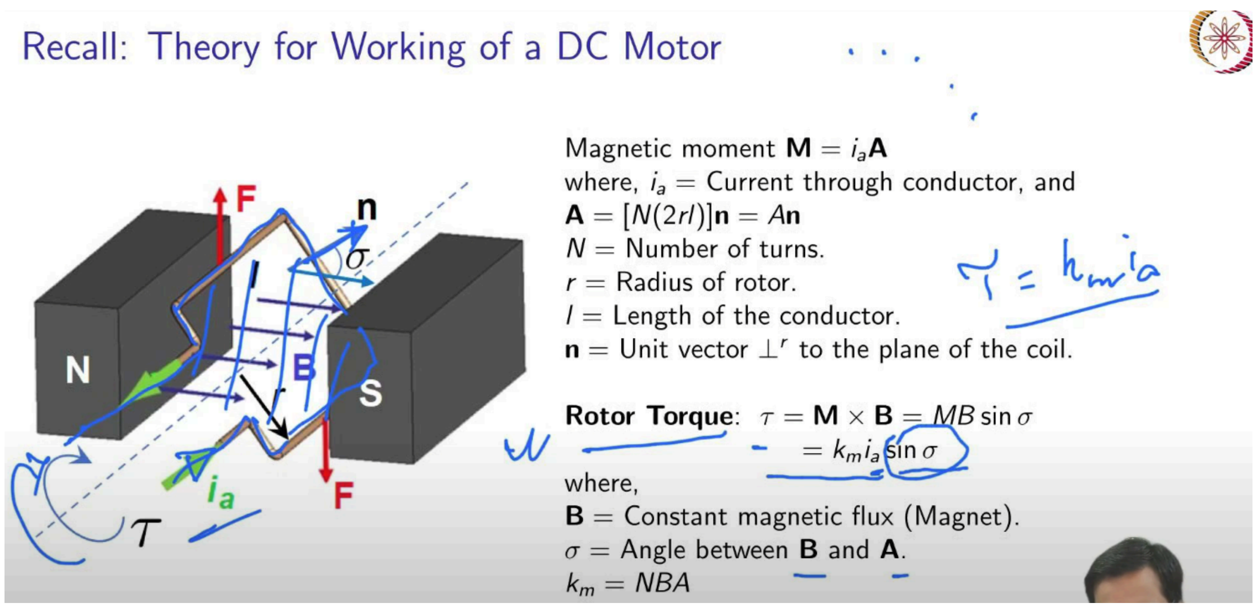
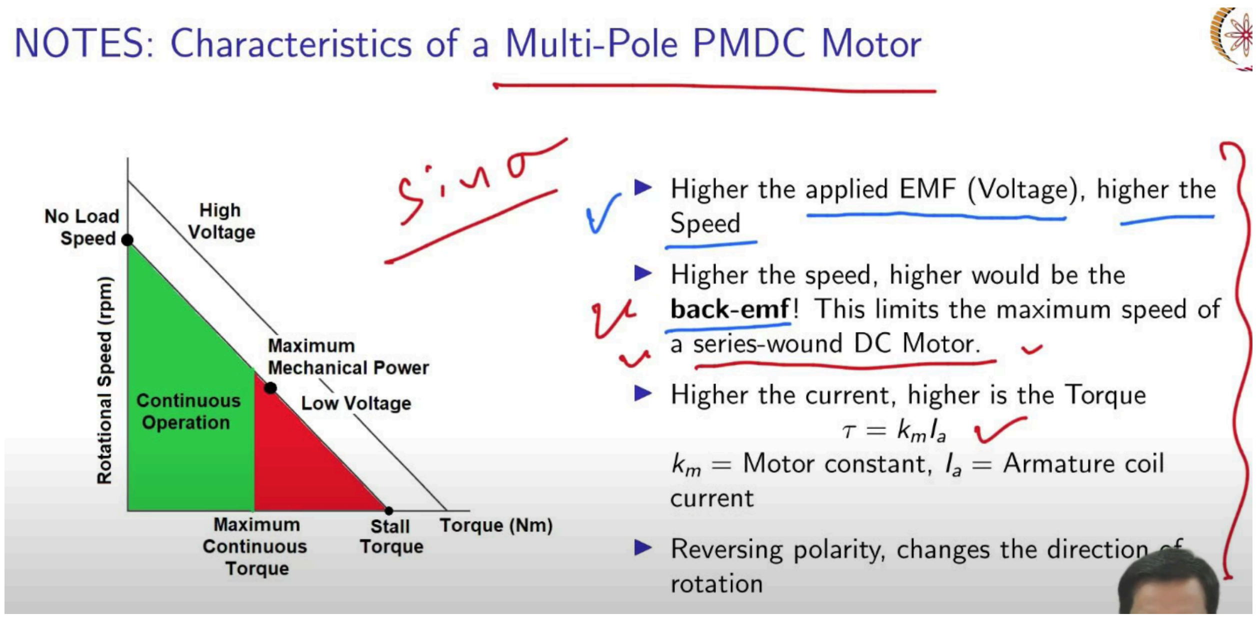
NPTEL Online Certification Courses
Industrial Robotics: Theories for Implementation
Dr Arun Dayal Udai
Department of Mechanical Engineering
Indian Institute of Technology (ISM) Dhanbad
Week: 11
Lecture 46

A Robot Joint: DC Motor Model

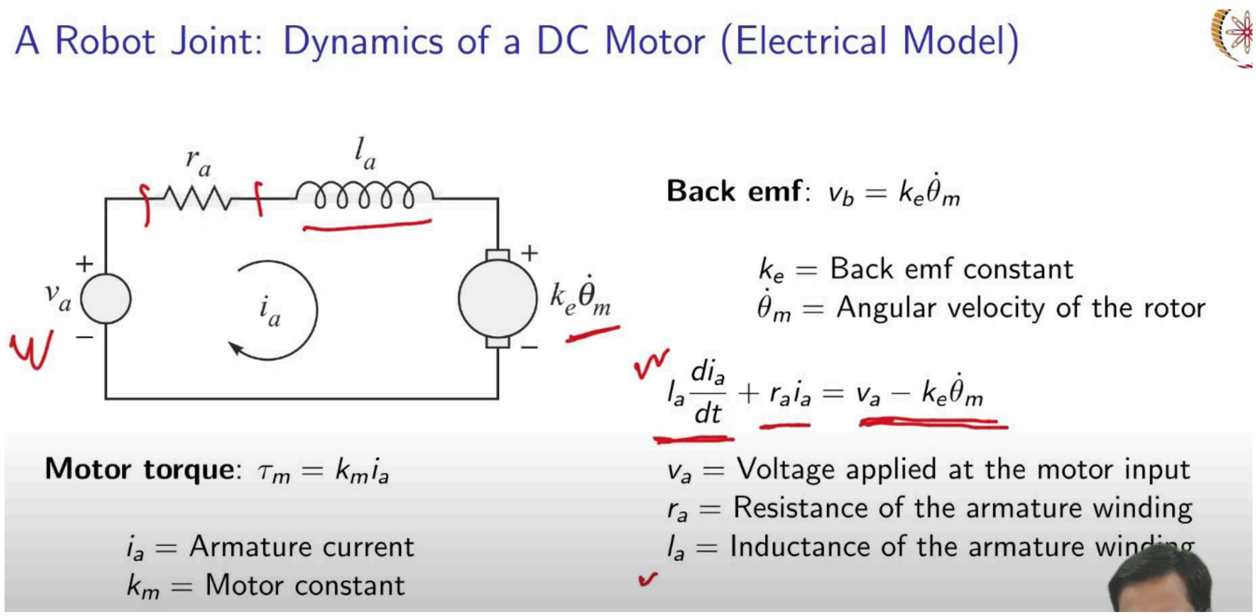
Hi, we are now ready with the tools to study control. So, we have so far studied linear control systems, specifically linear second-order systems. We have seen how to study its behaviour using the roots of its characteristic equation. We have seen how to analyse such systems using transfer functions and state space representations. So, at least, we are now ready to study robot control. So, let us now study controlling any robotic joint to begin with. So, let us start.



So, let us just recall the theory of working of a DC motor that we have seen earlier in module 2. So, yes, this is the first thing that we have learnt. So, we saw the structure of a DC motor. Basically, it contains a simple armature coil that is shown here, correct? And you see, it has a current that gets into that that is given by i_a. That is, the armature current comes out from here. It develops a torque, which is given by tau. Not many parameters are here to observe. The first thing that I want to point out is this equation: that is robot torque, that is the joint torque, that the rotor of a motor that is seen is proportional to the current, and obviously, there is something called sine sigma. Sigma is the angle between B and A. B is the field direction, and A is the normal to this area, That is the direction of normal to the area. If this is the area, so normal is this, so this is the area. So, you see, if you have multiple coils, which are there, which are at, let us say, you have multiple poles and multiple coils. So, this becomes almost insignificant, and in that case, tau is proportional to the current that is flowing through that. So, this torque, basically, varies in a sinusoidal manner. It comes up, it goes down, and if you have multiple coils, so by the time one of them dies off, the next one gets in. So, you have, let us say, 32 poles or maybe 20 poles or something like that. So, if at all, you have a multiple number of poles. So, that evens out the torque which is coming due to this change. So, tau is equal to Km into i_a. So, this is the first thing that we have seen.

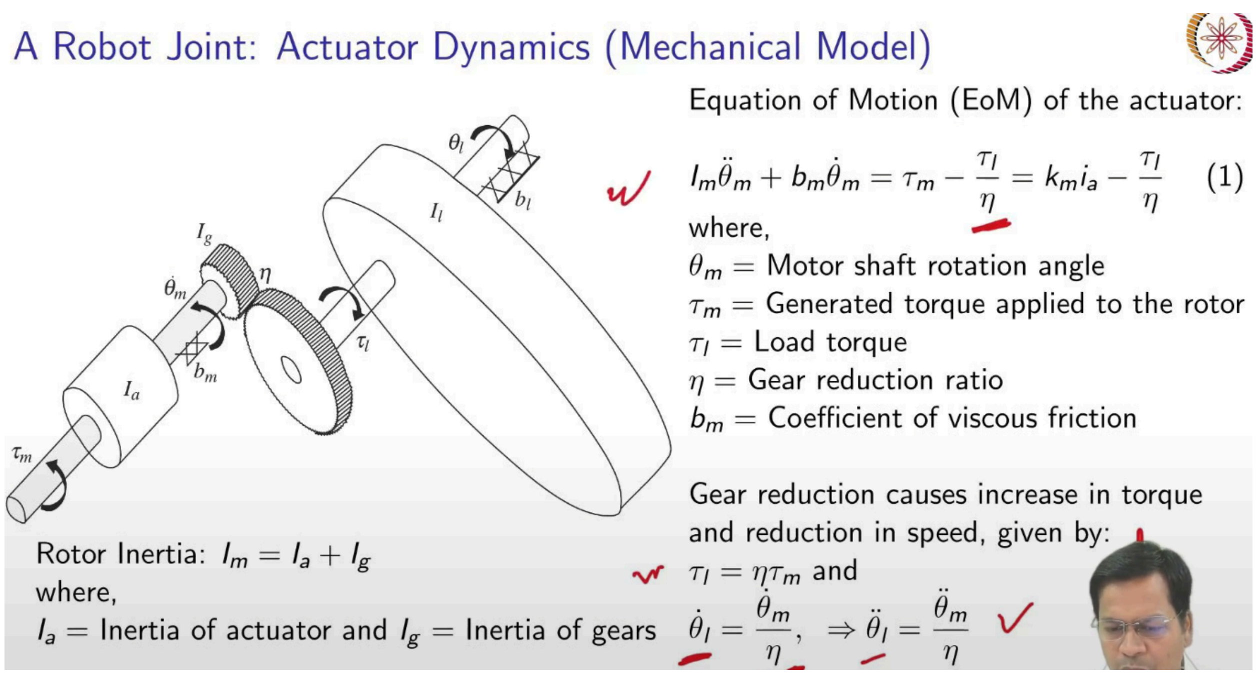


Also, the higher the applied EMF, that is, the voltage, the higher the speed that we have seen. We will see now why it is happening also. So, this is the first thing that I have pointed out earlier. And the higher the speed, the higher would be the back EMF. Now, what is the back EMF? Just let us go back to the slide once again. So, if you are providing current from here, so you see, you should feel forces, that is, the couple of forces which are there, that is creating a couple, basically. So, that is effectively creating the torque, which is aligned like this. So, if at all you are providing current, it produces torque. If you do not provide any current and the rotor keeps on rotating due to its inertia, let us say you want to stop your motor. So, in that case, what happens as soon as you stop flowing this current? This is the conductor which is moving inside this field. So, the field is always there. So, you see, you have a field which is always there. So, the field is there and you have conductors which are like this. So, conductors are there. So, if that is moving, it automatically induces a current in this conductor. So, that is there. So, even if you do not provide any current from outside, due to its motion within the magnetic field causes some current to be induced. So, that current is in a reverse direction. So, cause and effect. So, if this is the cause which is creating this current, the effect will be the current that creates a reverse torque to stop the motion, correct? So, it is known as a back EMF. So, the higher the speed, the higher would be the back EMF. So, this basically limits the maximum speed of any series wound DC motor. In the case of a series wound DC motor, if back EMF is there, that effectively reduces the incoming current. So, the total current flowing through the armature greatly reduces. At higher RPM, it is very, very significant, and that causes reduce in torque. Torque getting reduced. So, that is basically limiting the maximum speed of this. So, that is the reason why speed is limited. In series wound it is more significant because the same current flows through the armature also. So, that is there. The higher the current, the higher the torque. As you have already seen in the case of multiple DC motors. It is directly related, without any relation with sine sigma, that you have seen in the case of that. Reversing the polarity would change the direction of rotation because that would create the forces that are induced in the coils. So, these forces, this force direction, will be reversed if you reverse the direction of the current and that will create a reverse torque. So, these points should be noted before we begin.



So, let us begin with the robot joint. So, we will discuss the dynamics of a DC motor. So, this one in particular is the electrical model that I am going to discuss. So, yes, this is the applied voltage across the terminals of your motor. Let's say it has its own internal resistance. Coil resistance is given by r_a, which is armature resistance. Because it is a coil, it is a copper which is wound over a core, so, yes, it behaves very much like an inductor also. So, this has an inductance of l_a, and I know there is a back EMF. So, this is a back EMF constant, and that is giving a reverse voltage, which is proportional to, you see, back EMF voltage. This gets positive opposite to this. So, back EMF is there. So, this is back EMF constant and theta m dot. What is theta m? Theta m is the speed of your armature, the speed of armature. So, the higher the speed, the higher the back EMF. That is what I told you. So, these are the basic relations. That you already know from your previous slide. So, i_a is the current flowing through the armature. The motor constant is k_m. Now, the back EMF voltage is proportional to the speed and angular velocity of the rotor, and the back EMF constant is given by k_e. So, now, putting them all together, what I can write is this: closed circuits have potential across each of them. So, summing them all together should count to 0. So, yes, this is the applied voltage, applied EMF that is the driving voltage, and this is your back EMF. So, subtracting them should give you the effective voltage which is present in this loop. That is what is causing the potentials to be generated across resistance and across the inductance. So, across inductance, it is la, that is, inductance of the armature, winding into di_a by dt, rate of change of current. So, voltage induced across the inductor is inductance, that is, the constant into the rate of change of current. So, the higher is the rate of change, the higher is the voltage that is induced across this inductance. So, this is first. These are basic physics equations that hopefully you have gone through during your higher school also, and the voltage across this resistance is r_a into i_a. This is simply by Ohms' law. So, yes, r_a into i_a plus l_a di_a by dt should be equal to a total potential which is there across. This. So, this is the first fundamental relation, that is, the electrical relation of this DC motor.

$$l_a \frac{di_a}{dt} + r_a i_a = v_a - k_e \dot{\theta}_m$$



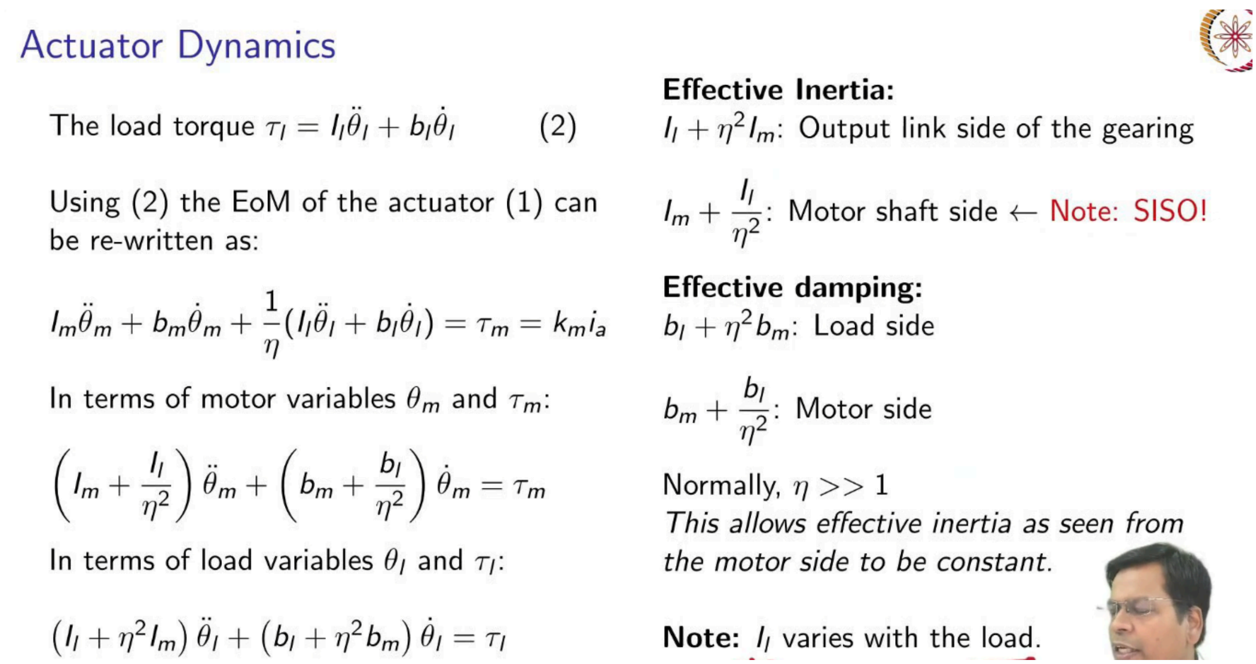
Now, let us look into the mechanical model, also what we call the Actuator Dynamics of the Robot Joint. So, yes, it is constructed like this. So, you have tau m that comes here, that is through the motor, and Ia is basically the moment of inertia of this armature whole, of the system, including the coil. All solid armature core is here, including the shaft; everything is here. And you have theta m, that is, the joint angle at an instant of time. You have bearings. That goes here, here, here, here, all together. Let us say you have a bearing constant that is, basically, the damping coefficient given by bm, and you have a gear ratio. Over here, there is a transmission. This is the smaller one, and this is the bigger one overall; it reduces the speed when it comes to this and enhances the torque. So, yes, that torque, which is transferred to the link joint, is tau l, so tau m is the motor torque. Tau l (τl) is the shaft that connects your link. Link has its own moment of inertia given by I L and the shaft rotates by an angle theta l. You also have a damping coefficient over here that is bl. So, these are the parameters. Overall, the whole of the mechanical system can be visualised like this. So, now let us start forming the dynamic equation for this. So, rotor inertia will be the armature plus the inertia of the gears. So, gears are also attached to the armature. So, total inertia, Im, that is, the inertia of the actuator should be equal to Ig plus Ia, armature plus gear. Total inertia is like this.

$$I_m = I_a + I_g$$

The axis of rotation remains the same so that it can be simply scalar added. So, the equation of motion of the actuator can now be given as so effective torque that is coming from the link side of it. So, if this has a torque which is tau l, which is here. So, the torque which is coming, resistance torque which is coming on to the motor shaft, is tau l by eta. So, yes, this is tau l by eta. That is reflected here. And tau m is the driving torque. So, the net driving torque due to both of these is tau m minus. This is the reaction torque which is coming on to this. That is tau l by eta, tau l by eta. So, that comes here. So, that is actually driving your shaft. So, it is Im theta m double dot moment of inertia into angular acceleration, plus bm theta m dot. bm is the damping coefficient into angular velocity. So, this is the total torque which is here. So yes, this is the torque which is developed due to the inertia. This is the torque, which is developed due to the damping. Total torque should balance out each other. So, that is here. Also, tau m is k_m into i_a. You already know the torque which is generated out of your motor is Km into the current that is flowing through that, and the rest remains the same. So, these are the parameters I have already discussed there. So, theta m is the motor shaft rotation angle, and tau m is the generated torque applied to the rotor. Tau l is the load torque, eta is the gear reduction ratio, and bm is the coefficient of viscous friction. So, yes, the reason why this is here is very much clear from here, hopefully. So, yes, angular velocity is eta time reduced. So, if angular velocity is reduced, torque is increased from the l side of it. Because you know tau l, theta l dot should be equal to tau m, theta m dot. This is the angular velocity in torque. Angular velocity into torque, that is, work done at the input, should be equal to work done at the output. So, if it is theta l dot is theta m by eta, theta m by eta, theta l dot is theta m by eta. That is there. So, effectively, tau l becomes equal to eta times of tau m or tau m side of it. We will see if the rotor is static. But if it is moving, you see tau l by eta times coming to the motor side of it. That is as a reaction torque. So, yes, I am wiping them all to make it very, very clear now. And same happens to the acceleration also. You see, theta l double dot is equal to theta m double dot by eta, and tau l is eta times of tau m.

$$\dot{\theta}_1 = \frac{\dot{\theta}_m}{\eta}, \Rightarrow \ddot{\theta}_1 = \frac{\dot{\theta}_m}{\eta}$$

So, These are some fundamental mechanical relations which should be quite clear. They are simple work input and work output relations. That is shown here, and this is the dynamic equation of motion.



So now, moving ahead, let us see what else we can put in here. So, load torque also drives something. It drives the link. So, it is a moment of inertia of the link into the angular acceleration of the link, got it? And you also have damping over there. So, it should give you a damping coefficient for the angular velocity of the link. So, total torque is driving all these. So, yes, now equation 2 can be put into your actuator equation that was equation 1. So, it looks like this, if you remember. So, I will just put tau l to this, or this, and I will bring it over here. So, what should I see now? So, it quickly can give me. So, if I substitute tau l over there, it should be put over here, and that is equal to tau m or km into i_a.

$$I_m\ddot{\theta}_m + b_m\dot{\theta}_m + \frac{1}{\eta}(l_1\ddot{\theta}_1 + b_1\dot{\theta}_1) = \tau_m = k_m i_a$$

So, this is the overall relation that is there. So, in terms of motor variables, theta m, and tau m. I will substitute these two equations once again. So, I will put theta l is equal to theta m dot by eta over here. So, theta l, wherever I see, I will put theta m dot by eta. So, that should give me, if I take all of them common and bring them out. Similarly, theta l double dot should be equal to theta m double dot by eta. So, effectively, eta square is going to come. So, you see this equation.

$$(I_m + \frac{I_l}{\eta^2})\ddot{\theta}_m + (b_m + \frac{b_l}{\eta^2})\dot{\theta}_m = \tau_m$$

So, it is Im plus Il by eta square into theta m double dot plus bm plus bl by eta square, and theta m dot is equal to tau m. So, this is quite a clear, straight relation. You can try getting it yourself. Also. Okay in terms of load variables, if I convert. So, I will substitute. Otherwise, that is theta l I will substitute. I have substituted earlier that theta l dot is equal to theta m dot by eta and theta l double dot is equal to theta m double dot by eta. So, I will substitute it. Otherwise, this time, I will substitute: theta m dot is equal to eta times of theta l dot. So, that going here okay and here okay. So, it should be theta l into eta is equal to theta m double dot. So, these two values I will put here and can get this relation. So, this is the load side variables, and this is in terms of motor side variables. Now, let us see what does these two equations tell me. So, effective inertia is Il, eta square Im. So, this is the effective inertia which goes to the motor. So, that is what actually comes. So, the output link side of the gearing. So, the moment of inertia that the output link side will feel is due to its own inertia and due to the rotor shaft inertia. That is multiplied, you see. So, more inertia you will feel. So, total inertia is eta square Im plus Il. So, output link side, due to the gearing, you can see, and similarly, on the motor side of it, the motor will see the link inertia is reduced by eta square times. If eta, eta is normally the ranging-in case of industrial robots it ranges from 100 to 300 gear ratio is there. So, in that case you can imagine it is reduced up to eta square times. So, all the rotors plus this inertia. Now, you will see, that this is almost like missing from your inertia. So, motor shaft inertia will only feel this one. Hardly there is any effect due to the link moment of inertia due to the gearing. So, that is the reason I told you earlier. Also, that is the reason why we can drive industrial robots using single input, single output system.

So, yes. Now, looking at the damping side of it, the same will happen to the damping also. So, from the load side, you see you have damping which is here, it has more damping, it is very much damp. So, naturally damped, and you also see from the motor side again, you see you have less damping due to the load side of it, and total in damping has increased also, but to a very lesser extent is coming from the link side of it. So, the motor side will see this much damping. That is here.

$$b_m + \frac{b_1}{\eta^2}$$

So, this is the damping and normally, you know, eta is very, very greater than 1. So, this allows effective inertia, as seen from the motor side, to be constant. So, from the motor side, the inertia which is there is almost constant. It is hardly getting affected by the inertia of the links. So, this is the reason SISO is quite good to be adopted in controlling industrial robots. Il varies with the load link. The moment of inertia varies with the load due to the supplementary load. Even with the end effector load, if it is carrying, a moment of inertia is going to change. So, this should be noted.

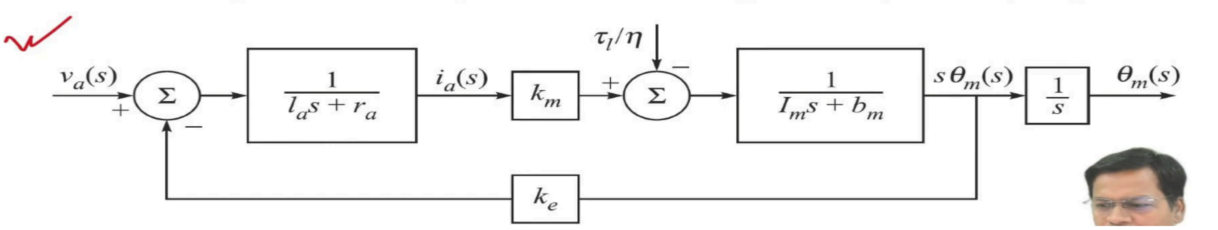
Transfer Function of the Joint with DC Motor

Taking Laplace of the electro-mechanical models (with zero initial conditions):

$$l_a \frac{di_a}{dt} + r_a i_a = v_a - k_e \dot{\theta}_m \text{ and} \quad \rightarrow (l_a s + r_a) i_a(s) = v_a(s) - k_e s \theta_m(s) \text{ and}$$

$$I_m \ddot{\theta}_m + b_m \dot{\theta}_m = k_m i_a - \frac{\tau_l}{\eta} \quad \rightarrow (I_m s^2 + b_m s)\theta_m(s) = k_m i_a(s) - \frac{\tau_l(s)}{\eta}$$

The block diagram form of the joint with motor and gear train system may be given as:



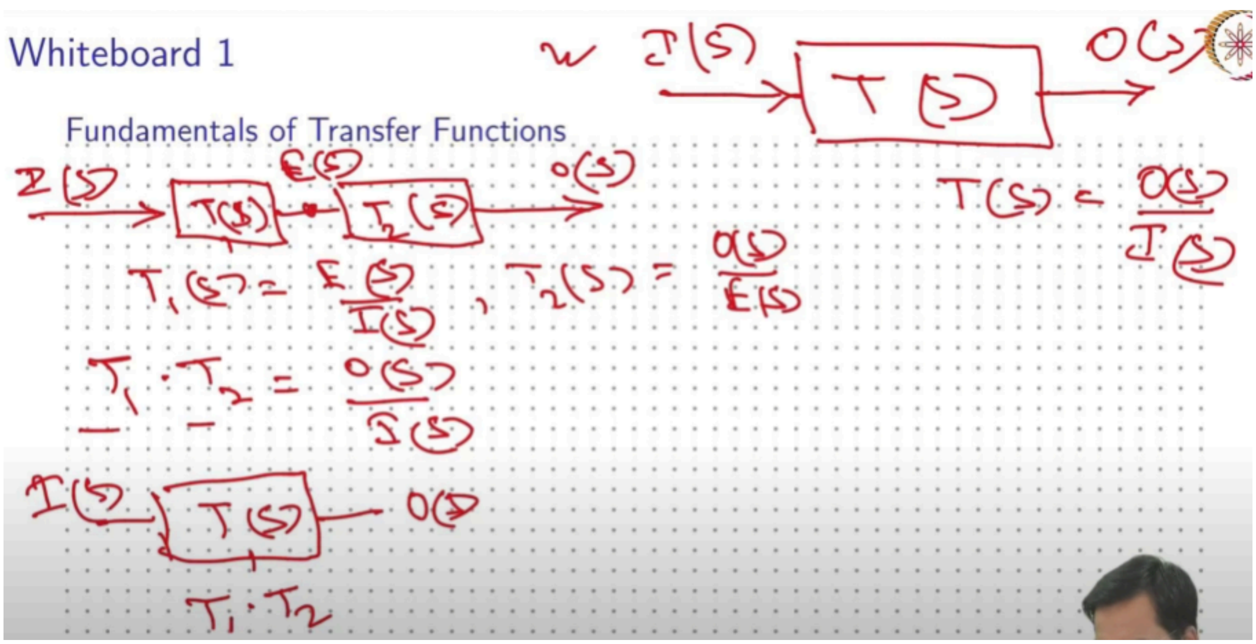
Now, let us get the transfer function of the joint so that I can get to the input and output relation for this DC motor system. That is the whole of the actuator here. So, these two are the fundamental equations that we have derived just now. This was the electrical relation.

$$l_a \frac{di_a}{dt} + r_a i_a = v_a - k_e \dot{\theta}_m$$

And this is your mechanical relation,

$$I_m \ddot{\theta}_m + b_m \dot{\theta}_m = k_m i_a - \frac{\tau_1}{\eta}$$

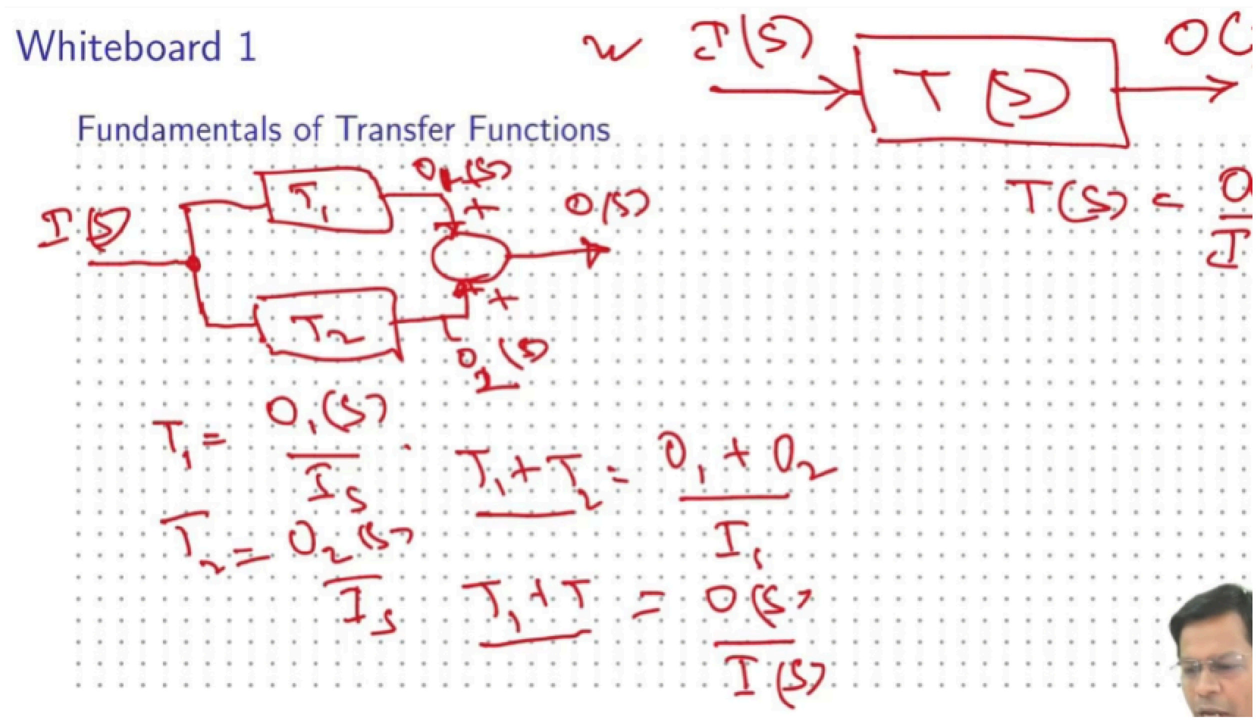
So, I call it electromechanical models. So, taking Laplace of this, and this will lead to this, so you can quickly see it here. So, Ia can be written as Ia s times of Ia is constant, current is varying, so Ia s into s, s goes here. So, s is now taken out with this, and similarly, Laplace of Ia will be Ia(s). So, taking common. I get to this right-hand side of it: v_a becomes $v_a(s)$, and k is constant. So, again, because it is theta m dot, it is s theta ms. Taking all the initial conditions as zero, the rotor has started from rest, zero velocity, from zero position, right, so that is the reason this is valid. And then taking Laplace of this will give me this, so it is theta double dot. It should give me Im s square, and theta ms will come out from both of these. Similarly, from the right-hand side, you have constants, which are there, so theta ls is constant, so tau ls is constant, and you see this. So, these are the two. So, can I draw the whole of this? Before I do this, let me just start with a little fundamentals first. Fundamentals of the transfer function. Before I can make you understand this one, I'll discuss it a bit, so let us start.



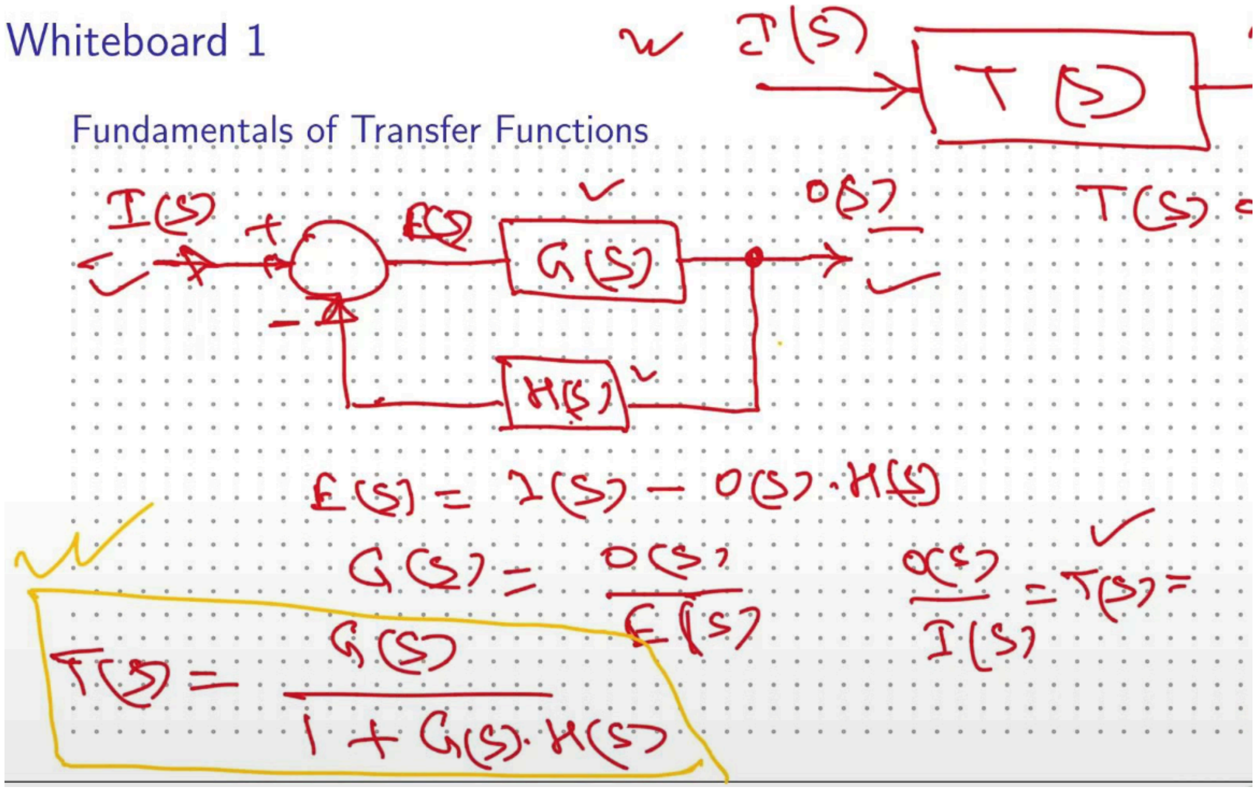
So, transfer function you know it is input, that I(s), and you have a transfer function. As you have seen, it is Laplace of output by input, so it is output Laplace, so output by input, the transfer function is given as output by input. This is quite okay.

$$T(s) = \frac{O(s)}{I(s)}$$

So, now again. So, I'll extend this idea further. So, if it is in series, you have two blocks coming one after the other. So, this becomes your input, and this becomes your output. You have transfer function one, and transfer function two. So, you can write it as, let us say, there is something that signal which is tapped over here is this: so T1 s is equal to ES by Is. Similarly, T2 s becomes equal to OS by Es. So, multiplying them together will give me T1. Let us forget about s now. T1 into T2 should give me E by I and O by E. So, effectively, Es will get cancelled. So, it gives me output by input. So, you see, if they are in series, they can be multiplied, so you can directly get an equivalent block diagram of the equivalent transfer function as T, that is, T1 into T2. That will directly relate the output to the input. This is the first thing that you should know. So, this is yours. If they are in series, what will happen if it is in parallel?



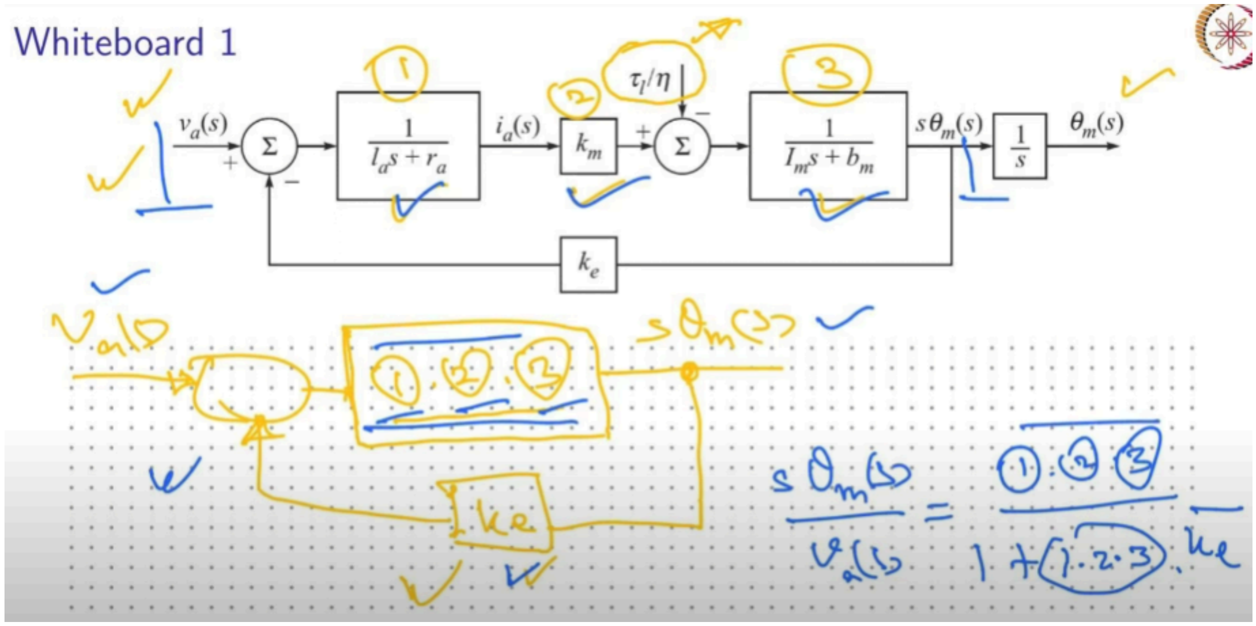
So, if it is in Parallel, let me just wipe it off so I can draw it like this. So, if it is in parallel, you cannot draw the way it is drawn for serial. This is your block. This is another block, and signals from both of them are now added over here, and it gives you a further output. So, this is added from here. This signal coming from here is added. You have T1, and you have T2. This is your output, and this is your input. So, this time you see, there is a signal which is from here. So, let us say this is O1, this is O2s, O1s and O2s. So, T1 will be equal to this O2 and this one as O1. So, it is O1s by Is, and similarly, T2 will be equal to O2s by Is. So, if I add them together, what will I get? T1 plus T2 will be equal to O1 plus O2 by Is. So, O1 plus O2 is nothing but output by input. So, that becomes the equivalent transfer function in this case. So, this is your T1 plus T2 is output by input. So, in the case of parallel, it gets added like this.



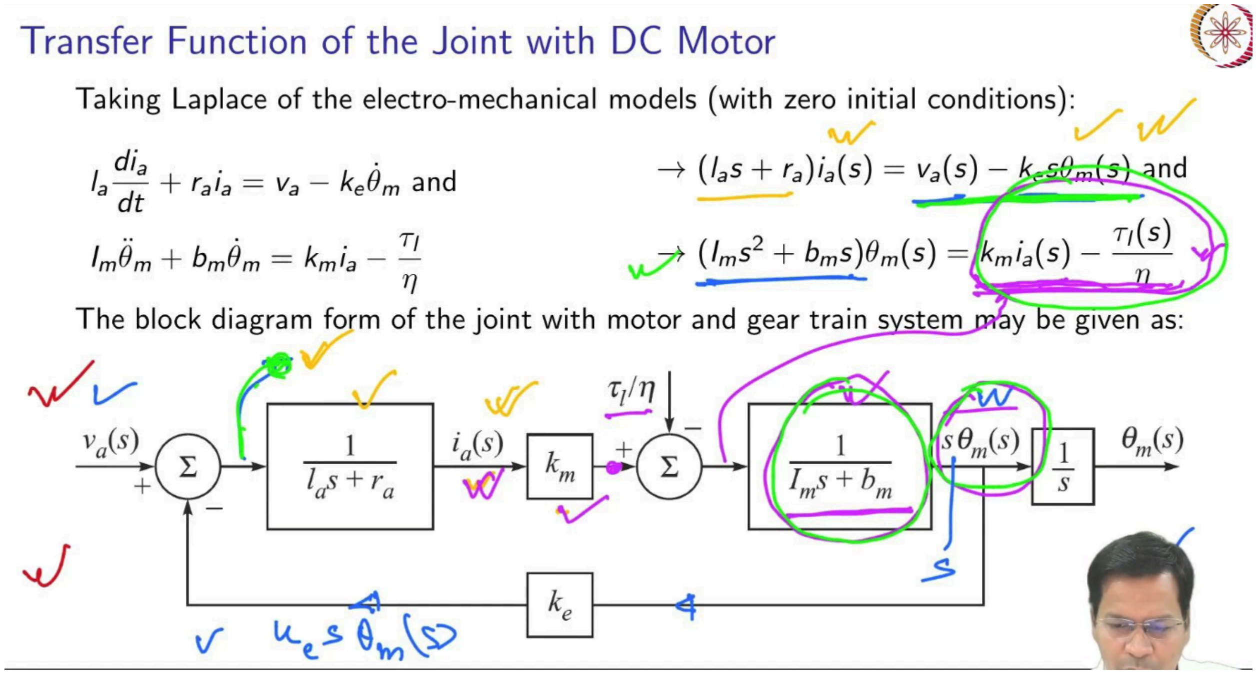
Now, let me again wipe it off and see you show you one more example. Now, let me draw it in a closed loop way, because the way we want to do it now is a closed loop, a loop publication. So, this becomes your input. That comes here to the summing block over here. That is a plus. This goes to the transfer function, which has G(S), and that is your output. You have a tapping from here that comes through and a feedback line, which is here, that is, H(S), and that comes here with a negative sign. So, this is your input, and this is your output. Now, let me see how what should be the equivalent transfer function that connects the output to the input. So, the signal which is seen here now, let us say if it is E(s), so E(s) is nothing, but Es is equal to the difference of these two. So, I(s) minus. O(s) signal is getting multiplied here is coming here. So, O(s) into H(S). So, that comes here. So, that is getting multiplied, and G(S) may be written as O(s) by E(s). So, substituting this E(s) here. So, what I should see now and rearrange that will give me a direct transfer function that relates the output to the input. Output by input, as that is the equivalent transfer function of this, so that is equal to. So, I will put it here so that it is clearly visible to you. So, the transfer function can now be evaluated as G(S) by 1 plus I(S) into H(S). So this is known as a feedback transfer function. This is the system transfer function input and output. So, the equivalent transfer function this time is this: you need to remember this.

$$T(s) = \frac{G(s)}{1+ G(s).H(s)}$$

We will be using this now, so this is the reason I have discussed this in so much detail. So, yes, this is your G(S) by 1 plus G(S) into H(S). The system structure is very much like this. There are many other transfer function rules. I won't discuss it here. But, yes, this is how you can handle them in blocks. So, that is the beauty of arranging them in transfer functions. So yes, now let us come to our problem. Now, move here.

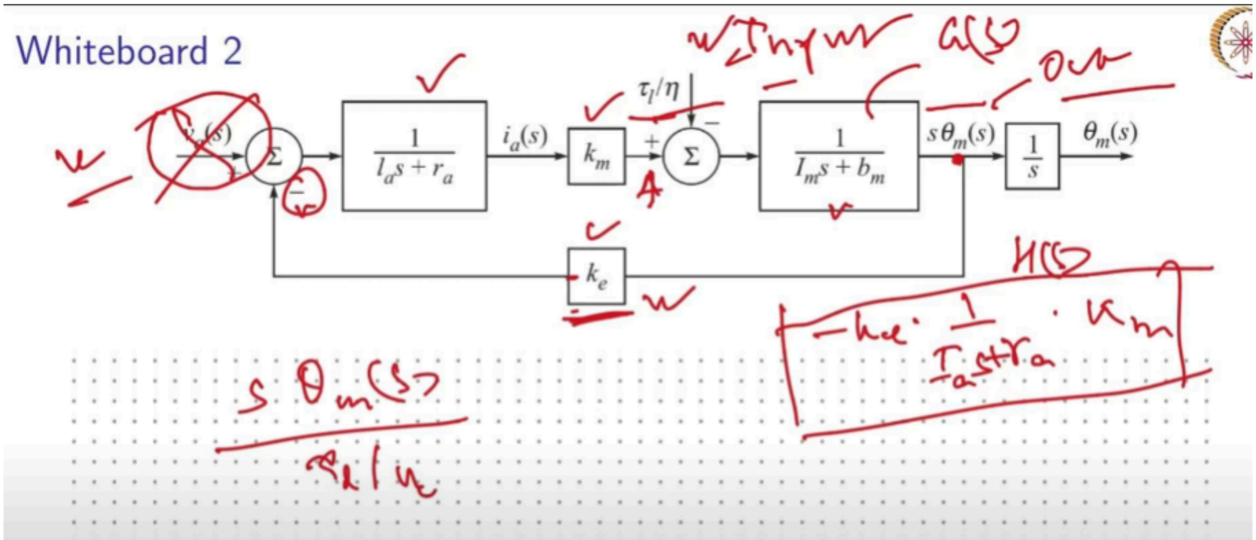


So yes, this is the first thing that I want to discuss. Let me come back to my slide also. So, this was there. You saw till here. I want to come to this now. How these two equations are holding true, and how they can be clubbed in this manner. They can be clubbed in this manner. So, that is what I am going to discuss now. So now, when this part is there directly, you have input here, you have output here, and let us forget there is any disturbance which is coming from the link side of it. So, if this is not there, I'll just write one by one what all system that comes in. So, yes, you see, if I leave this out for now, this, this, and this is in series so that you can draw equivalent transfer functions for all of these together as this. So, it is the product of 1, 2 & 3 terms that will come here. 1 into 2, into 3 will come here, and that is going out. You see, you have s theta ms ($s\boldsymbol{\theta}_m(s)$), which is written here, and you have k_e, which is here, and you have a block that is shown here. This is your input, was, that is the voltage from to the armature. So, that comes here. So, this looks very much like a closed-loop system with feedback as k_e, and this is coming here, and this is the whole. So, yes, the total transfer function for all of this system starts from here till. from here to here. The equivalent transfer function can be given by this, and that can be calculated as output. That is, s, theta, ms by input. Vas should be equal to is, so it is the product of all these three. So, it is 1 into 2, into the third term, by 1 plus 1 into 2, into 3, into k_e, got it? So I'll just make myself vanish a bit. So, this is what I am talking about. So, output by input should be equal to the product of the terms 1, 2 & 3, which are shown here, here and here divided by 1 plus G(s)H(S). So, that is k_e that comes here and this one that goes here and here. So, the output is this, and the input is this. So, this is the first thing that is there. Now, again, you see what? All equations that I want to put here? Okay, the first equation.

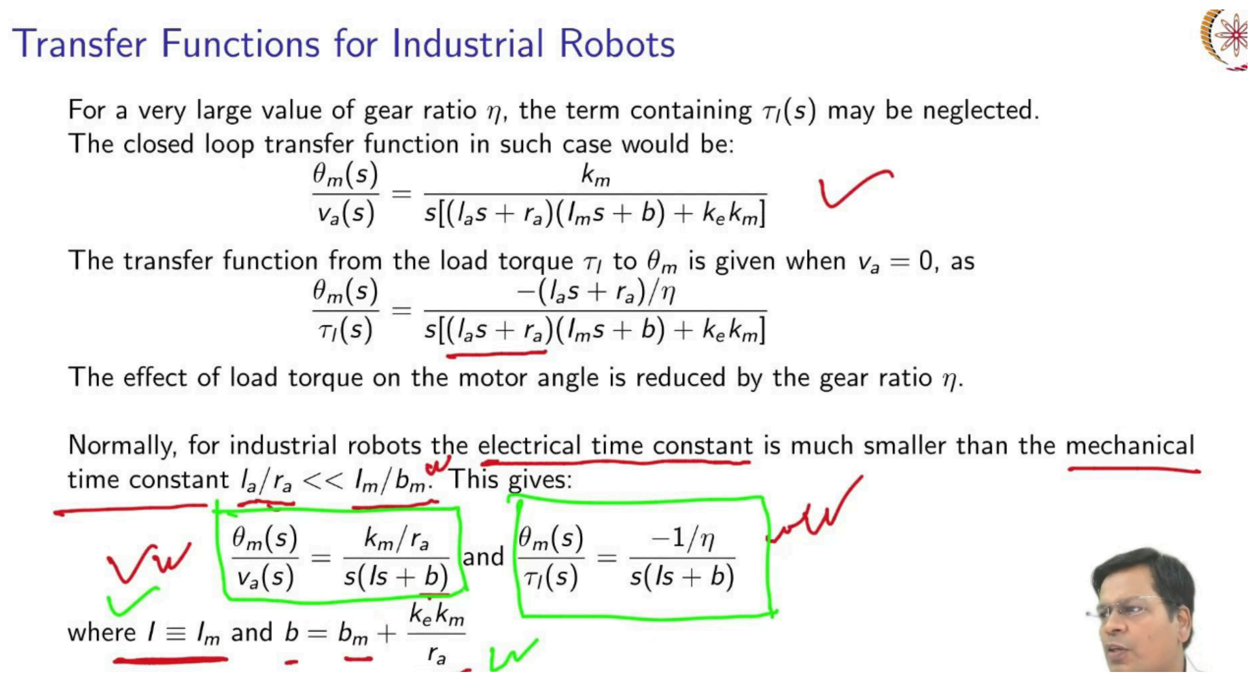


So, let me just come back to the previous slide. So, this is what was there. Just see if it is correct or not. First, let us try and get to this block diagram first. So, the first equation: where is it located in the transfer function system. So, this is Va(s) minus k_e s theta ms. So, s theta ms signal is coming to this feedback. So, this is getting multiplied with this. So, over here you see, you have k_e into s theta ms that comes here. So, this minus this is getting into this line. So, here you see, you have exactly this one. So, I'll just mark them here. So, this value is visible here. So, this into this gain. Now, the value which is here in this gain should be equal to this. So, that is what is written here. So, if you multiply the value which is here, that is here, into 1 by I_as ra. 1 by I_as ra gives me I_as, so that is Ial. So, you have understood how this equation is put to the variables which are lying in this line.

Now, let us see the model which is coming next. Now, km let me use another ink this time. So, km into I_as that is getting here, minus tau l eta, tau l by eta. So, km I_as minus tau l by eta. is now coming here. So, this signal is this one from here to here. So, this is into 1 by I_as plus by. So, this is here, and the term which is here is s theta ms. So, this into this should give me this. Got it stepwise, so I'll tell you once again. So, this into this should give me this. So, that is what is put here, and I have brought in this s which is inside to this. The s which is here. So, I put that inside this bracket. And I should see I_as square plus by s. So, that is what is here. So, these two equations are clearly put to this transfer function and the whole of this can now be compacted using the approach that I have discussed here. So, I will make a transfer function of all of this. So, in the first case, I am excluding the disturbance which is coming from outside. That is this term. Taking the product of all these three and creating a closed loop over here with this one as the system, electromechanical system. So, this is electrical, and this is the mechanical part of it. So, that is sitting here. This is your output of the closed loop that comes here, and this is your input. That is v_as this is your feedback. So, an equivalent transfer function can now be given as this one. That is this part.



Again, look at otherwise. This time, let us say it is driving by itself; that is, it is being driven by the load. So, in that case, you are not applying any external voltage. So, this is totally absent. So, in that case, load torque will be visible with the gear reduction ratio. It is coming onto the system, mind it. So, this is always there. So, this is the driving torque and the whole of the system is now getting driven by that. So, what? All things that go in series are s times of theta ms into Ke. So, this is your output. So, in the feedback, what all things are getting in. So, this becomes your input, and this becomes your output. So, the output is s theta ms. So, s theta ms by input, you have tau l by eta. That comes here. So, that is visible from here. Output by input: feedback is this time positive. So, if you apply 1 by 1 plus is by 1 plus is, HS, so HS is this. So, that is basically over here. It goes with a negative sign. So, you have to take the product of this coming to this and this. So, the whole of this is in series. Now, the whole of this is in series. So, the minus of Ke will be visible because you see a negative sign over here. So, Ke into. So, you have total terms which are there in the HS side of it. That is, the feedback side will be Ke into 1 by minus Ke into 1 by I_as ra plus ra into km. So, all these will be there as HS block. This is your input, and this is your system. This becomes your G(s). So, G(s) is here, if you remember. So, this is your G(s), and that is your output and input. So, now you can write the closed-loop transfer function, got it? So there are two approaches to it. First, the robot is getting driven by the input voltage, which is appearing over here, and in the second one, you have stopped giving any voltage, and the robot is now getting driven by the link weights that is the torque that is coming from the link side of it. It is maybe because of the external force, or due to the payload, or due to the weight of the links itself, so that is coming here.



So, there are two things, as I have said. This is the first one. When you see, there is motor displacement due to the voltage that is there, and this is the forward transfer function that is, the motor is driving your system. In this case, for a very large value of gear ratio eta, the term containing tau l (s) may be neglected. That is what is neglected, as I have said. So, anything that is coming from here is now neglected. So, that is neglected, and the closed-loop transfer function is given by this.

$$\frac{\Theta_m(s)}{v_a(s)} = \frac{k_m}{s[(I_a s+r_a)(I_m s+b)+k_e k_m]}$$

This is the first case. In the second case, you don't apply any voltage, Va becomes equal to 0, and it is getting driven by the load torque itself. In that case, this is your transfer function.

$$\frac{\Theta_m(s)}{\tau_1(s)} = \frac{-(I_a s+r_a)/\eta}{s[(I_a s+r_a)(I_m s+b)+k_e k_m]}$$

Ke and Km, you see, both were appearing in the product, so that is visible here, and that is this one. So, the effect of load torque on the motor angle is now reduced by the gear ratio eta. So, that is always there. This, this we have seen earlier. So, normally, for industrial robots, the electrical time constant is very much smaller as compared to the mechanical time constant. Time constants are the terms which are written here. la by ra is the electrical time constant. It is very, very smaller than I_m by b_m. So, that is your moment of inertia by damping coefficient. So, that is very, very high, significant as compared to this. So, I can ignore that. Dividing the whole of this by ra and ignoring the terms that contain la by ra, you get to this and this from the second one. So, this is from the first one, and this is from the second one. Got it, okay. When I moment of inertia is only Im, b is, which is here is the sum of moment of inertia. It is the sum of bm and Ke km by ra. So, those terms, if you substitute, you can get to a very simple term which is like this. So, I will just highlight them. So, the first equation is this.

$$\frac{\Theta_m(s)}{v_a(s)} = \frac{k_m/r_a}{s(Is+b)}$$

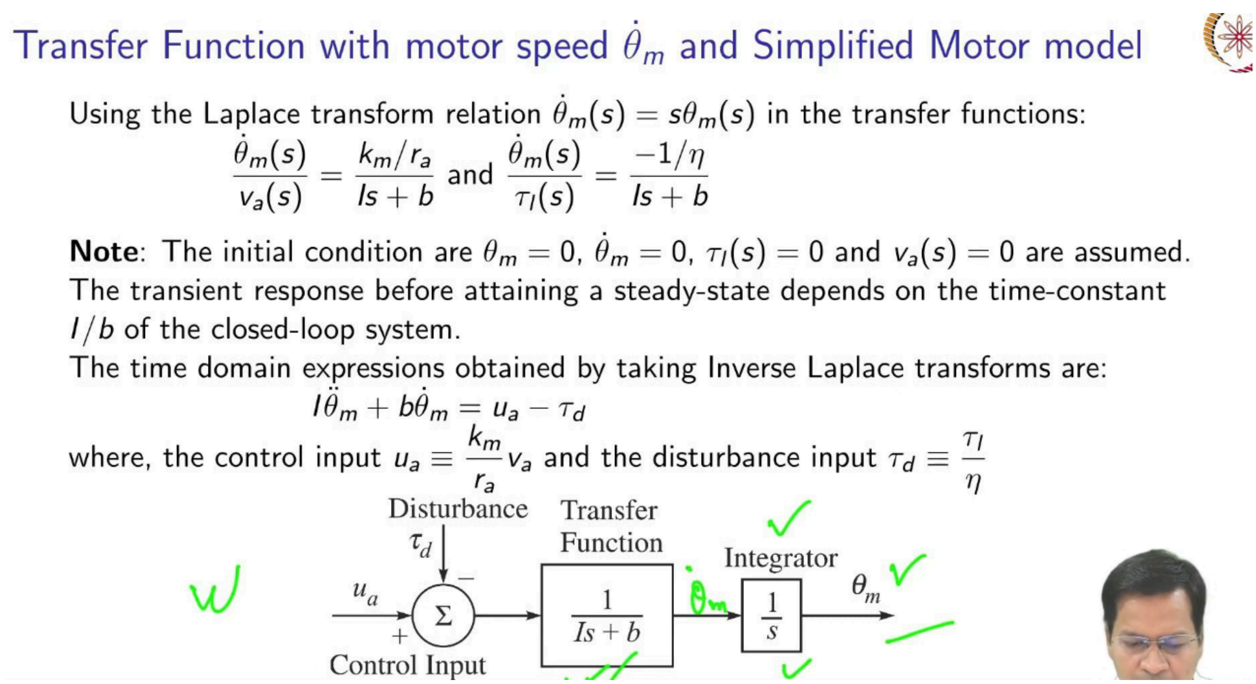
The next equation is this.

$$\frac{\Theta_m(s)}{\tau_1\ (s)} = \frac{-1/\eta}{s(Is+b)}$$

Where I is I_m and b is this,

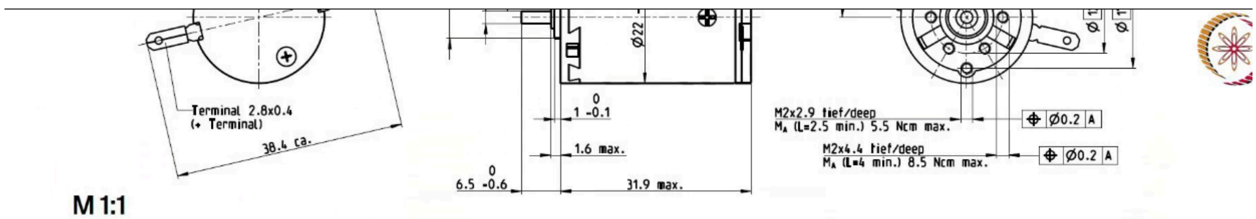
$$b = b_m + \frac{k_e k_m}{r_a}$$

This becomes your very fundamental equation for any industrial robot. This is your transfer function in both cases when you're driving using the voltage and you are getting driven by the load and not applying any voltage to the motor. So, these are the two transfer functions.



Now, using the Laplace transform of these two transfer functions that we have obtained, this one and this one. So, I can get to this. In the transfer function, taking initial conditions as 0, all these initial conditions are assumed to be 0, and this is your relation. So, the transient response before attaining a steady state depends on the time constant. I by b of the closed loop. That is basically the part of the equation which is there over here. That becomes your characteristics equation for this transfer function that is in the denominator. So, I by b. Basically, if you take b, common, that gives you the root and that only governs your system that is the input and output relation. So, time domain expression for this may be obtained by taking inverse Laplace. You know you can obtain the time dimension solution. So, this is your differential equation for that, and you now have u_a minus t_d. What are those? t_d is the disturbance input that is from the load side of it. I see this load as a disturbance. This can have a payload, this can have a supplementary load, or it can have some sort of external force such as noise. So, that is now my disturbance force. This is your control input, which is there. That is what should be desired by the voltage which is here and this is your system which is getting driven. So, now effective inertia I, you know already what was that and the b terms. So, that was written here. So, I and b are here. So, that is what is here, and now you can write your system in a very simple form like this. So, here you were, getting theta m dot. So, if I integrate it, that is 1 by s. that, as a Laplace integrator, can be written as 1 by s. So, it should give me an angle. So, this gives me my angle. This is my transfer function of the system. Inertia and damping are present here. This is your disturbance, and you have control input u, which is shown over here. So, this is your u, which is here. So, yes, this is your system there. That is very, very simple. It becomes a single input, single output system. You have ignored the load that is coming due to the torque, that is coming due to the load side of it.

So now, let us take one small example by taking a data sheet of the Maxon motor.

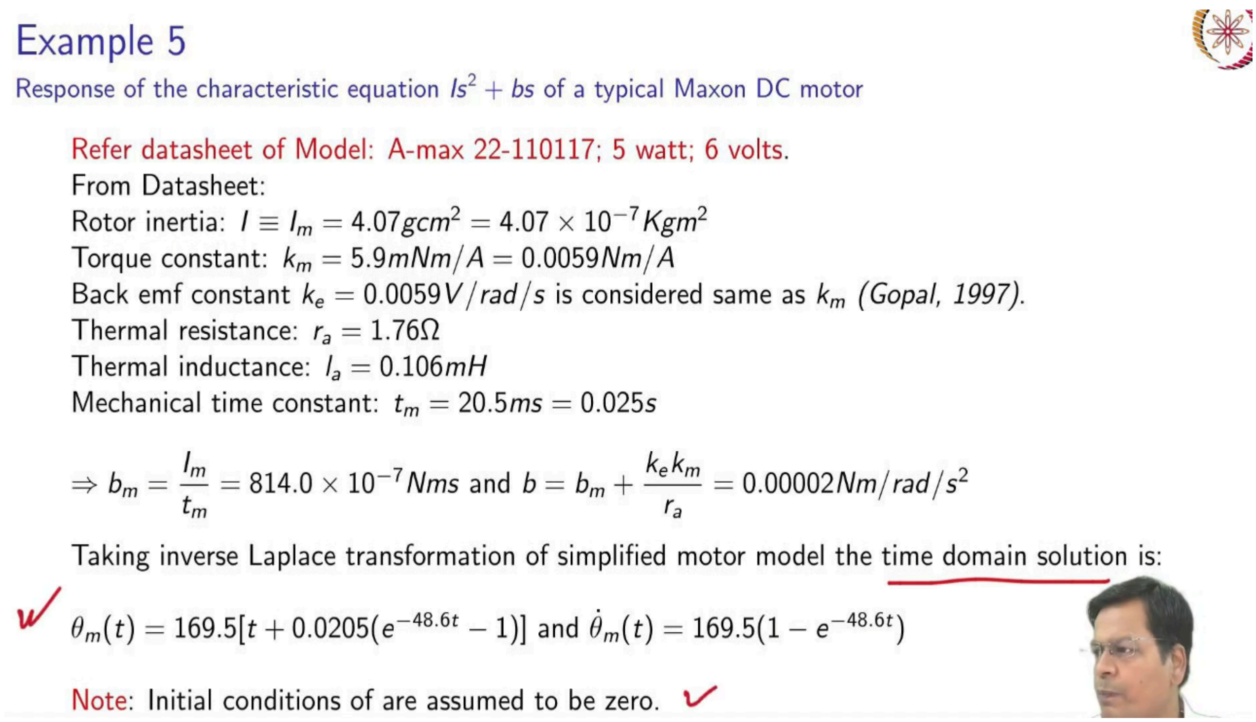


M 1:1

Stock program
Standard program
Special program (on request)

Part Numbers		1	2	3	4	5	6	7	8	9	10	11	12
with terminals		110117	110119	110120	110121	110122	110123	110124	110125	110126	110127	110128	110129
with cables		139838	218799	238798	202413	258367	137255	134267	134666	267423	137476	310003	342390
Motor Data													
Values at nominal voltage													
1 Nominal voltage	V	6	9	9	12	12	15	18	24	30	36	48	48
2 No load speed	rpm	9630	9970	8760	10400	9400	10300	9970	10700	10800	9800	9280	8370
3 No load current	mA	29.5	20.8	16.8	16.8	14.2	13.1	10.4	8.81	7.18	5.06	3.47	2.93
4 Nominal speed	rpm	7390	7300	6100	7770	6700	7530	7220	7970	8070	7000	6420	5520
5 Nominal torque (max. continuous torque)	mNm	4.81	6.22	6.3	6.24	6.18	6.1	6.05	6.02	5.98	5.94	5.83	5.9
6 Nominal current (max. continuous current)	A	0.84	0.745	0.661	0.586	0.523	0.451	0.362	0.291	0.234	0.175	0.122	0.111
7 Stall torque	mNm	20.1	22.9	20.5	24.3	21.4	22.9	22	23.5	23.5	20.8	19	17.4
8 Stall current	A	3.42	2.68	2.11	2.23	1.77	1.65	1.28	1.11	0.894	0.599	0.387	0.32
9 Max. efficiency	%	83	84	83	84	83	83	83	83	83	83	82	82
Characteristics													
10 Terminal resistance	Ω	1.76	3.36	4.27	5.39	6.78	9.07	14	21.6	33.5	60.1	124	150
11 Terminal inductance	mH	0.106	0.222	0.288	0.362	0.445	0.584	0.89	1.37	2.1	3.68	7.29	8.95
12 Torque constant	mNm/A	5.9	8.55	9.73	10.9	12.1	13.9	17.1	21.2	26.2	34.8	48.9	54.3
13 Speed constant	rpm/V	1620	1120	981	875	790	689	558	450	364	274	195	176
14 Speed / torque gradient	rpm/mNm	482	438	430	432	443	451	458	459	465	474	494	486
15 Mechanical time constant	ms	20.5	19.8	19.7	19.7	19.8	20.2	20.1	20.2	20.3	20.3	20.5	20.4
16 Rotor inertia	gcm^2	4.07	4.32	4.37	4.36	4.26	4.27	4.2	4.2	4.16	4.09	3.97	4.01

So, let me just show you that datasheet first. So, this is the data sheet that I am using now. So, you see, there are many motors which are there. The parameters that we have just discussed are mentioned here against different motors which are here. So, I will be talking about the motor, which is 110117. Let me see where it is. So, it is the first one, 110117. So, it has various constants, which are mentioned here. The specific one which I will be using is the moment of inertia of the shaft. Where is it? So you see, the time constant is here. Torque constant km is here. Terminal resistance. Rotor inertia is the moment of inertia, but you see, you have to mind the units which are here. Convert it to the appropriate SI system if you are working with that. Terminal inductance and resistance: these are inductance, l, and r_a is the resistance torque constant km speed constant. So, you see, this is the data sheet that I am using. That is for the motor, which is manufactured by Maxon.



Now, let me come back to my slide. So, this is it. So, I have just copied all the values converted to appropriate units. Torque constant, back, EMF constant: all these parameters were there. You can just come back and see that. So, this is considered based on the torque constant, which is given in Goal's book, that is, the controls book and r_a. If it is not given, you can take some standard value. Thermal resistance: that is the resistance only. Inductance is here. Mechanical time constant: Tm is here. This is the mechanical time constant, so you know it is 20.5 milliseconds, which was there. So, now let me use this, so it is. bm is nothing, but I am by the mechanical time constant. So, the time constant is I buy b, you know that. So, bm, you can obtain bm, which is the damping equivalent. Damping like this: and total b will be bm plus Ke, Km by r_a. That is what you have derived just now. So, that is calculated like this. So, Ke and Km are here. And resistance is known, so the total viscous coefficient is here. So, now, taking inverse Laplace of the simplified motor model that we have seen just now in the time domain solution, it can be written as this:

$$\theta_m(t) = 169.5[t + 0.0205(e^{-48.6t} - 1)] \text{ and } \dot{\theta}_m(t) = 169.5(1 - e^{-48.6t})$$

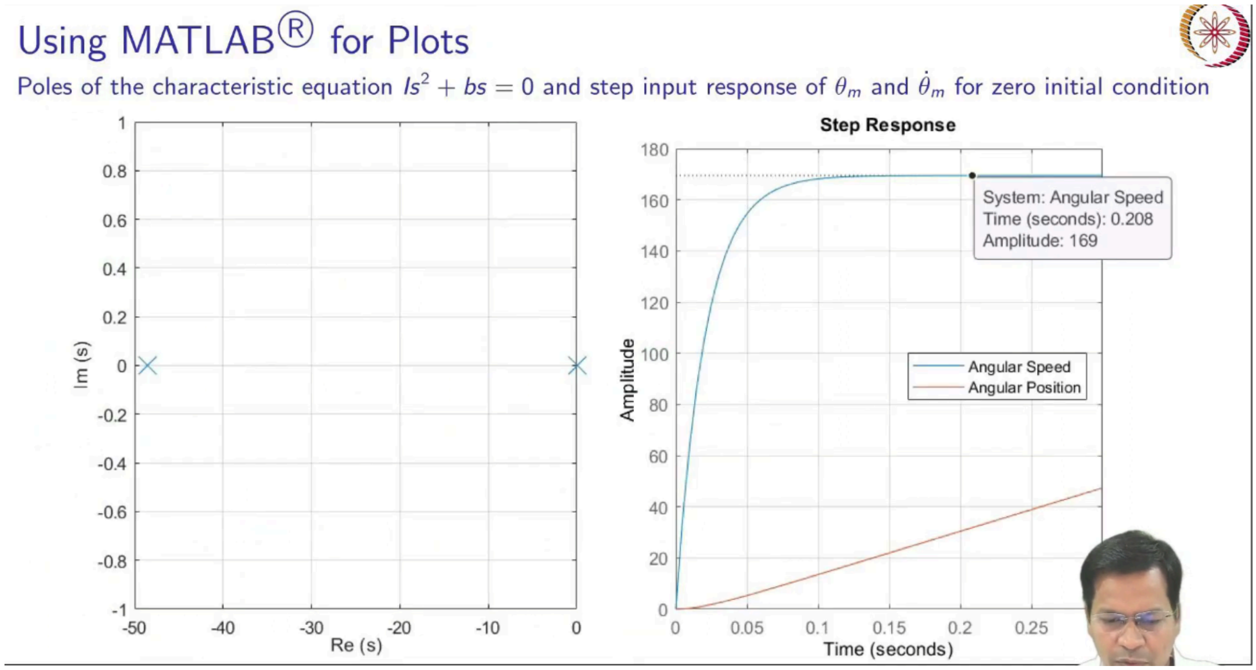
Initial conditions are assumed as zero when you take inverse Laplace. So, this is the behaviour of your system. You can just plot this, and you can see.

Demonstration: MATLAB® code

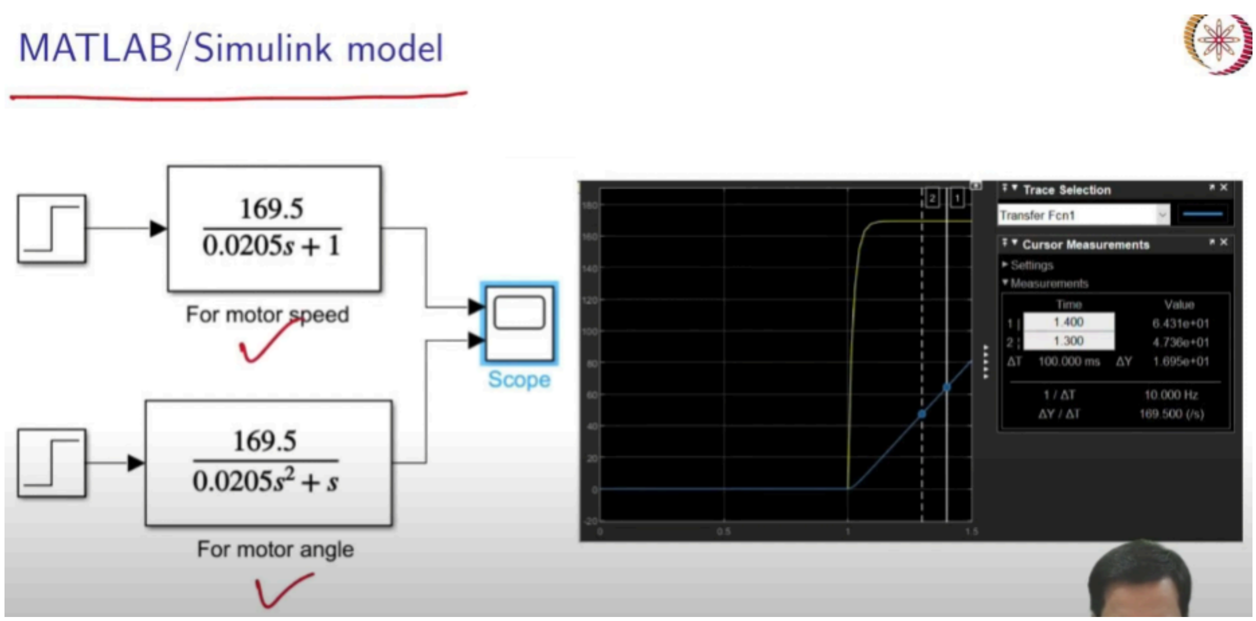
```
%% Response of a DC Motor
%Defining the parameters
t=0:0.005:0.3; ten7 = 10000000; i_m=4.07/ten7; k_m=5.9/1000;
k_e=k_m; la=0.106/1000; ra=1.76; i_eff=i_m; tau_mech = 20.5/1000;
b_m=i_m/10*tau_mech; b_eff=b_m+k_e*k_m/ra;
%defining the transfer functions
num=[k_m/ra];
den_w=[i_eff b_eff]; % for speed
den = [i_eff b_eff 0]; % for shaft angle
%Plotting the roots
p = roots(den); pr=real(p);pim=imag(p);
subplot(1,2,1),plot(pr,pim,'X','MarkerSize',15)
xlabel('Re (s)'); ylabel('Im (s)'); grid on;
%Plotting the step response
sys_w=tf(num,den_w); sys = tf(num,den);
subplot(1,2,2), step(sys_w,sys,t); grid on;
```

Note: The response can also be plotted using time-domain solution.

So, using MATLAB, you can define the systems. All the constants are defined to be effective. bm is here. Everything that we did analytical just now, we can put it to MATLAB and get the things done. So, the numerator has the coefficient, which is like this: okay, Km by ra, the denominator is for the speed. You have a denominator that is given by the polynomial I and b. Similarly, the denominator for the shaft angle is Ib, and 0 was there. So, these are there. So, you have roots of the denominator which is here. That is for the soft angle I am storing here. Real and imaginary parts are segregated here, and I have done the plots for real and imaginary, that is, the poles plotting I have done. Again, for the Omega part, that is d theta by dt part, that is theta dot part, you have numerator, you have denominator which is given by for the speed it is given by this. You have created your system using the transfer function with a numerator and denominator like this and plotted the step input of that. So, this time, it is step input for your system for speed and system for angle against time. Time is varying from 0 to 0.3 seconds only. This is your step, and let us see what it looks like.



So, you have poles given by one of them will be here. Other ways here. Both are on the real number line towards the negative side of the complex plane, and you see you have for system angular speed, it goes something like this, and in case of angular position, as soon as you apply the voltage, it gradually picks up and keeps on changing, almost like linear. So, it goes like this: and in the case of angular speed, it picks up and goes there. So, this is how you can analyse any motor using your standard analytical, analytical techniques like the transfer function method I have used. I have drawn the poles and saw everything is okay, it is not unstable and all. So, it is done.



Using MATLAB Simulink also, you can define your system, motor speed and the motor angle transfer function, put the step inputs and plot them here, and you can study it this way also, without getting into programming. You can do this, so, but this one is very well convertible to the Python code also, so that can be done and yes.

So, that's all for today. So, you see, we have now studied the way how transfer functions and Laplace transforms. Those tools can be used to study the robot joint, you know. So, your robot joint, you use. How does it behave? It is basically. I have considered the DC motor model here, but you know the fundamental principle tau is equal to km into I is the same because the conductors are lying in the same manner, even in synchronous motors. So, the motor equation should not vary much. There are other parameters also which can go a little different, but yes, the analysis should be somewhere around this. So, okay, that's all for today. Thanks a lot.