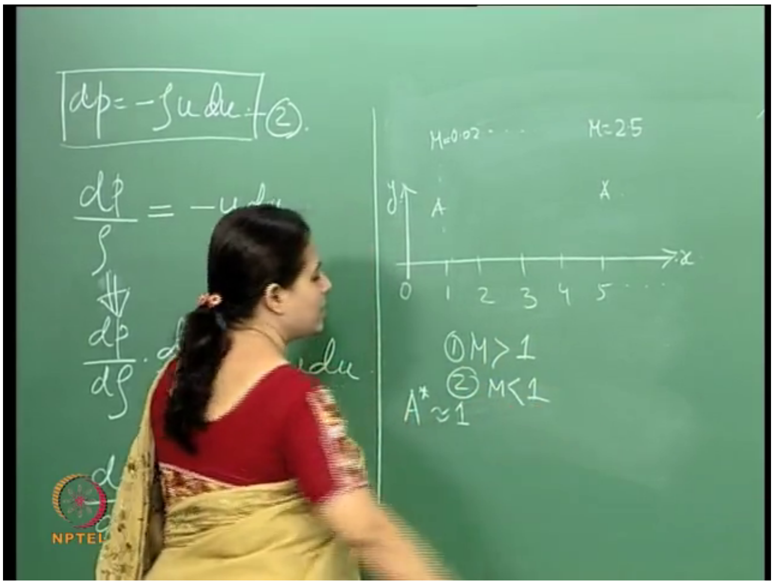

So, I will go along say this direction say from 1 to 2 to 3 to 4 to 5 and so on and so forth. I will go along x, and at each section I have a different Mach number right.

Now, this is a Mach number which I am going to give. So, see this is Mach 0 point say is 0 2, and then it goes up like that. So, say here Mach is equal to 2.5. So, as I move along say this distance, right. I am also I am going to change I am going to pass my I am going change the Mach number of my flow, and this is something then I am doing. And corresponding to this change in Mach number and calculating the area change, right. Area change, and let us just say for conjunct just sake that A star is equal to unity. So, then I can get. So, say area at the throat say area at the throat, say area at the throat this is just for a just case.

So, we will we will we will then be able to calculate the area at each location. So, area here and area here. So, what I am going to do here is what I have done is do 2 cases. One is Mach is supersonic, and the other case the flow is subsonic. So, we will do the these things. So, what I have done again is gone that the scilab and plotted this. And let us see what we get now we said that we will come back and talk about this is to what this physically means right. So, in this you can see that I am going to like a said here, I will go make Mach subsonic and supersonic.

So, if I do that this have a certain medical certain mathematical implication. So, we will come back and look at this. Now before that let us go and look at the scilab code here.

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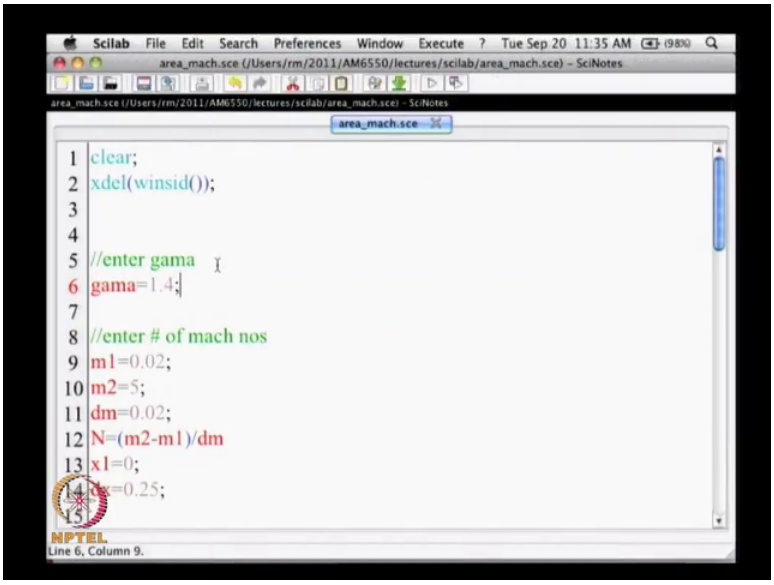

Now, what you see in this basically this is the code that I have write. So, what I am doing here is that.

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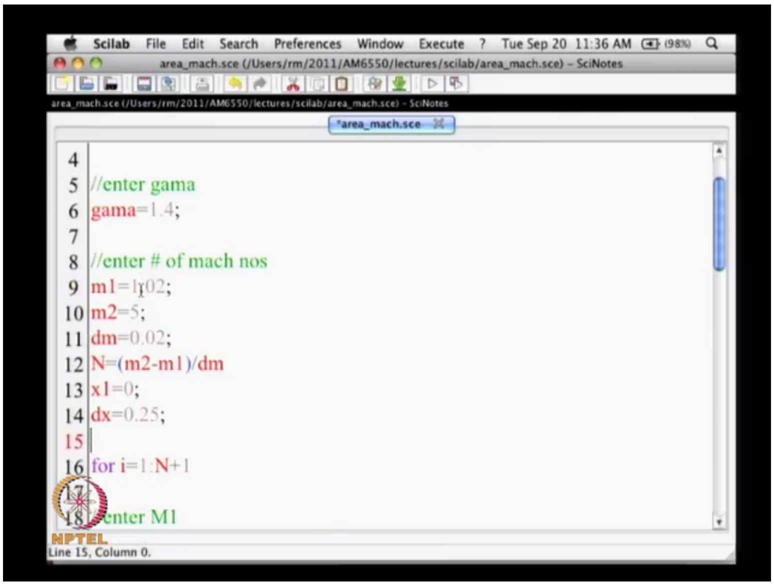

For a gamma is you can see it is a gamma 1.4 right. So, then if I come here. So, what I am going do is go from you know, do this for a supersonic flow right. So, I am going to go from say starting Mach 1.02, and go up to 5. And the Mach number increases by 0.02. And the locations this is something that this I have come up with.

So, locations I have basically go from say 0, and I increment by 0.25. So, at a every 0.25 location, I have a different Mach number, and what I am going to do is calculate the corresponding area change right.

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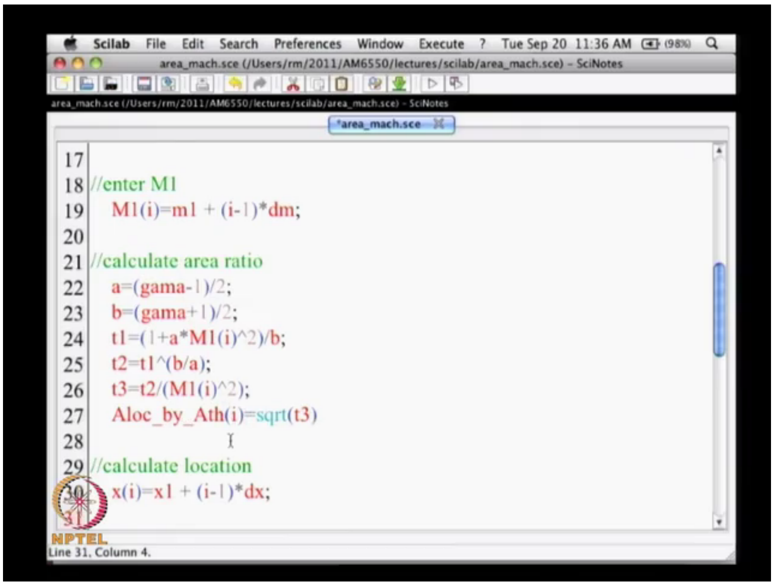

So, I enter the Mach number here, and then I calculate the area change. So, a location by a throat. So, we get that and hence you know we will. So, we will go ahead and do this.

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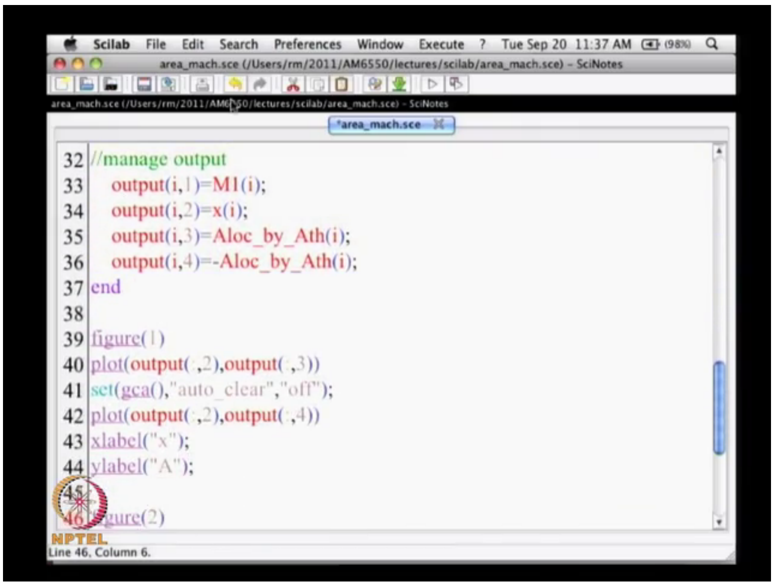

So, having done that, what I am going to do is run this.

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So, what you can see over here, that as I go this is this is my x direction, you can see this is my x, and the corresponding change in mach.

So, the Mach is you know, I am basically changing linearly in this case right. So, what you can see this is gone up to 5. So, if I may sort of a increase over here, now may what we will do that. So, I think if I do this then I should be able to this right. So, for some reason I am not able to do that. So, basically, we are going from 1.02 here, right. And I have not done any sonic condition that is what I wanted to show actually. So, if I do that so, then this is the increase in the Mach number.

Now, let us look at this. Now what you see over here is that this is the change in the Mach number, as you go from 0 to 50, this is along the distance.

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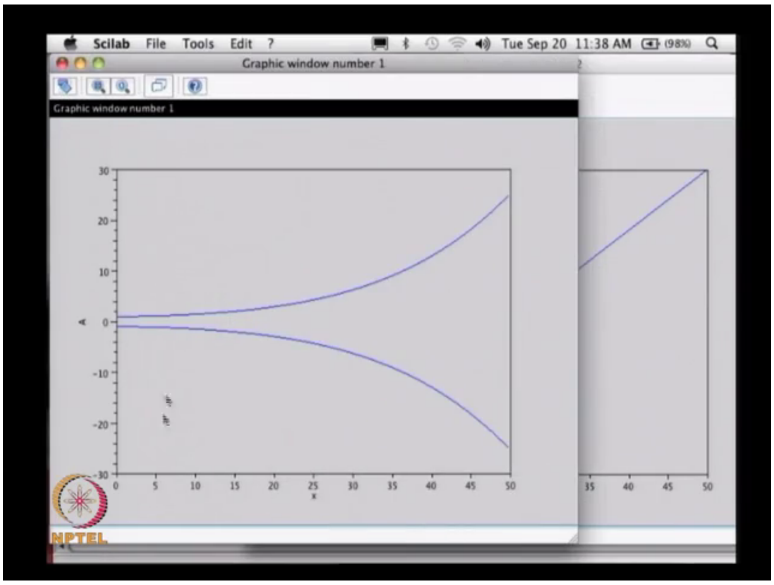

So, along x as you go in this case, right. What you have here is that as you increase the Mach number, the area has also increased, right. As the as you increase the Mach number the area has also increased, but this Mach number is the entire flow here is; however, supersonic right. So, for supersonic flow therefore, as I increase the area as I, an as I increase the Mach number the area is also increasing; which means that that the flow speed is increasing. So, this is supersonic here right.

So, for a increase in area there is a increase in the velocity. Let us go back to this equation over here. Just look at this if Mach number is positive, right. If this is a supersonic therefore, this is positive, which means for a positive change in the area there is a corresponding positive change in the velocity. So, which means therefore, a supersonic flow which means that for supersonic flow, right. With area increasing the velocity also increases. Now let us then go back, let us we run, this let is we run this. So, let us go say back over here, or say let us go here. So, let us go back to the code, and let us say we will run this only for subsonic cases. We will run this for subsonic cases.

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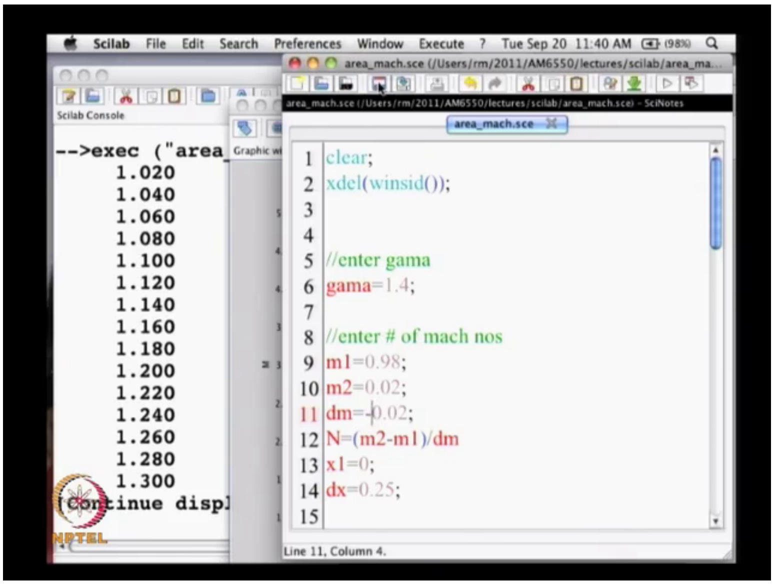

So, say this is running from just below, this and it goes to write and that decreases. So, this is what we get over here. So, we will go from you know the flow is subsonic in the entire region. So, if I do that, now if I do that. So, let us do this, we will run this again that scilab for you essentially right.

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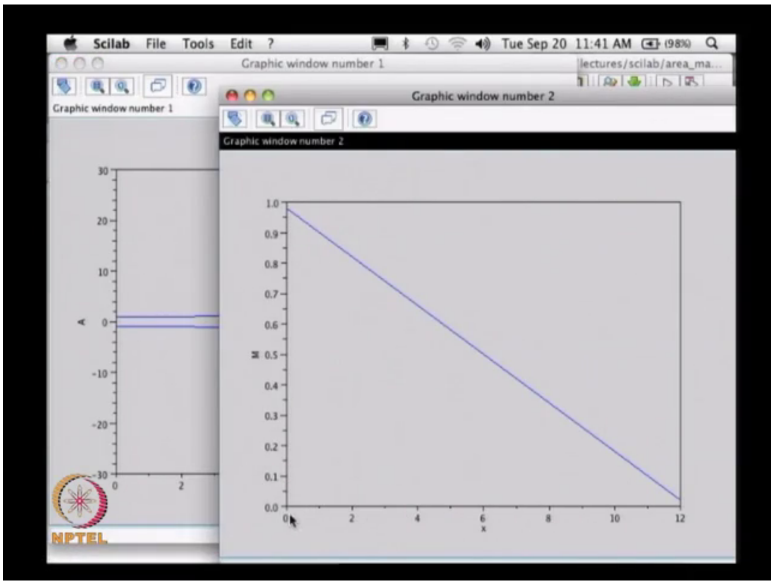

So, if I do this. So, again as you can see that over this range basically over this range.

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We have the Mach number which is decreasing right.

Now, you can see previously where we went from 0 to 50; when we went from say 1.0225. In this case we are going from 0.98 to 0.02. So, this is 0.98 2.02, this is my change in Mach number, let us look at the corresponding change in the change in the area. So, what you see over here is that the flow is subsonic. The flow is subsonic in the entire region, now as you increase as you increase, the area have the area increases you can see is call gradual, right. That is area increases quite gradual, and there is you know area also increases right.

So, in this case so, therefore, the velocity is decreasing, isn’t it? Because in here as you see the Mach number is decreasing. The Mach number decreases in a subsonic flow. So, if the Mach number is decreasing, then in the area is increasing, but Mach number is decreasing which means the flow is slowing down. So, therefore, in a subsonic flow, if the flow is totally subsonic, right. There is say and increase in the area right, but the velocity decreases, right as you can see. So, therefore, this is essentially the difference between supersonic and subsonic. Now let us come back to the equation, now if this is subsonic, right. That will make this as negative.

So, which means that if that for a positive change in area; which is the plus d a this becomes the negative d u which means that there is a slowing down of the flow. In other words, if this is a negative then say this becomes negative. So, which means that if I increase the flow velocity d u, then there is a negative a which means the area has to decrease. So, therefore, e if the flow is subsonic, if the flow is subsonic, and you need to increase the flow velocity, then you need to decrease the area, right whereas, in a supersonic clay supersonic case, if you need to increase the velocity you need to increase the areas well.

There is what we see from the equation here, and as I plotted and I plotted this you know in this scilab. So, you can see for yourself how these 2 sort of give us a lot of information. Now what we will do is we will make this thing run for a series of we will let it let it go from say subsonic to supersonic will let this run from 0, right.

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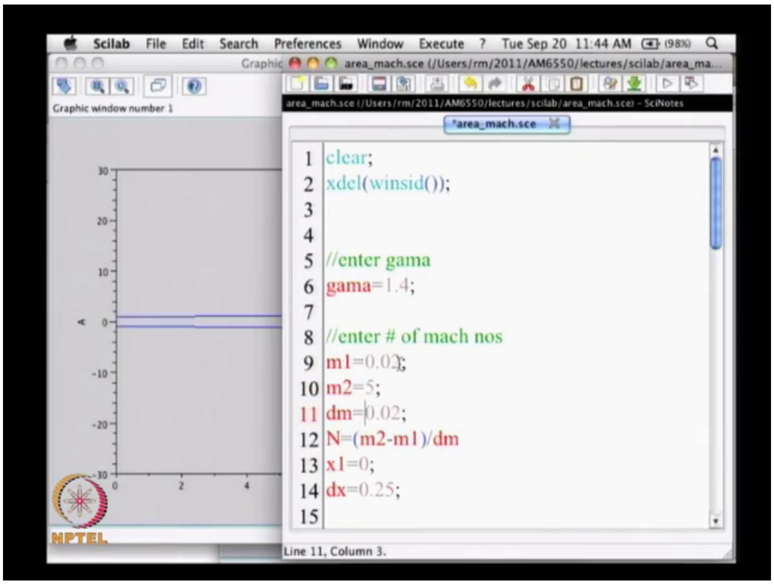

We will let this run from 0 2 to 5, and this will go 2.02 right. So, this is what we will make this do it goes from subsonic Mach number of 0.02 and it goes to Mach number of 5 in steps of 0.2.

If I do that let me go and run that. So, if I do that now if I do that. So, what I get is this.

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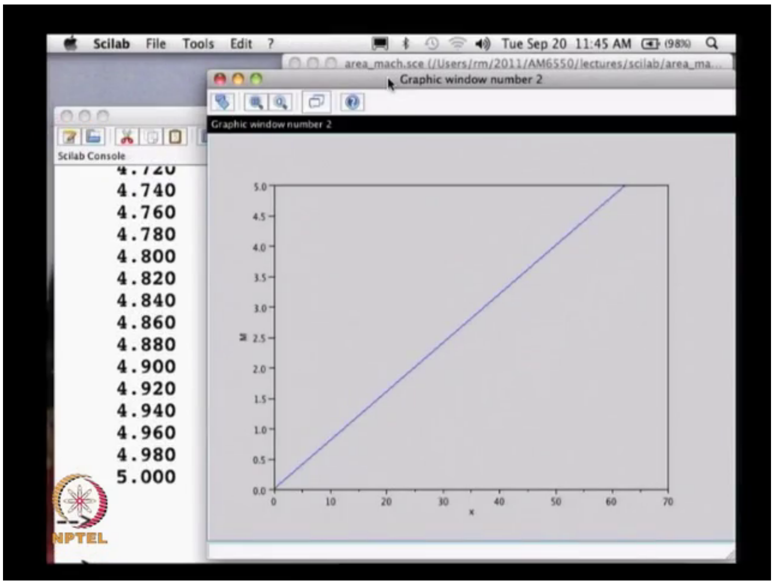

So, what you can see now that when I do this it goes from 0.02 to 5. So, this is my throat condition which is being which around say the distance 10. So, what I see over here is this right.

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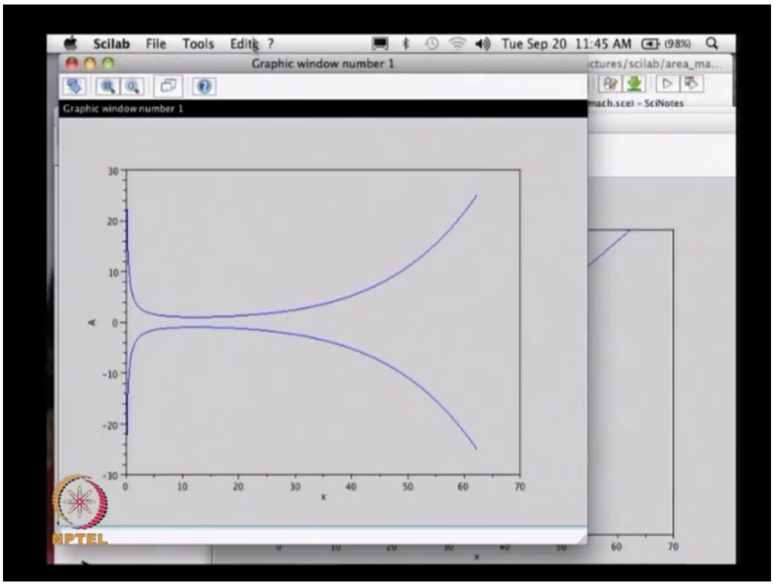

So, what you see over here is that beyond this 10. So, beyond this distance flow is supersonic. So, for supersonic as you increase and increase the disk as you increase the area the Mach number is increasing right, but in the which means the flow is you know travelling faster. In here however, we are going from with the there is a decrease in the area as you increase the Mach number.

And this is the subsonic zone. Now let us talk about this region. So, we went we divided this region we said this part. So, this part say beyond 10 with this part here is the supersonic zone, and this part is the subsonic zone. So, what do we have over here, right? So, we went from 0.98 and one point over 2. So, what happens in this region? Now this here is basically as you can see from this diagram also, it is the basically the least area. Least area in the entire distribution right. So, the this is where it goes through a Mach 1, it goes through a Mach 1. So, if you come here if you come here, and you will do this for a Mach one you will get a corresponding.

Now, that is essentially the throat region. Essentially the throat region now just taken back you know in this case to be basically the least area. So, based on that we will you know get the areas of the rest of the duct so to speak. So now, if you look at the if you put M is equal to 1 here, it kind of does not you know make too much sense. You know, this relationship does not whole. So, what essentially this equation here would mean? That physically what that means, that it is the least area in the distribution, and that is the least area in the distribution. And so, the flow when it is going from say, the flow when it is going from converging to diverging, passes through this point right.

It passes through this point, and what we get is you know supersonic flow. So, what essentially you are having here is that a subsonic flow it goes through this kind of an area change and becomes supersonic. So, this is what is you know something that (Refer Time: 36:33) had done you know long term back right. So, this is essentially your converging and diverging nozzle. Now based on that let us do a quick problem, and we will see how you are going to use this. Before I move forward what you can see here is that for a given Mach number this can be given, which is exactly so because this is also available in the tables.

So, like I was getting over here. So, in the showing.

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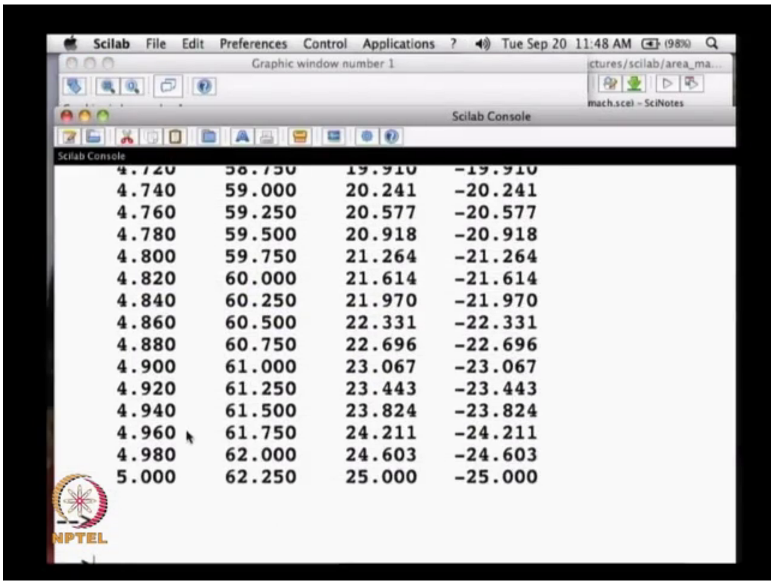

Yes so, this is this this is so what you will get in the tables is from point say 0 2 to 5; you will get the corresponding areas. You will get the corresponding area changes like this. So, that is what we get. So now, let us do a quick problem and sort of wind up our understanding of this. So, what we have is so we have an isentropic flow, through a convergent divergent nozzle with an exist throat ratio of 2.

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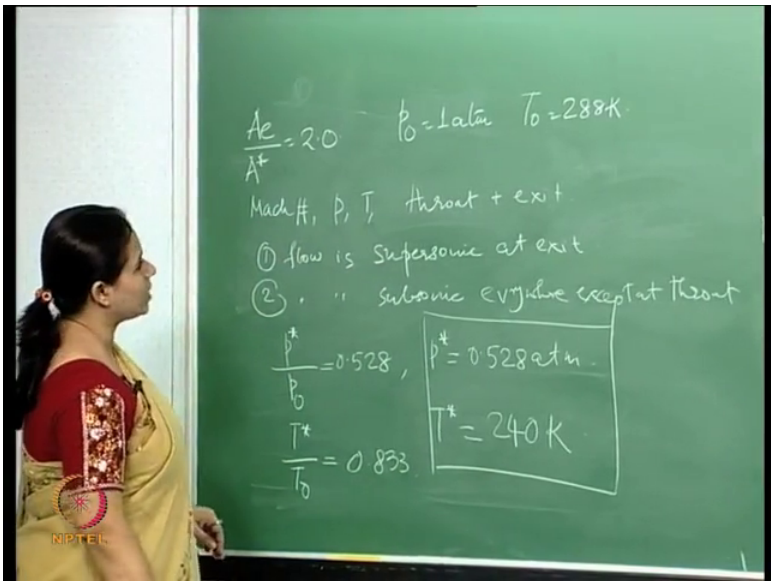

So, we have an exit to throat ratio of 2, we have this area right.

Now, whether you want to do this. So, over a length of 50 or one is really up to you, but when we will have the will come to a place where we will also talk about minimum length nozzle and that is you know a theory that we will talk about later on the reservoir pressure and temperature are one atmospheres and 288 kelvin respectively. So, basically the chamber in which you are developing a producing the fluid, right. The gas. So, the reservoir temperatures I am going to call that as is one atmospheres this is given and T naught is 288 kelvin. So, what we need to find out is calculate the Mach number right.

So, the pressure temperature at both the throat and the exit, so we need to find this out at the throat and exit for 2 conditions. Number 1 is the flow is supersonic at the exit the flow is supersonic at the exit. And number 2 is the flow is subsonic throughout the nozzle except that the throat, but it is 1, right. The flow is subsonic everywhere, right. Everywhere except of course, at the throat, right where it is equal to 1. So, how do we go about and do this. So, what we will do is this we will use this we will need to find out.

Now, we do not know any throat condition, this is all that is given to us the stagnation conditions are given and the area ratio is given. So, all we will do is find these out P star by this, right. How we will find out? So, basically in this case what we are going to get is that. So, this is so what is given in the tables is this. What you will get in the tables is this. So, this is the local Mach number. So, what we will do is find this out for at the throat. So, which means my local Mach number is 1. So, corresponding to that we get P star by P naught is 0.528 and T, I get as, I get this right?

So, therefore, A 2 or A 2 calculate the so, say we need to calculate the right. So, P naught is given and T naught is given. So, therefore, we get the throat conditions. So, we get P star as 0.528, right. We get a throat conditions, and T star is equal to 288 in to this. So, which we get at 240 Kelvin, these are my throat conditions. So, one is step we get this. So now, next is that we basically have to find out now supersonic flow. Now for the supersonic flow if it is the supersonic right.

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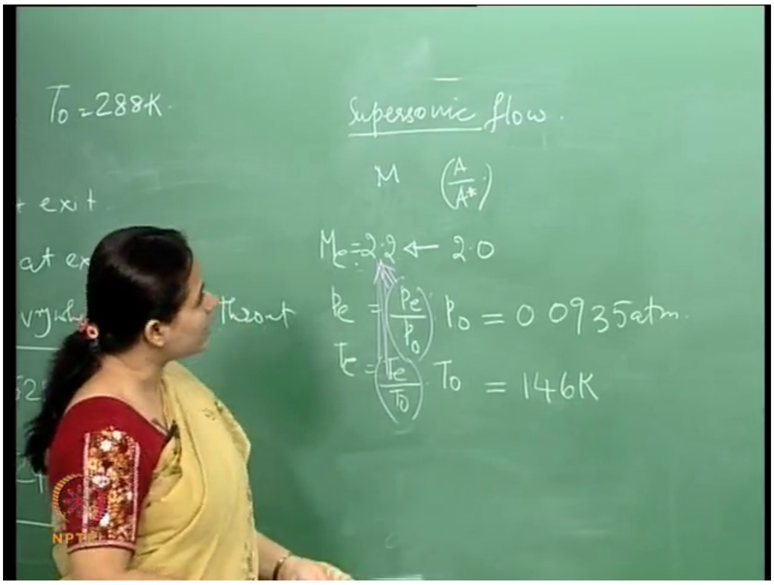

Now, if you look at the tables, what you will get is there is a Mach number, and there is a corresponding area ratio. So, local area by the throat area. So, this is what you will get, what we are given here is not this were what we giving here is this, right. We are not given and a incoming Mach number, we just know we just know the area ratio. We need the stagnation conditions from which we can calculated the throat conditions, right. And in here what we given is this. So, corresponding to this so corresponding to this being 2, we calculate the corresponding Mach number, and what we get that is 2 right.

So, what we need to calculate is Mach number pressure and temperature, right at the throat and exit. So, throat is something that we have calculated, what we need to calculate is at the exit. So now, corresponding to the exit area ratio which is of 2 with this throat we will get a Mach number of 2.2. So, the Mach number of the exit is 2.2. So, this is equal to my Mach number at the exit. So, what we need to now calculate is pressure at the exit and temperature at the exit. What we know at the stagnation condition? So, what we will have to calculate is P e by P naught into P naught right.

Now, P e by P naught is something that we will get again from the isentropic tables, right corresponding to this Mach number. So, corresponding to this Mach number we will get that and what is this comes out to be right. So, what we get over here. So, this and P naught is equal to 1. So, what we get over here is 0.0395 atmosphere. So, similarly for this here temperature. So, T e by T naught into T naught, right.

So, again this is something that we will get, this is something we will get in corresponding to this Mach number, and this comes out to be 146 kelvin. So, therefore, now since we know now if you go to the tables, you will see that it runs from 0.02 to 5. So, which means it goes from subsonic to supersonic. So, when we calculate this Mach number, we have to look at the supersonic part of the table, right. Because there will be 2 values. So, for this area ratio for this area ratio we go to the supersonic part of the tables, and that is why we get corresponding Mach number which is 2.2 and corresponding to that when from the isentropic tables, we can find out this.

Now, for the second case which is subsonic, right?

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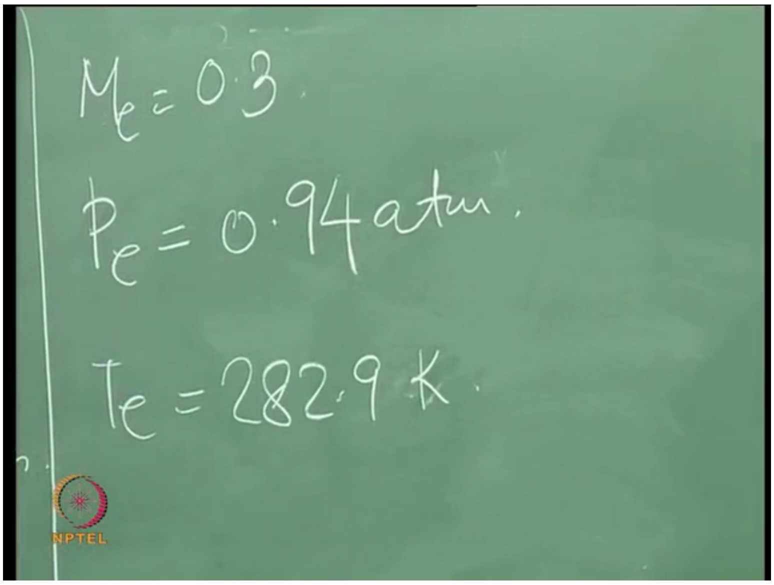

So, what we do now is go to the subsonic part of the table and calculate the corresponding Mach number. So, and the Mach number in that case comes out to be exit comes out to be 0.3, right. And again, we do the same procedure. Corresponding to that Mach number we will now calculate the pressure and temperature, right. Using the same way, and what I get here is this. That and the corresponding temperature is 282.9.

So, that is it. So, what essentially, we have done here; is there from just this information all we have, see all we have here is that we know the pressure and temperature of the chamber in which the gas is being developed. And all we know is the physical you know dimensions the area of the exit by area of the throat that is all we know, but we have been able to calculate host of information from using the very simple tables by using our knowledge. And so, basically what we have so, that we were able to basically use these relationships, right. Develop the tables and use it for something you know something simple, but interesting all, right. There should be on.

Thank you.