NPTEL Online Certification Courses
Industrial Robotics: Theories for Implementation
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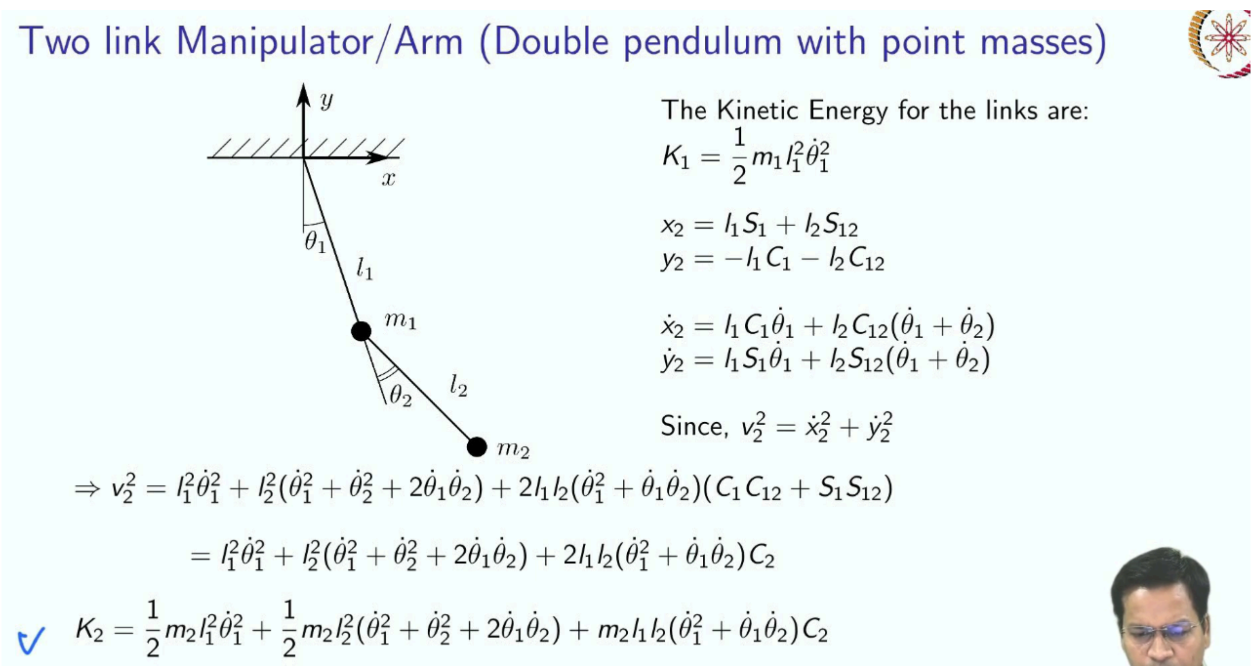
Week: 09
Lecture 37

Equation of Motion (EOM) for a Two-Link Manipulator using LE

Welcome back to the module Robot Dynamics. In the last class, we started by deriving a dynamic equation of motion using Lagrange-Euler equations. I gave you the example of one degree of freedom system, including an oscillating spring-mass system. Today, we will continue further with Lagrange-Euler formulation and do a two-degree-of-freedom planar manipulator, just like a pendulum treatment, the way we did it in the last class, and do some interpretation of any dynamic equation of motion. So, let us continue with today's class.

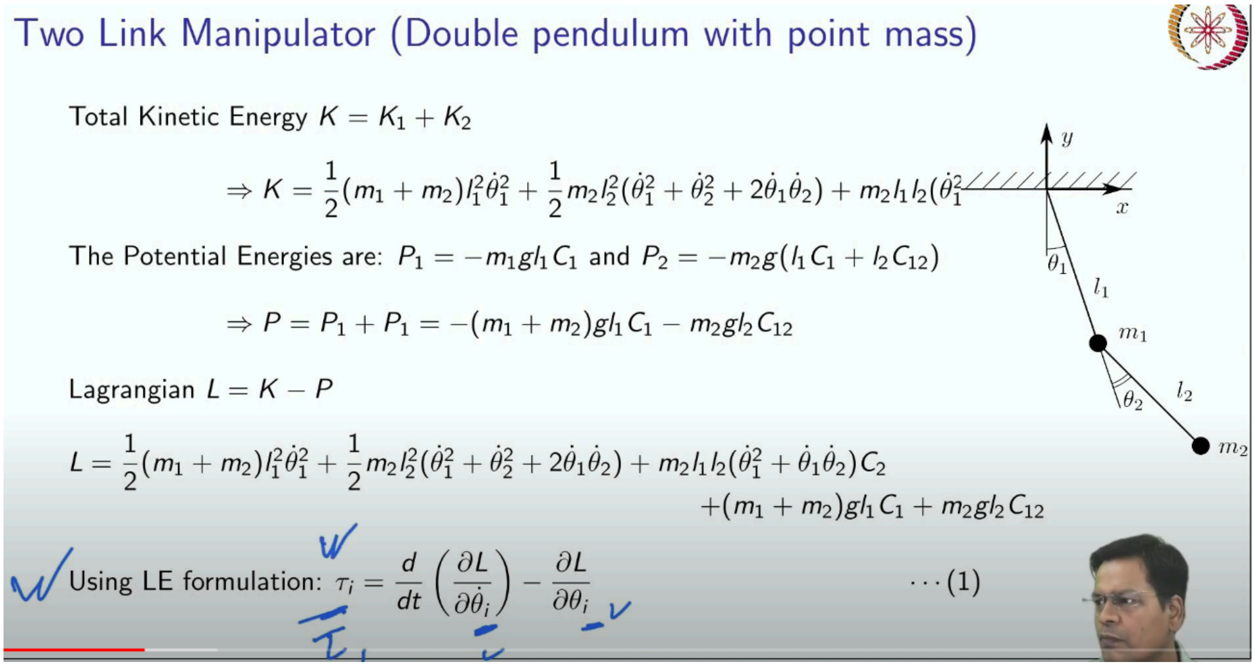


So here is my system, which is exactly similar to a double pendulum. It's a point mass system again. So, you see it has a bob which is hanging here. That is the first link, link 1, ending with a point mass, which is here, and then you have a joint over here. You have a joint over here. So, two joints, two links. So, the second joint is here l2. That also ends with a bob, which is here, of mass m2. This one was of mass m1. So, the relative angle you see with the frame: first link substance and angle with the frame from the vertical, theta 1, second joint angle is theta 2, that is the relative angle you know. So, exactly in a similar manner, you can have a two-link manipulator or an arm also if it is a planar manipulator. It is exactly similar and similarly referenced also, but they are powered at the joint. In the case of a double pendulum, it is not powered. So, we assume it is powered, and we start our derivation using the Lagrange-Euler formulation. Later on, we may make torque equal to zero, and we can get the dynamic equation of motion of a simple oscillating pendulum like this. This becomes a double pendulum in that case. So, yes, let us begin. So, we will have to find out kinetic energy and potential energy in that case also. So, the kinetic energy of the links is that there are two links. So, the first link is exactly similar if you just exclude the second link from here. So, what you see is exactly similar to the way we did it in the last class. It was a simple pendulum. So, for the second part of the link, that is the bottom link. So, that it can be written as x2. x2 is nothing but a displacement of point mass m2 along the x-axis, so that becomes x2. So, it is l1 sine theta 1. So, it is an acronym. S1 is sine theta 1. You assume you remember the way we did it in our forward and inverse kinematics classes. So, exactly in a similar way. So, it is l1 sine theta one plus l2, sine theta 1 plus theta 2, so this is your theta 1. So, effectively, this is your sine theta 1 plus theta 2, so it is that. So, from here to here, it is l1, s1. So, the total is this.

Similarly, y2 can be written as a cosine component, but it is directed downwards, opposite to the positive y direction, so it is negative over here. So, it is l1, c1 plus l2, c12, so downward it is negative. So, x and y, that is, the coordinates along x, coordinates along y. So, taking the derivative of that, I get the velocity directly. Velocity along x, and velocity along y can be written like this.

$$\dot{x}_2 = l_1 C_1 \dot{\theta}_1 + l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y}_2 = l_1 S_1 \dot{\theta}_1 + l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

It is nothing but the derivative of this squaring and adding. I get the resultant velocity of this, that is, v2. That is x velocity square plus y velocity. The square root of that should give me v2. So, v2 is this. You can halt it here, and you can just look at this, or you can derive it using your own pen and paper and use this to derive this so that you can be very much conversant with these types of systems if you do all alone to any other system. So, this is your velocity. So, putting that over here in kinetic energy form, so it is the kinetic energy of the second link, the second mass system should be 1 by 2 m2 v2 square, 1 by 2 m2 v2 square. So, that includes everything from here to here when we take v2 square. So, combined can? It can be written like this: so you have the kinetic energy of the first link and the mass. This is the kinetic energy of the second link and the mass.



So, total kinetic energy will be the sum of those two. That is k1 plus k2.

$$K = K_1 + K_2$$

So, that gives you this. So, you got the kinetic energy. I have taken some common in the bracket also so that it looks quite compact. So, now, taking the potential energy, potential energy can be calculated as this and this. Let me just show you the figure once again. So, this is how it was. So, you see, the projection along negative y should be: for the first link, it is m1g l1 cos theta 1, and for the second link, it is the sum of l1 and l2 projected towards the negative of the y-axis, and that becomes your potential energy. So, it is negative potential energy if the top one is the reference frame; that is, the origin is over here where the x and y-axis intersect. So, you have negative potential energy with respect to that. So, if you give, you get to this. So, this is your potential energy, and kinetic energy is already there. So, now you can quickly obtain your Lagrangian. Lagrangian is kinetic energy minus potential energy.

$$\text{Lagrangian } L = K - P$$

So, this minus this should give you this. So, now that you have Lagrangian, you can quickly use the Lagrange LE formulation to obtain the torque at each joint. This time it is two degrees of freedom. So, when we want to obtain the first torque at the joint one, you have to use theta one over here, and you get what is tau one, you get tau one. So, in case you want to obtain the torque at joint two, it should be theta two, which is here. Theta two should be used here. That is the joint angle, which is there, and you get what you get: tau two, you get what you get, exactly the torque at the joint two. So, that is how it is, and then I will just use one of them one by one now.

Two Link Manipulator (Double pendulum with point mass)

Lagrangian $L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)C_2$

For the first joint variable:

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2m_2 l_1 l_2 C_2\dot{\theta}_1 + m_2 l_1 l_2 C_2\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2 - 2m_2 l_1 l_2 S_2\dot{\theta}_1\dot{\theta}_2 - m_2 l_1 l_2 S_2\dot{\theta}_2^2$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2)g l_1 S_1 - m_2 g l_2 S_{12}$$

From equation (1): $\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$

$$\tau_1 = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2 - 2m_2 l_1 l_2 S_2\dot{\theta}_1\dot{\theta}_2 - m_2 l_1 l_2 S_2\dot{\theta}_2^2 + (m_1 + m_2)g l_1 S_1 + m_2 g l_2 S_{12}$$

So starting with Lagrangian here and for the first joint variable, first joint variable using the Lagrange-Euler formulation. So, you know that all that was tau L by tau theta one dot, so it was against velocity. You can just see it here once again. So, you should use this. So, going one by one. So, I have taken a derivative of this, a partial derivative of this, with theta one dot, and I obtained this.

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + 2m_2 l_1 l_2 C_2\dot{\theta}_1 + m_2 l_1 l_2 C_2\dot{\theta}_2$$

You also can try doing it and now the time derivative of this is to be done. So, the time derivative of this. Again, using the first equation, I directly obtain this.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2$$

So, this is what I obtained. That is the first part of the Lagrange-Euler formulation.

Now, the second part. The second part is doL by dotheta dot. So, it is directly the joint angle variable, so quickly everything becomes constant wherever you see theta dot, so wherever you have only theta, it is only this. So, only this derivative is included here. So, it gives me this one directly: the first part of the Lagrange-Euler and the second part of the Lagrange-Euler formulation. So, you have to take this minus this, that gives you Tau one. So, tau one is using this. Tau one is the whole of this. Minus this, it gives you this. Minus this, it gives you this. So, this is your torque at joint one, or this is the torque which is required to drive joint one.

Two Link Manipulator (Double pendulum with point mass)

Lagrangian $L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)C_2 + (m_1 + m_2)gl_1C_1 + m_2 g l_2 C_{12}$

Similarly, for second joint:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 C_2 \dot{\theta}_1$$

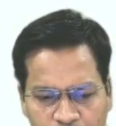
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 S_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) - m_2 g l_2 S_{12}$$

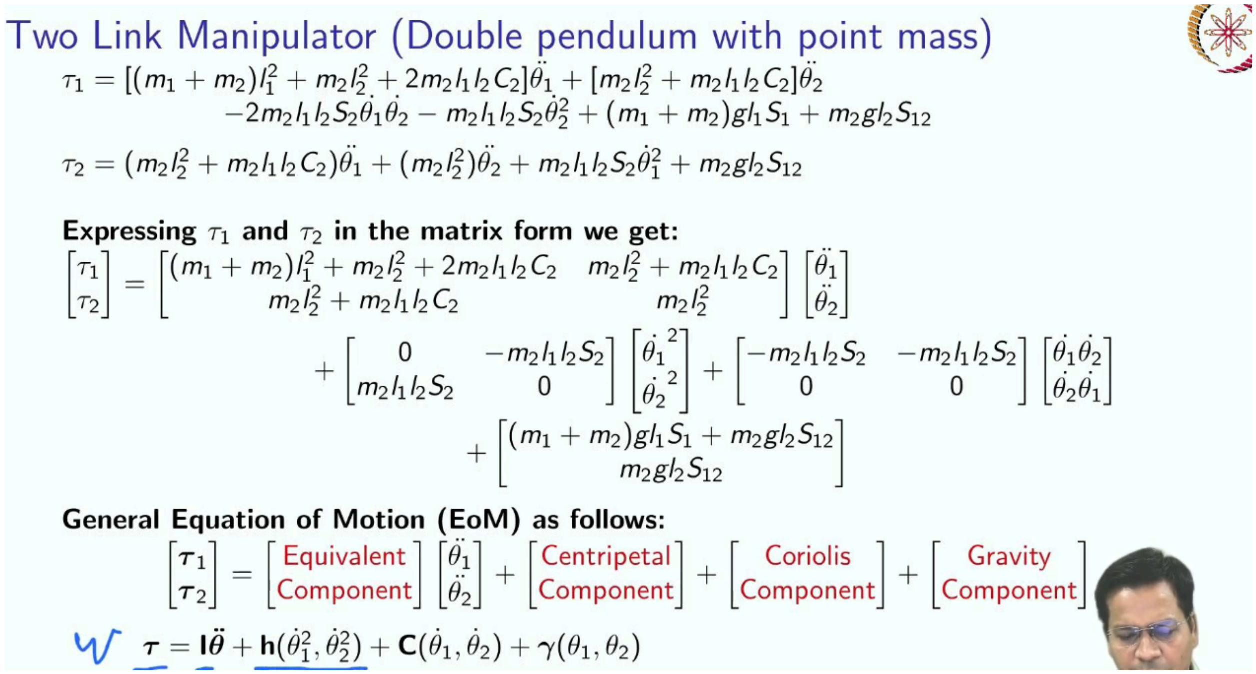
From equation (1): $\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$

$$\tau_2 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_1\dot{\theta}_2 + m_2 l_1 l_2 S_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) + m_2 g l_2 S_{12}$$

$$= (m_2 l_2^2 + m_2 l_1 l_2 C_2)\ddot{\theta}_1 + (m_2 l_2^2)\ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + m_2 g l_2 S_{12}$$

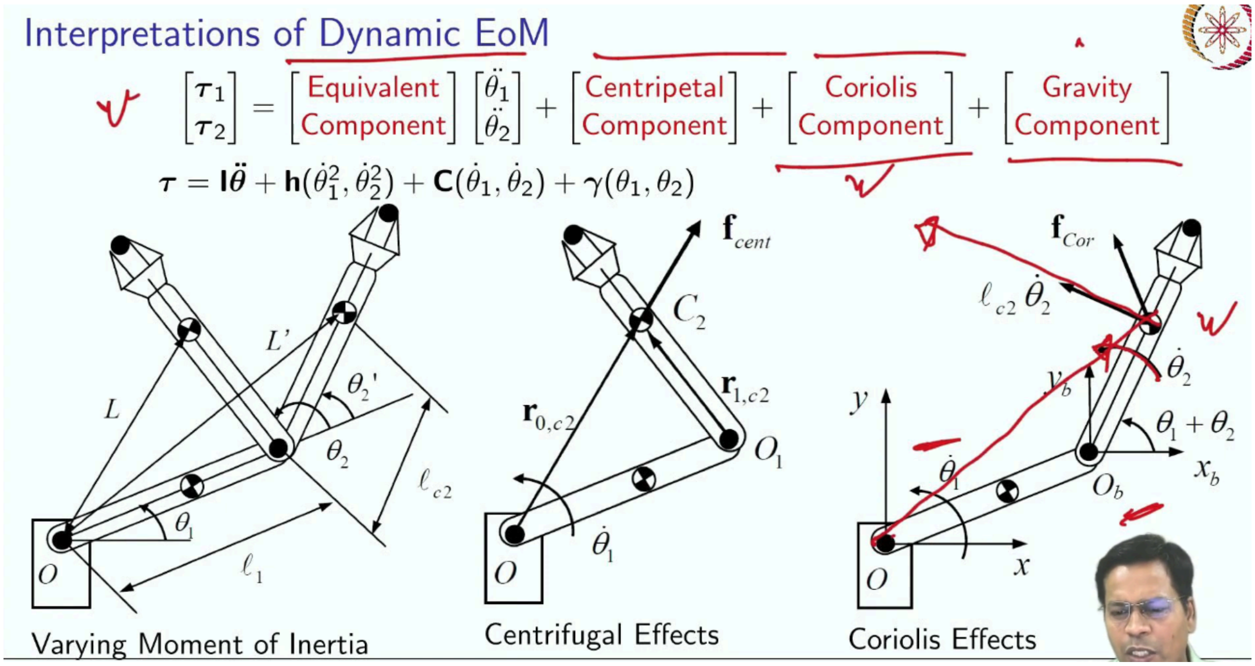


In a similar way, I will start here once again for the second link and the second joint. So, again, you have to take the derivative of this Lagrangian with respect to joint velocity and second joint velocity. So, again, wherever you see velocity, that is to be included. Only the two are to be included. Rest all becomes constant. So, using that, I quickly can obtain this again. The time derivative of this gives me this. So, this becomes the first part of the Lagrange-Euler formulation once again, and the second part is the derivative of Lagrangian with respect to the joint angle. It gives me this: only the last part has got the variable that restt, everything has got velocity, so those are treated as constant. So, that comes here directly. So, now using these two, this and this, so this and this, I can complete my dynamic equation of motion that is the Lagrange-Euler formulation. So, using this once again, this minus this, it gives me tau two is equal to the whole of this minus this. So, finally, I arrived at this. So, this is your torque two.



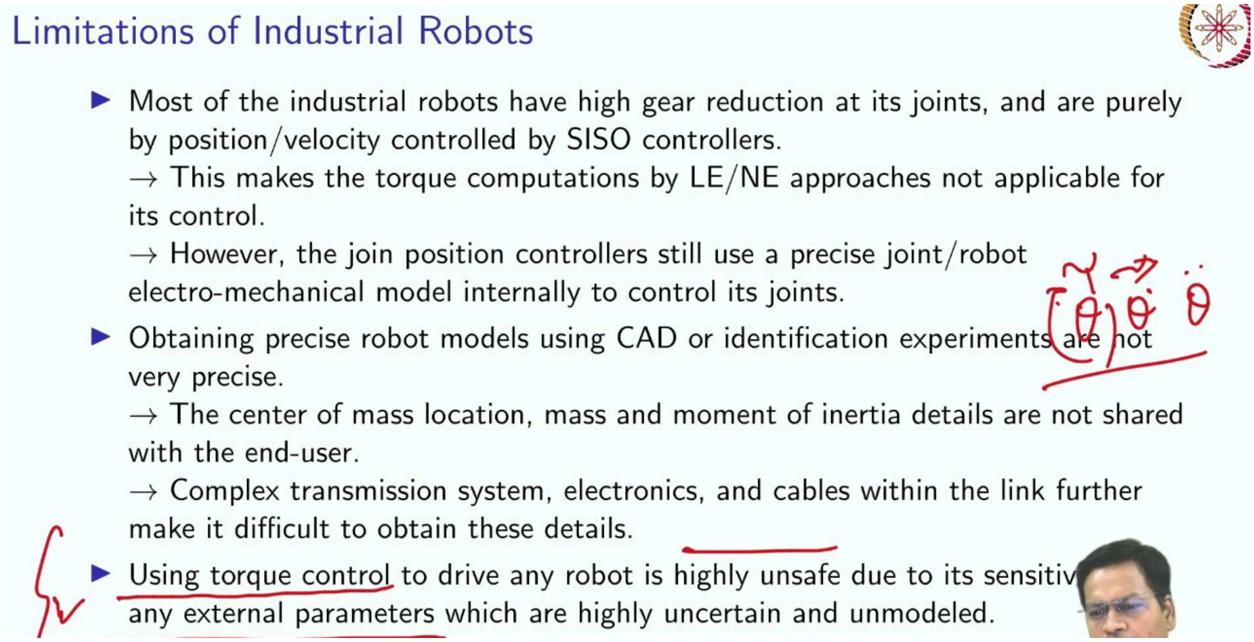
So using both, using both the torques, writing them together here. So, this is tau one, this is tau two. So, torque at joint one, torque at joint two, in order to obtain the velocities which are here: theta one, theta one dot, theta one double dot, and similarly for theta two, so it should have this much torque that is required to get to those velocities and accelerations. So, this is torque one and torque two. So, this is just a scalar representation. Let me put it in a compact way. So, if you make it, you can write it in matrix form. So, tau one and tau two are clubbed here, and wherever you see theta one double dot and theta two double dots, those are the joint acceleration terms. So, they can be written as this into two cross two matrix forms, which is this: which is this? Again, you have joint and rate terms, which is this one: this is theta one dot square, theta two dot square. That is pure single joint terms. So, it can be again written as two cross-two matrices. That is this one and the column. You can write this. The third term is a product of two angular velocities, that is, theta one dot and theta two dots. So, that can be again written as two cross-two matrices into the product of angular velocities. That goes here, and finally, the last term doesn't have any acceleration. The velocity term at the joint is just the torque, which is due to the gravity component. You see, only g are visible. So, those are the four terms which are major terms of any joint torque. provided you don't have any external forces which are acting on the robot. So, this is the dynamic equation of motion for the robot. So, this, in general, can be written as not necessarily for a two-degree-of-freedom robot. In general, it is like this. So, torque will go here, all the torques. you have an equivalent component which is similar to your moment of inertia term. This is known as the Generalised Inertia Matrix, GIM, and this is your acceleration term. So, all the acceleration is in the vertical column. It can be written as a matrix like this. So, it is having all the inertia terms Joint accelerations will go here. Next part: you see, it has theta one dot square, theta one dot square and theta two dot square, so it is mr omega square, similar to that. So, it is known as a centripetal component, it is known as a centripetal component. So, it is.

this is the torque which is generated by the centrifugal forces. Centrifugal forces which are created due to the motion of the links. So, all the torques, they come here. So, it is known as the centripetal component. So, that is this one. Now, the third part of this torque. So, what is that? You see, it is a product of angular velocities. Theta one dot and theta two dots, or theta two dots and theta one dot. Both are equivalent. So, i omega one into omega two. You see, you have I term moment of inertia, similar to this. So, you have exactly the way it is in any Coriolis torque component. If you have seen. So, it is. this term is known as the Coriolis component. So, it has the product of angular velocities, and the last term is nothing, but we have already derived it earlier using statics, in the module statics. We have already derived this. That is the torque which is due to the gravity. If you make the whole of joint velocities and acceleration, that is equal to zero. That means if the robot is at rest. So, this is the torque which is going to come at the joint. So, this is exactly the same as what we have derived earlier in module statics, that is torque compensation, or compensation, gravity compensation, we call it. So, it is exactly that. So, these are the four major terms which form any dynamic equation of motion for any robot. So, the only thing is that the size of these matrices will change. So, this should be n by n; that is, the number of degrees of freedom cross the number of degrees of freedom. That is here. Similar should be this and this: the last one is n cross one, and torque is also n cross one. So, this is a general equation of motion. So, this is the. Let us just interpret. How does it look like if it is put into a physical system? So, in compact notation, it can also be written as so tau is equal to i. This is a generalised inertia matrix. Theta double dot h, theta one dot square, theta two dot square. It is a function of those two. So, that is the centripetal component, and this one is the Coriolis component. So, that is theta one dot into theta.Two dots, so it is that term which is there, so it has all the terms which have this, and the final one is nothing, but it is just the function of joint angles. So, all of them are dynamic in nature. They keep changing with the pose of different velocities it changes. So, even if it is static, at different angles you see different torques. So, that is dynamic in nature.



So now let us put all these two physical robots. So, this is the first term. So, this is your equivalent component to the moment of inertia. So, that comes here. So, you see, at each and every state of the robot for a given joint angle, it has its own structure. So, if it is fixed, if you freeze this joint angle, the robot looks like this, or the robot looks like this: at different angles, it will have a different shape. So, if at all you want to analyse what the torque is over here, it treats the whole of the body frozen. So, the inertia component will have a value over here that treats, that basically has the moment of inertia, which is treating the rest of the robot frozen, so that term will come here. So, this term will assume the whole of the robot is frozen and that gets multiplied by the term theta one double dot. I theta double dot. So, this is theta one double dot. This is the moment of inertia, considering the whole of the robot is frozen, only the joint one is moving, so that term comes here. Similarly, the last term, which is here, will assume there is no other link after that. So, that is the only joint that exists, the joint, last joint which is there. So, that gets multiplied by the last joint, acceleration. This is the moment of inertia of the last link only. So, all the intermediate terms, the diagonal terms, are moment of inertia, primary moment of inertia, and principal moment of inertia about the link itself. And considering the links which are after this, the last joint is frozen, so that comes here. And all the moments of inertia should be represented in a single frame. We prefer putting them in the first frame, which is attached to the ground. So, using, the kinetic energy, using the potential energy. Everything should be calculated with respect to the frame which is attached to the ground. So, you already know how the moment of inertia can be transferred to any frame. So, it is Q, i, Q transpose. So, it is. It is exactly in a similar way. You can calculate it to convert. If it is in the moment of inertia normally is expressed in its own frame. So, it is. It can be transferred to the fixed frame using the transformation matrix Q. That is nothing but it. You can extract it from the forward kinematic transformation so that that helps you to transfer it to the fixed frame. That is attached to the ground. So, that is what you can do. And also potential energy. You have to use vector-matrix form to convert it all to the fixed frame. So, and then you can do all this. So, kinetic energy, potential energy, everything can be converted to the root frame, and then you have to calculate them and take the Lagrangian differential of that and find out the torques. So, this is how it is done. So, this is the physical meaning of any generalised inertia matrix. So, that is what it means.

And then the second one, the second term, that is the centripetal component. So, that again assumes it is for any particular angle. So, you see, if it is for this angle, it assumes the whole of the structure is fixed in this case. So, this is the mass, so that becomes the centripetal force which is acting, trying to pull it here. Centrifugal force is extended like this, and then that creates the torque, which is over here. So, this is the centrifugal force and this is the arm which is there. So, r1 and c2 into absent. So, it gives me the torque that comes here due to this centripetal force.
Similarly, if it is not passing through the origin, that will also create torque at joint one. So, this is, these are the forces that will create torque at any joint if it is moving. Similarly, the last component is the Coriolis component.
How does it arise? So the first link is, but the second link, you see what kind of motion iS this. It has a revolute motion or rotating motion as well as, because of the motion of joint one. It has a linear velocity also at any instant of time, because this is the distance. This moves with theta one dot, so it has a value which is like this, and it has a rotating thing also. So, this has a complex motion of linear and rotating motion. So, this is on the rotating reference frame. So, you have a Coriolis component that comes here and this is the last thing you have analysed already in very much detail in our earlier module. What was that? That is statics. You saw how gravity compensation torque can be generated. So, if there is no velocity and acceleration, all these terms will become equal to zero, and the torque that will remain is only due to the gravity component. So, that is the physical meaning of all the terms which are here.



So, there are some limitations. Even though we have arrived at a dynamic equation of motion that gives you a torque for any kind of motion which a robot makes. but in the case of industrial robots, it has a high gear reduction ratio. So, if at all, you have first linked, and then you have a joint which has having high gear ratio, and then you have a second link which comes here. So, a small amount of torque is quite good enough to drive this link because you have a high gear ratio ranging from 300 to 100. So, that is the normal gear ratio which is there at each joint normally in case of any industrial robot. So, you get a very poor torque reflectance. So, whatever torque is generated due to this is reflected very badly over here. So, you see a very small amount of torque that comes here that physically isolates all the joints. All the links from the previous link can be treated independently, treated independently. So, it is. This type of robot is not back-drivable most of the time. You cannot apply forces over here and try to rotate it because it is a very high gear reduction ratio, and they have friction also inbuilt into that because of the friction between the metal elements, which are in surface-to-surface contact, even after the huge amount of lubrication, are there. They have friction, and they don't allow any back-drivable thing. Due to its own weight, it will poorly fall very slowly. Ultimately, it will touch the ground. But that is the reason it normally isolates the second link from the first link to the 0th link so that they can be treated like single input and single output controllers. So, they are a dynamically decoupled system. So, that is what is advantageous and allows us to run this type of robot directly in position and velocity control mode. You just give the joint velocity and position command, and the robot goes to the commanded position. You can calculate the joint velocity and positions using inverse kinematics and Jacobians. So you can directly feed in those rates over here, and you see the end effector rates and the positions varying. So, it will, it won't depend too much on dynamics in that case. So, as compared to the dynamic torque that actually comes on to the motor is very, very less. So you only need to drive the joints in this. So, most of the industrial robots support commanding for position and velocity only. You cannot command it to use a particular joint torque and drive the end effector or even the link. So having a torque using the dynamic equation of motion doesn't help much when you are using a ready-made kind of industrial robot in your lab or the industry.

So, this makes the torque computation using LE or Newton Euler formulation not much applicable for its control. However, the position joint position controllers still use precise joint model robot electro-mechanical model internally to control its joints. That doesn't mean the whole of the exercise that we did is meaningless. So, in order to achieve a very high amount of accuracy and positional repeatability, these kinds of controllers are still there at the joints, at the joint level, but yes, there are you, the manufacturers. Normally, they don't give you access to run the robots using torque commands. So, you can only command for the position.

So, obtaining precise robot models using CAD or any identification experiments is not very precise. So, even because robot joint links are very, very complex, you hardly know the exact moment of inertia and the mass centre location or even the mass at times. Once it is assembled, it is very difficult to find out even the mass. Because you have a huge amount of electronics that are there inside the links, you have gearboxes, and you have belts, you have other transmission systems, concentric shafts and many other things. So, that makes it very, very complex. So, it is not a uniform density, so that you can get through the CAD.

The centre of mass location, mass, and moment of inertia details are not shared with the end-user. That is only for the manufacturer's purpose and maintenance. So, sometimes they do use it, but they don't allow users, end users, to drive the robot using these parameters. So, what we have in hand is only position and velocity control that we can do.

Again, due to the complex transmission, as I have said, electronic cables within the link further make it difficult to obtain these details. So, using torque control to drive any robot is also highly unsafe. It is unsafe because it is very, very sensitive. That type of running will make your robot very, very sensitive to any external parameters. Even if you put a small supplementary load on one of the links, the dynamics change, and your control things will change. So, whatever torque you are feeding, that should, that will not be good enough to obtain particular joint velocity or acceleration or any joint position. So, that directly affects the end effector parameters also, like end effector position, velocity and acceleration. So, you know your system is highly unmodeled, and it is highly uncertain in this case. So, that is the reason due to safety and other factors in industrial robots, it is not allowed by the manufacturer. So, yes, if at all you are using, you are making a robot or collaborative robot, this becomes very, very useful, this kind of knowledge, and if you make your own robot, you want to make one of your own. This knowledge will be very, very useful. And even if you want to understand the behaviour of your robot, why putting a load, supplementary load, beyond a certain value which is prescribed by the manufacturer goes beyond that value, why your robot should not behave well, why load should be within the prescribed limits, why wire should be last properly. So, there are many reasons and many answers that you will get out of this knowledge.

So, that's all for today's class. In the next class, we will start with another approach for obtaining an equation of motion. So, that is Newton Euler's approach that we will be doing. For PhD students and research students, for UG students who want to go for research, they should quickly follow any book like Professor Sahas's book, like Introduction to Robots should be good enough, or any book you feel like. So, you can follow that book, and I would definitely suggest go for the vector-matrix approach for obtaining an equation of motion using the Lagrange-Euler formulation. Why? Because of the way we did it, we directly took a derivative. It is good for symbolic understanding of your system and expressing at least for two degrees of freedom it was trivial. But if the system goes in three dimensions, and you have to program it for dynamic or dynamically obtaining the torques, it becomes very, very difficult. So, this method is not suitable for programming for equation of motion, to obtain equation of motion and dynamic torque. So, I would suggest go for the vector-matrix approach to obtain the torque to obtain an equation of motion. So, that will also support programming your robot for this kind of approach to control your robot. So, that is an additional reading I suppose you should go for. So, that's all for today. Thanks a lot, you.