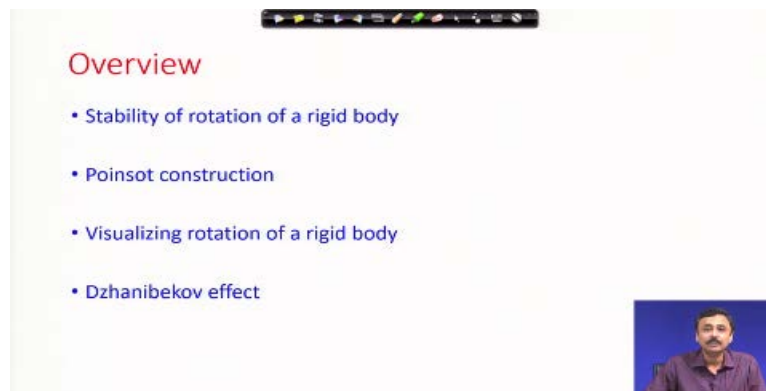


Advanced Dynamics
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Lecture – 60
Intermediate Axis Theorem

In this lecture, we will revisit the intermediate axis theorem discussed in one of the previous lectures.

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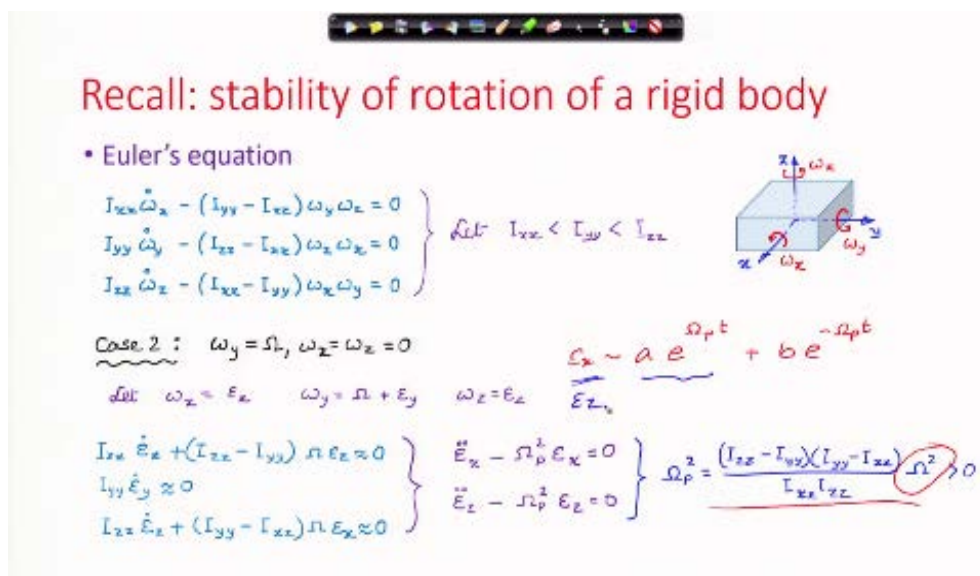


Overview

- Stability of rotation of a rigid body
- Poincaré construction
- Visualizing rotation of a rigid body
- Dzhanibekov effect

We are going to look at the stability of rotating rigid bodies and understanding the dynamics in terms of what are known as Poincaré construction. Then, we are going to see the Dzhanibekov effect which was observed by a Russian cosmonaut in space.

(Refer Slide Time: 01:22)



Recall: stability of rotation of a rigid body

- Euler's equation

$$\left. \begin{aligned} I_{xx} \ddot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z &= 0 \\ I_{yy} \ddot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x &= 0 \\ I_{zz} \ddot{\omega}_z - (I_{xx} - I_{yy}) \omega_x \omega_y &= 0 \end{aligned} \right\} \text{Let } I_{xx} < I_{yy} < I_{zz}$$

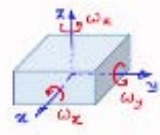
Case 2: $\omega_y = \Omega, \omega_z = \omega_x = 0$

Let $\omega_x = \epsilon_x, \omega_y = \Omega + \epsilon_y, \omega_z = \epsilon_z$

$$\left. \begin{aligned} I_{xx} \ddot{\epsilon}_x + (I_{zz} - I_{yy}) \Omega \epsilon_z &\approx 0 \\ I_{yy} \ddot{\epsilon}_y &\approx 0 \\ I_{zz} \ddot{\epsilon}_z + (I_{yy} - I_{xx}) \Omega \epsilon_x &\approx 0 \end{aligned} \right\} \left. \begin{aligned} \ddot{\epsilon}_x - \Omega_p^2 \epsilon_x &= 0 \\ \ddot{\epsilon}_z + \Omega_p^2 \epsilon_z &= 0 \end{aligned} \right\} \Omega_p^2 = \frac{(I_{zz} - I_{yy})(I_{yy} - I_{xx})}{I_{xx} I_{zz}} \Omega^2 > 0$$

$$\epsilon_x \sim a e^{\Omega_p t} + b e^{-\Omega_p t}$$

$$\epsilon_z \sim \frac{\Omega_p}{\Omega} (a e^{\Omega_p t} - b e^{-\Omega_p t})$$



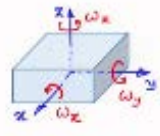
The linear stability of an unsymmetric spinning rigid body is revisited in the above slide.

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Recall: stability of rotation of a rigid body

- Euler's equation

$$\left. \begin{aligned} I_{xx} \dot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z &= 0 \\ I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x &= 0 \\ I_{zz} \dot{\omega}_z - (I_{xx} - I_{yy}) \omega_x \omega_y &= 0 \end{aligned} \right\} \text{Let } I_{xx} < I_{yy} < I_{zz}$$




Case 2 : $\omega_y = \Omega, \omega_z = \omega_x = 0$

Let $\omega_x = \epsilon_x, \omega_y = \Omega + \epsilon_y, \omega_z = \epsilon_z$

$$\left. \begin{aligned} I_{xx} \dot{\epsilon}_x + (I_{zz} - I_{yy}) \Omega \epsilon_z &\approx 0 \\ I_{yy} \dot{\epsilon}_y &\approx 0 \\ I_{zz} \dot{\epsilon}_z + (I_{yy} - I_{xx}) \Omega \epsilon_x &\approx 0 \end{aligned} \right\} \begin{aligned} \ddot{\epsilon}_x - \Omega_p^2 \epsilon_x &= 0 \\ \ddot{\epsilon}_z - \Omega_p^2 \epsilon_z &= 0 \end{aligned} \quad \Omega_p^2 = \frac{(I_{zz} - I_{yy}) \Omega}{I_x}$$

⇒ Unbounded solution (unstable)

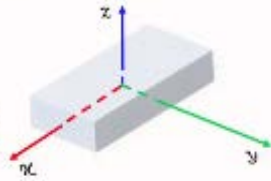


This linear stability analysis says that the spin about the intermediate axis has unbounded perturbation dynamics.


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Intermediate axis theorem

The spin of a body with unequal principal moments of inertia about the intermediate moment of inertia axis is unstable.



- Known since 1834 (Louis Poincaré)
- Experiment in space: Russian cosmonaut Vladimir Dzhanibekov observed this in 1985 (**Dzhanibekov effect**)



The intermediate axis theorem is stated in the above slide.

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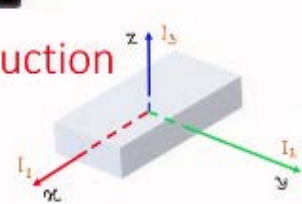

Visualizing rotation: Poinsot construction

Force and torque-free motion

⇒ Conservation of energy and angular momentum

$$E = \frac{1}{2} \vec{\omega} \cdot I_G \vec{\omega} = \frac{1}{2} [I_1 \omega_x^2 + I_2 \omega_y^2 + I_3 \omega_z^2] \leftarrow$$

$$\vec{H} = I_G \vec{\omega} = I_1 \omega_x \hat{i} + I_2 \omega_y \hat{j} + I_3 \omega_z \hat{k}$$

$$\Rightarrow H^2 = I_1^2 \omega_x^2 + I_2^2 \omega_y^2 + I_3^2 \omega_z^2 \leftarrow$$



Poinsot devised an interesting way of visualizing the motion of a spinning rigid body. We start with the consideration that energy and angular momentum have to be conserved. These appear as constraints as presented in the slide above.

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Visualizing rotation: Poinsot construction

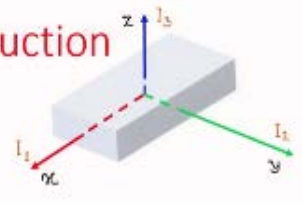

$$E = \frac{1}{2} \vec{\omega} \cdot I_G \vec{\omega} = \frac{1}{2} [I_1 \omega_x^2 + I_2 \omega_y^2 + I_3 \omega_z^2]$$

$$H^2 = I_1^2 \omega_x^2 + I_2^2 \omega_y^2 + I_3^2 \omega_z^2$$

Define $h_x = \frac{I_1 \omega_x}{\sqrt{2E}}$ $h_y = \frac{I_2 \omega_y}{\sqrt{2E}}$ $h_z = \frac{I_3 \omega_z}{\sqrt{2E}}$ $(\vec{h} = \frac{\vec{H}}{\sqrt{2E}})$

$$\Rightarrow \frac{h_x^2}{I_1} + \frac{h_y^2}{I_2} + \frac{h_z^2}{I_3} = 1 \quad (\text{energy})$$

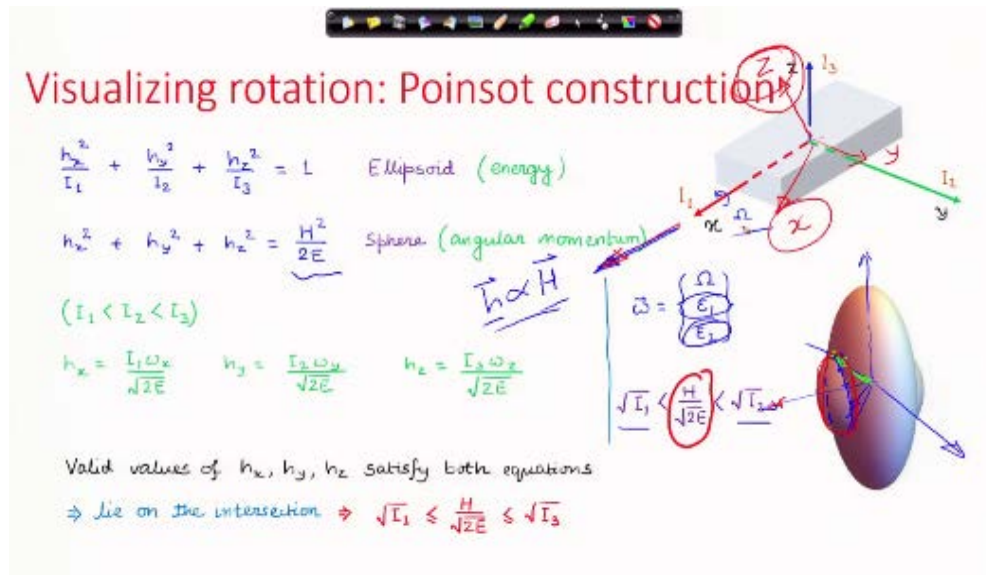
and $h_x^2 + h_y^2 + h_z^2 = \frac{H^2}{2E} \quad (\text{angular momentum})$

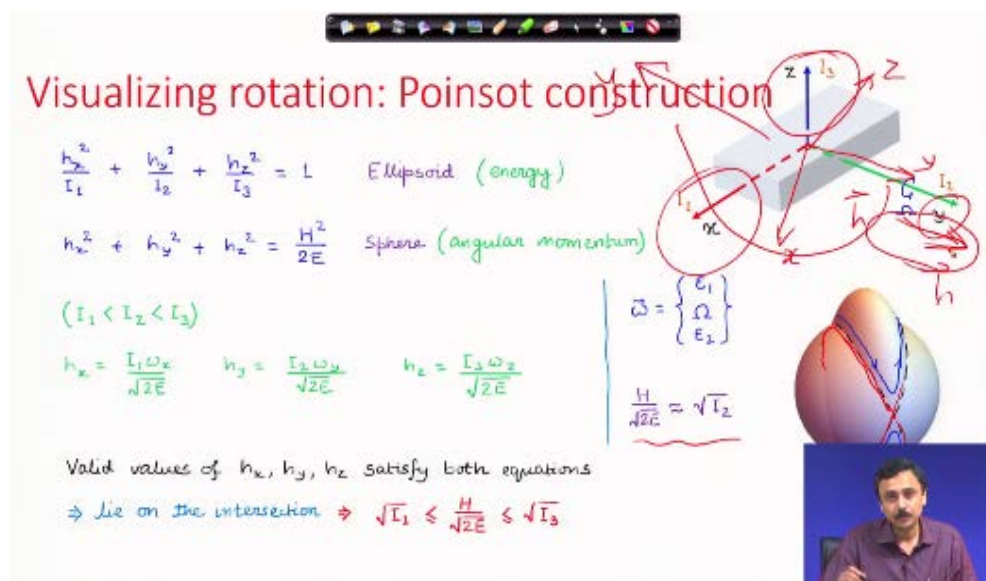
With a redefinition, we can recast the energy conservation constraint as an ellipsoid, and the angular momentum conservation constraint as a sphere, as presented in the above slide.

The Poincaré construction for the three cases of rotation, namely about the minimum, intermediate and maximum moment of inertia axes are presented in the following three slides.

(Refer Slide Time: 16:45)



(Refer Slide Time: 23:47)



(Refer Slide Time: 26:54)

Visualizing rotation: Poincaré construction

$$\frac{h_x^2}{I_1} + \frac{h_y^2}{I_2} + \frac{h_z^2}{I_3} = 1 \quad \text{Ellipsoid (Energy)}$$

$$h_x^2 + h_y^2 + h_z^2 = \frac{H^2}{2E} \quad \text{Sphere (Angular momentum)}$$

$(I_1 < I_2 < I_3)$

$$h_x = \frac{I_1 \omega_x}{\sqrt{2E}} \quad h_y = \frac{I_2 \omega_y}{\sqrt{2E}} \quad h_z = \frac{I_3 \omega_z}{\sqrt{2E}}$$

$$\vec{\omega} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$\sqrt{I_2} < \frac{H}{\sqrt{2E}} < \sqrt{I_3}$$

Valid values of h_x, h_y, h_z satisfy both equations
 \Rightarrow lie on the intersection $\Rightarrow \sqrt{I_1} \leq \frac{H}{\sqrt{2E}} \leq \sqrt{I_3}$

The maximum axis theorem is discussed in the following slide. This happens when there is dissipation of energy due to flexibility and internal damping.

(Refer Slide Time: 28:52)

Visualizing rotation: Poincaré construction

$$\frac{h_x^2}{I_1} + \frac{h_y^2}{I_2} + \frac{h_z^2}{I_3} = 1 \quad \text{Ellipsoid (Energy)}$$

$$h_x^2 + h_y^2 + h_z^2 = \frac{H^2}{2E} \quad \text{Sphere (Angular momentum)}$$

$(I_1 < I_2 < I_3)$

$$h_x = \frac{I_1 \omega_x}{\sqrt{2E}} \quad h_y = \frac{I_2 \omega_y}{\sqrt{2E}} \quad h_z = \frac{I_3 \omega_z}{\sqrt{2E}}$$

$$\vec{\omega} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$\sqrt{I_2} < \frac{H}{\sqrt{2E}} < \sqrt{I_3}$$

Effect of flexibility of the body (internal dissipation)

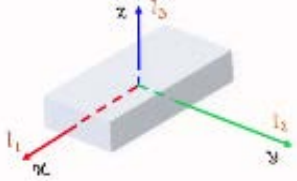
- Reduction of E , conservation of $H \Rightarrow \frac{H^2}{2E} \sim \sqrt{I_2}$
- Spin moves to I_3 axis (Major axis theorem)

Now, we visualize the rotation through animations. The formulation is presented in the following 4 slides.

(Refer Slide Time: 32:39)

Visualizing rotation: simulation

Generalized coordinates: (ψ, θ, ϕ)
Yaw-Pitch-Roll




Lagrangian: $\mathcal{L} = \frac{1}{2} \vec{\omega} \cdot \mathbf{I}_0 \vec{\omega}$ (In free space: no potential energy)

$$\mathbf{I}_0 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

(x-y-z principal body-fixed frame)
 $(I_1 < I_2 < I_3)$

$\vec{\omega}$ parametrized by yaw-pitch-roll coordinates (body-fixed frame)

$\vec{\omega} = \vec{\omega}(\psi, \theta, \phi, \dot{\psi}, \dot{\theta}, \dot{\phi})$



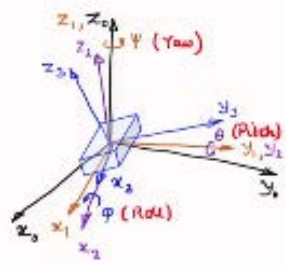

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Yaw-pitch-roll parametrization

Recall ${}^0R_3 = R_{z_0, \psi} R_{y_1, \theta} R_{x_2, \phi}$

$$R_{z_0, \psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{y_1, \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_{x_2, \phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

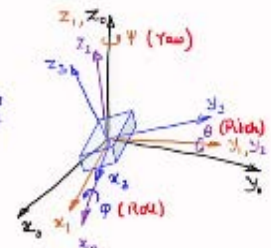

$${}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi s\theta + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$



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Angular velocity vector

Recall
Angular velocity (body frame): $\vec{\omega}^3 = \dot{\psi} \hat{z}_0^3 + \dot{\theta} \hat{y}_1^3 + \dot{\phi} \hat{x}_2^3$

$$\hat{z}_0^3 = \begin{Bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{Bmatrix} \quad \hat{y}_1^3 = \begin{Bmatrix} 0 \\ c\phi \\ -s\phi \end{Bmatrix} \quad \hat{x}_2^3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \underline{\vec{\omega}^3} = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} = \underline{[T]} \underline{\dot{\Psi}}$$



(Refer Slide Time: 35:01)

Visualizing rotation: simulation

Lagrangian: $\mathcal{L} = \frac{1}{2} \vec{\omega} \cdot \underline{I}_a \vec{\omega}$

$$\underline{I}_a = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} = \underline{[T]} \underline{\dot{\Psi}}$$

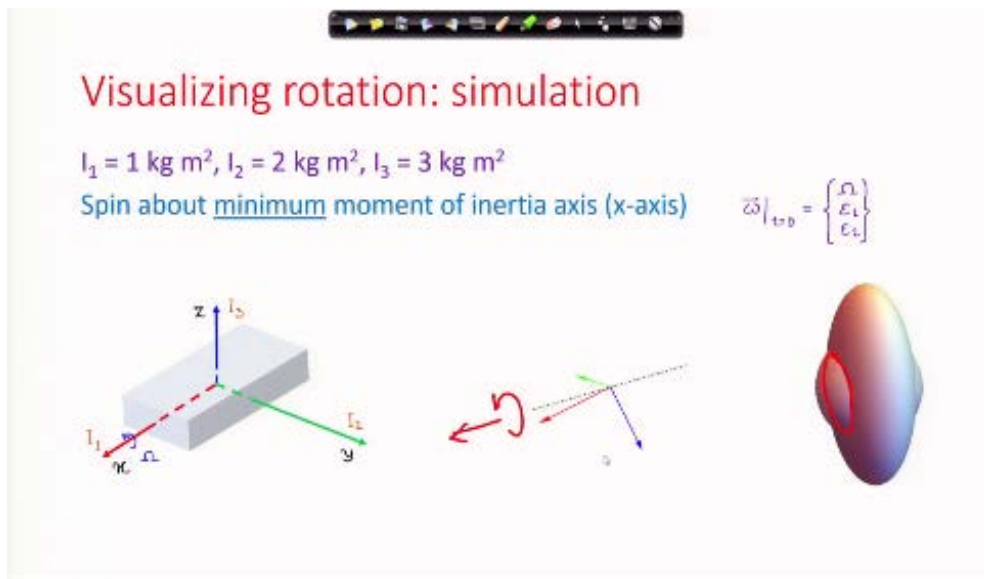
$$\mathcal{L} = \frac{1}{2} \left[I_1 (-\dot{\psi} s\theta + \dot{\phi})^2 + I_2 (\dot{\psi} c\theta s\phi + \dot{\theta} c\phi)^2 + I_3 (\dot{\psi} c\theta c\phi - \dot{\theta} s\phi)^2 \right]$$

Equations of motion:

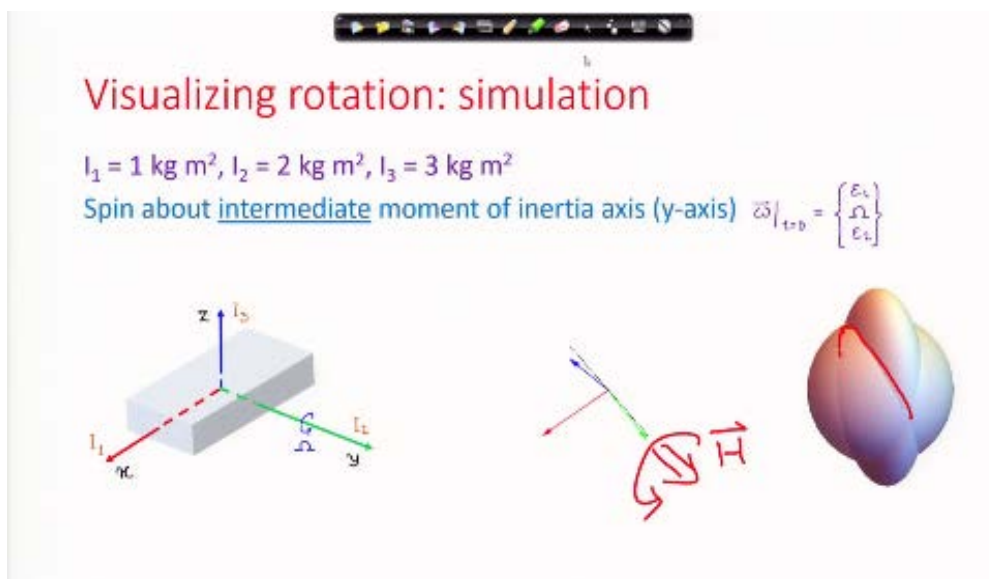
$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \underline{q} = \{\psi, \theta, \phi\} \quad \text{Initial velocity: } \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}_{t=0} = \underline{[T]}^{-1} \underline{\vec{\omega}}|_{t=0}$$

The simulation results are presented next in the lectures.

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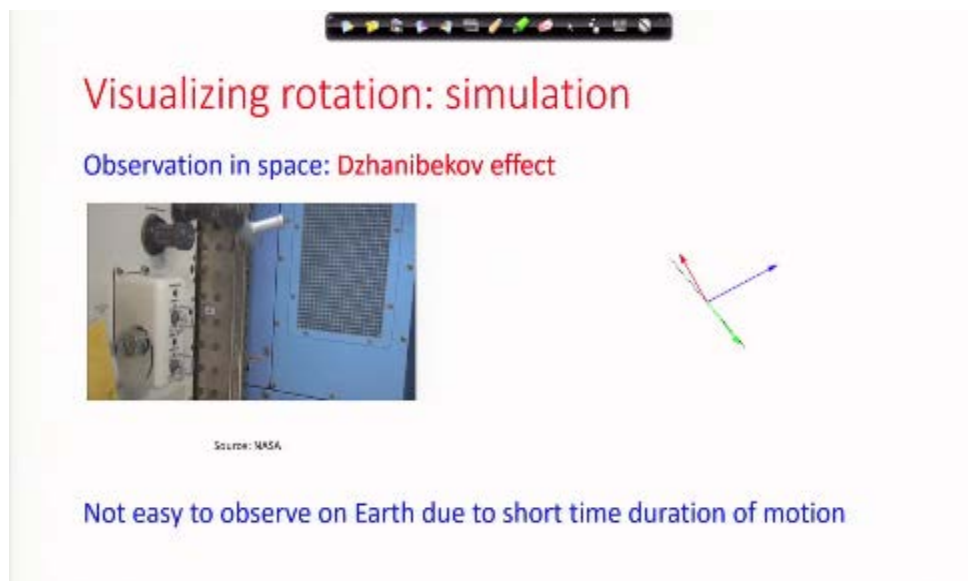


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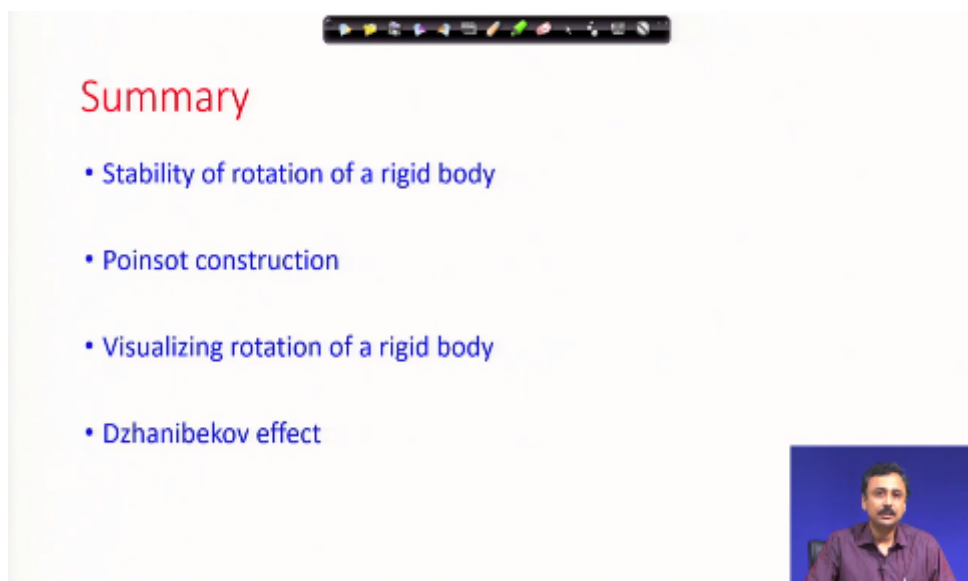


The Dzhanibekov effect is shown as a video in the lecture along with the simulation.

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(Refer Slide Time: 40:25)



The summary is provided above.