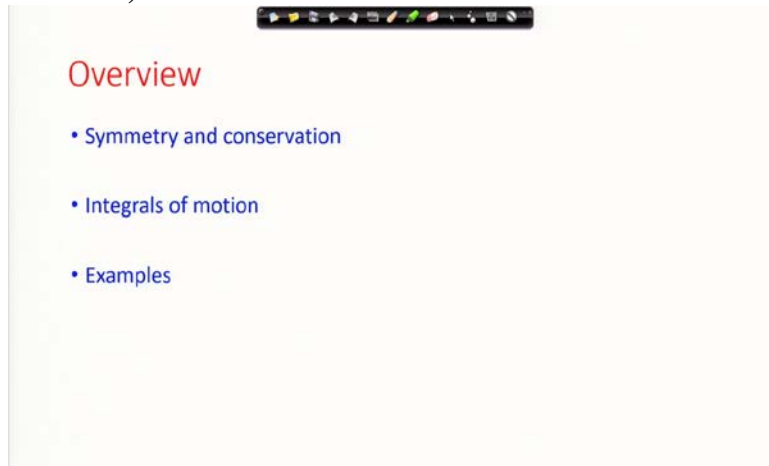


**Advanced Dynamics**  
**Prof. Anirvan DasGupta**  
**Department of Mechanical Engineering**  
**Indian Institute of Technology, Kharagpur**

**Lecture - 59**  
**Symmetries and Conservation Laws - IV**

In this lecture I am going to discuss some examples of symmetries and conservation laws that how we derive these conservation laws or constants of motion and how we use them to understand the dynamics of a system.

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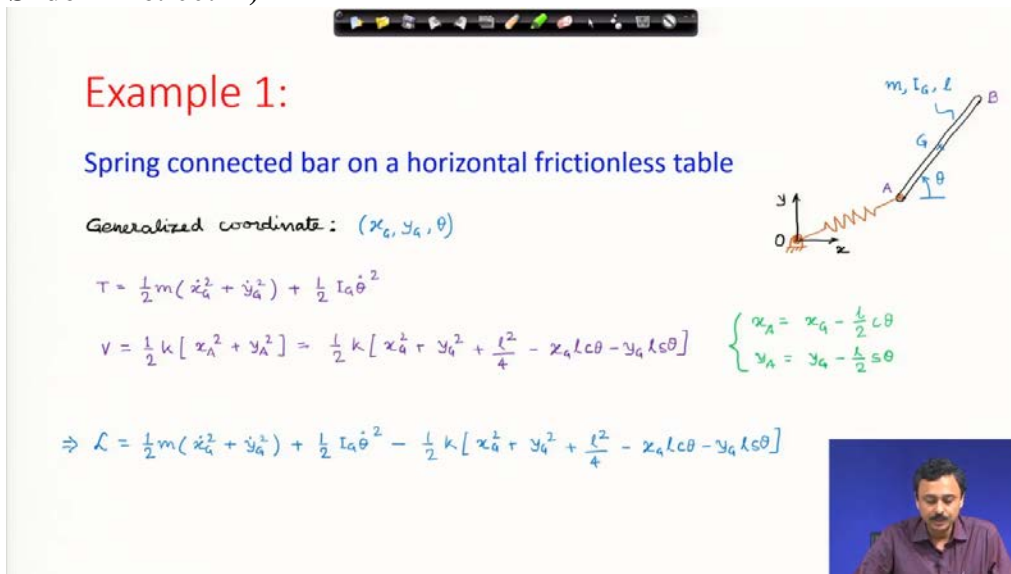


**Overview**

- Symmetry and conservation
- Integrals of motion
- Examples

We have been discussing about symmetries and the corresponding conservation we have looked at Noether's theorem we have discussed the integrals of motion and how to find out the integrals of motion here I am going to take some examples.

(Refer Slide Time: 00:44)



**Example 1:**

**Spring connected bar on a horizontal frictionless table**

Generalized coordinate:  $(x_G, y_G, \theta)$

$$T = \frac{1}{2} m (\dot{x}_G^2 + \dot{y}_G^2) + \frac{1}{2} I_G \dot{\theta}^2$$

$$V = \frac{1}{2} k [x_A^2 + y_A^2] = \frac{1}{2} k \left[ x_G^2 + y_G^2 + \frac{l^2}{4} - x_G l \cos \theta - y_G l \sin \theta \right]$$

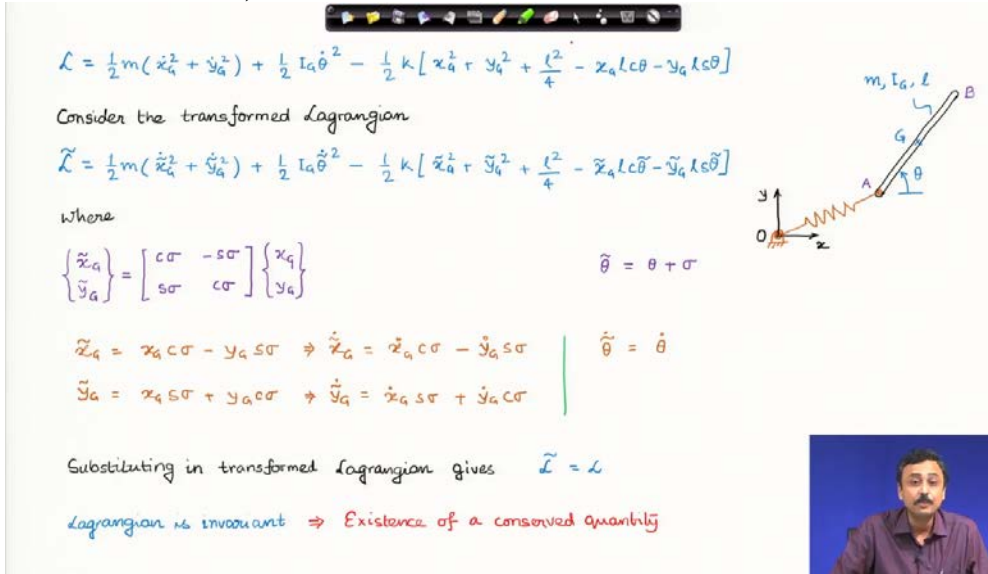
$$\Rightarrow \mathcal{L} = \frac{1}{2} m (\dot{x}_G^2 + \dot{y}_G^2) + \frac{1}{2} I_G \dot{\theta}^2 - \frac{1}{2} k \left[ x_G^2 + y_G^2 + \frac{l^2}{4} - x_G l \cos \theta - y_G l \sin \theta \right]$$

Diagram labels:  $m, I_G, l$ ,  $G$ ,  $A$ ,  $B$ ,  $\theta$ ,  $x$ ,  $y$ ,  $O$ .

Equations for point A:
$$\begin{cases} x_A = x_G - \frac{l}{2} \cos \theta \\ y_A = y_G - \frac{l}{2} \sin \theta \end{cases}$$

The first example is a spring connected bar on a horizontal frictionless table as shown in the slide above. The Lagrangian is also presented.

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$$\mathcal{L} = \frac{1}{2}m(\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2}I_A\dot{\theta}^2 - \frac{1}{2}k\left[x_A^2 + y_A^2 + \frac{l^2}{4} - x_A l \cos\theta - y_A l \sin\theta\right]$$

Consider the transformed Lagrangian

$$\tilde{\mathcal{L}} = \frac{1}{2}m(\dot{\tilde{x}}_A^2 + \dot{\tilde{y}}_A^2) + \frac{1}{2}I_A\dot{\tilde{\theta}}^2 - \frac{1}{2}k\left[\tilde{x}_A^2 + \tilde{y}_A^2 + \frac{l^2}{4} - \tilde{x}_A l \cos\tilde{\theta} - \tilde{y}_A l \sin\tilde{\theta}\right]$$

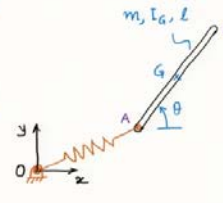
Where

$$\begin{Bmatrix} \tilde{x}_A \\ \tilde{y}_A \end{Bmatrix} = \begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix} \begin{Bmatrix} x_A \\ y_A \end{Bmatrix} \quad \tilde{\theta} = \theta + \sigma$$

$$\begin{aligned} \tilde{x}_A &= x_A \cos\sigma - y_A \sin\sigma \Rightarrow \dot{\tilde{x}}_A = \dot{x}_A \cos\sigma - \dot{y}_A \sin\sigma \\ \tilde{y}_A &= x_A \sin\sigma + y_A \cos\sigma \Rightarrow \dot{\tilde{y}}_A = \dot{x}_A \sin\sigma + \dot{y}_A \cos\sigma \end{aligned} \quad \dot{\tilde{\theta}} = \dot{\theta}$$

Substituting in transformed Lagrangian gives  $\tilde{\mathcal{L}} = \mathcal{L}$

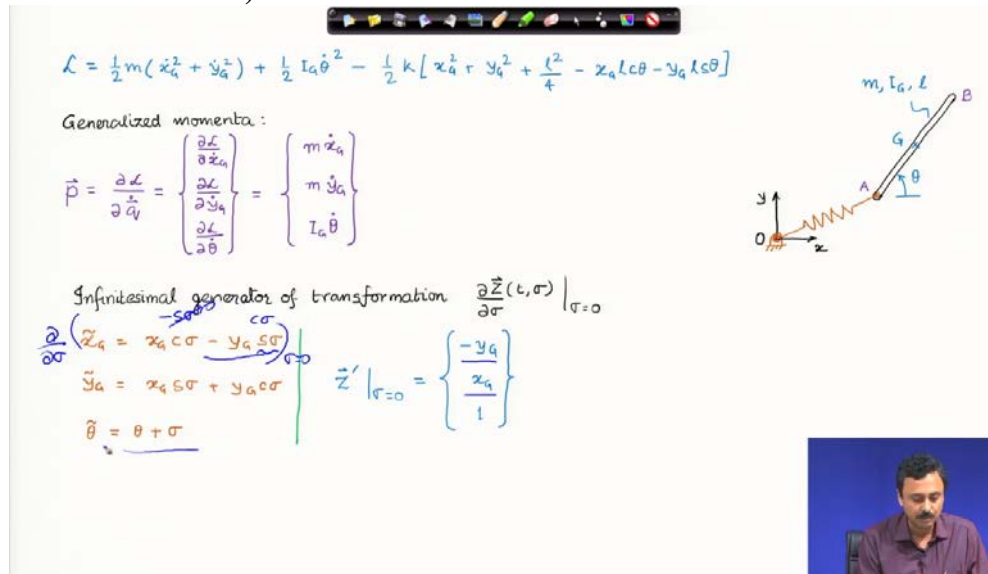
Lagrangian is invariant  $\Rightarrow$  Existence of a conserved quantity



In the above slide, an infinitesimal symmetry of the Lagrangian is determined. This leads to the conclusion, based on Noether's theorem, that there exists a conserved quantity.

Our next task is to find the conserved quantity. This is determined in the following 2 slides.

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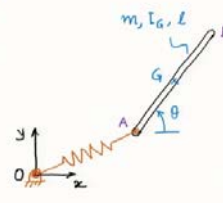
$$\mathcal{L} = \frac{1}{2}m(\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2}I_A\dot{\theta}^2 - \frac{1}{2}k\left[x_A^2 + y_A^2 + \frac{l^2}{4} - x_A l \cos\theta - y_A l \sin\theta\right]$$

Generalized momenta:

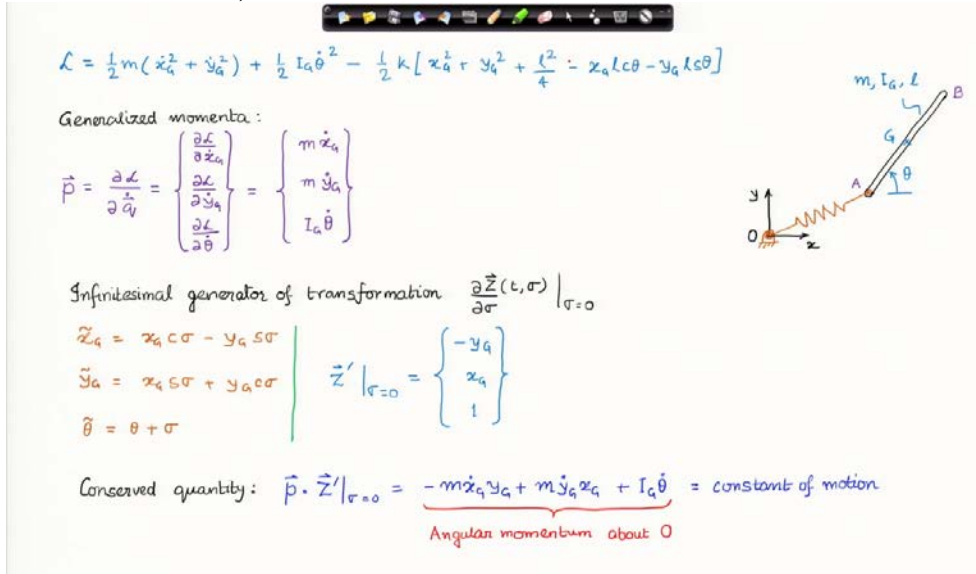
$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \begin{Bmatrix} \frac{\partial \mathcal{L}}{\partial \dot{x}_A} \\ \frac{\partial \mathcal{L}}{\partial \dot{y}_A} \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \end{Bmatrix} = \begin{Bmatrix} m \dot{x}_A \\ m \dot{y}_A \\ I_A \dot{\theta} \end{Bmatrix}$$

Infinitesimal generator of transformation  $\left. \frac{\partial \tilde{Z}}{\partial \sigma} \right|_{\sigma=0}$

$$\frac{\partial}{\partial \sigma} \begin{pmatrix} \tilde{x}_A = x_A \cos\sigma - y_A \sin\sigma \\ \tilde{y}_A = x_A \sin\sigma + y_A \cos\sigma \\ \tilde{\theta} = \theta + \sigma \end{pmatrix} \bigg|_{\sigma=0} = \begin{Bmatrix} -y_A \\ x_A \\ 1 \end{Bmatrix}$$



(Refer Slide Time: 08:56)



$$\mathcal{L} = \frac{1}{2} m (\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2} I_A \dot{\theta}^2 - \frac{1}{2} k \left[ x_A^2 + y_A^2 + \frac{l^2}{4} - x_A l \cos \theta - y_A l \sin \theta \right]$$

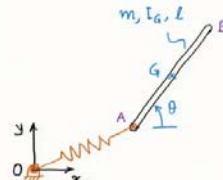
Generalized momenta:

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \begin{Bmatrix} \frac{\partial \mathcal{L}}{\partial \dot{x}_A} \\ \frac{\partial \mathcal{L}}{\partial \dot{y}_A} \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \end{Bmatrix} = \begin{Bmatrix} m \dot{x}_A \\ m \dot{y}_A \\ I_A \dot{\theta} \end{Bmatrix}$$

Infinitesimal generator of transformation  $\left. \frac{\partial \vec{Z}}{\partial \sigma} \right|_{\sigma=0}$

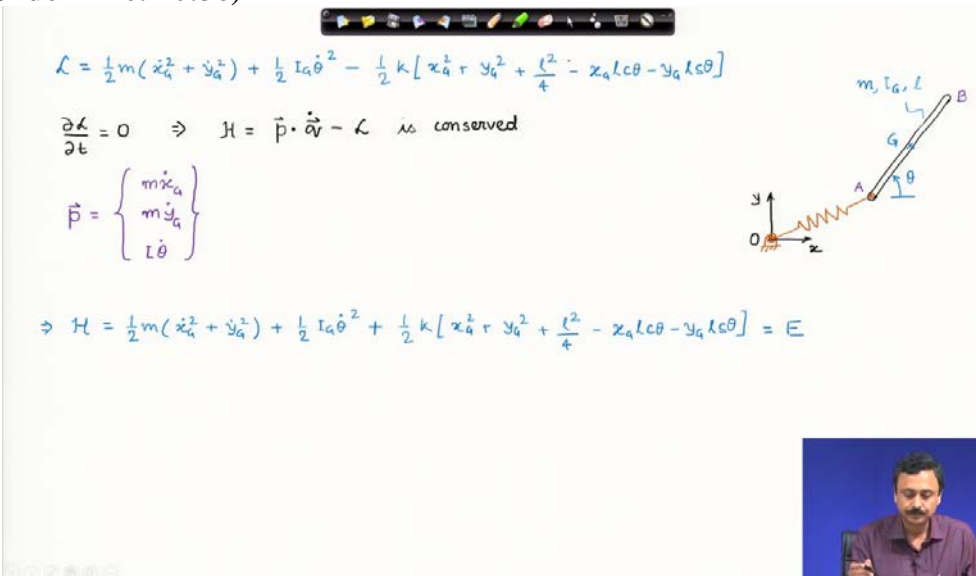
$$\begin{aligned} \vec{Z}_A &= x_A \cos \sigma - y_A \sin \sigma \\ \vec{y}_A &= x_A \sin \sigma + y_A \cos \sigma \\ \vec{\theta} &= \theta + \sigma \end{aligned} \quad \left. \vec{Z}' \right|_{\sigma=0} = \begin{Bmatrix} -y_A \\ x_A \\ 1 \end{Bmatrix}$$

Conserved quantity:  $\vec{p} \cdot \vec{Z}'|_{\sigma=0} = \underbrace{-m \dot{x}_A y_A + m \dot{y}_A x_A + I_A \dot{\theta}}_{\text{Angular momentum about O}} = \text{constant of motion}$



The conserved quantity is nothing but the angular momentum of the bar about the fixed point.

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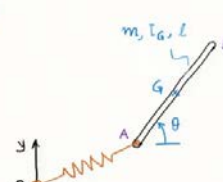



$$\mathcal{L} = \frac{1}{2} m (\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2} I_A \dot{\theta}^2 - \frac{1}{2} k \left[ x_A^2 + y_A^2 + \frac{l^2}{4} - x_A l \cos \theta - y_A l \sin \theta \right]$$

$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \mathcal{H} = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}$  is conserved

$$\vec{p} = \begin{Bmatrix} m \dot{x}_A \\ m \dot{y}_A \\ I \dot{\theta} \end{Bmatrix}$$

$$\Rightarrow \mathcal{H} = \frac{1}{2} m (\dot{x}_A^2 + \dot{y}_A^2) + \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} k \left[ x_A^2 + y_A^2 + \frac{l^2}{4} - x_A l \cos \theta - y_A l \sin \theta \right] = E$$

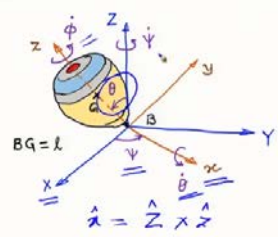
The second symmetry is obtained from the observation that the Lagrangian is not explicitly time dependent. In other words, the partial derivative of the Lagrangian with respect to time is 0, and that immediately tells us that the Hamiltonian is a conserved quantity. In this case, the Hamiltonian is the total mechanical energy of the system.

(Refer Slide Time: 11:55)


**Example 2:**

**Dynamics of a top**

Generalized coordinates:  $(\psi, \theta, \phi)$



The diagram shows a spinning top with a fixed point at the origin of a fixed coordinate system  $(X, Y, Z)$ . A body-fixed coordinate system  $(x, y, z)$  is also shown. The top's axis is at an angle  $\theta$  from the  $Z$ -axis. The precession angle is  $\psi$ , and the spin angle is  $\phi$ . The distance from the fixed point to the center of mass is  $BG = l$ . Angular velocities are indicated by curved arrows:  $\dot{\psi}$  for precession,  $\dot{\phi}$  for spin, and  $\dot{\theta}$  for nutation. The total angular velocity vector is  $\vec{\omega} = \dot{\psi}\hat{Z} + \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_\theta$ .



Next we look at this classic example of a spinning top. The generalized coordinates are noted in the slide above.

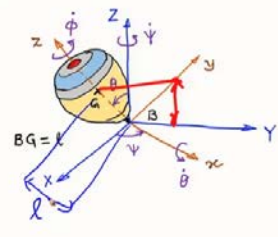
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**Example 2:**


**Dynamics of a top**

Generalized coordinates:  $(\psi, \theta, \phi)$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} \vec{\omega} \cdot I_0 \vec{\omega} - mgl \cos \theta \quad (\text{Fixed point motion})$$


The diagram is identical to the one in the previous slide, showing the spinning top with coordinate systems  $(X, Y, Z)$  and  $(x, y, z)$ , angles  $\theta$ ,  $\psi$ ,  $\phi$ , and distance  $BG = l$ . It illustrates the geometry for the Lagrangian formulation.



The Lagrangian can be written very easily as shown above.

(Refer Slide Time: 15:22)

**Example 2:**

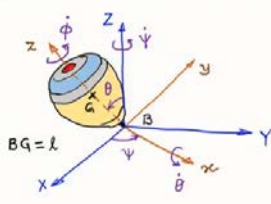

**Dynamics of a top**

Generalized coordinates:  $(\psi, \theta, \phi)$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} \vec{\omega} \cdot I_B \vec{\omega} - mgl \cos \theta \quad (\text{Fixed point motion})$$

$$\vec{\omega} = \dot{\theta} \hat{i} + \dot{\psi} \sin \theta \hat{j} + (\dot{\psi} \cos \theta + \dot{\phi}) \hat{k} \quad (\text{in } x-y-z \text{ frame}) \quad I_B = \begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} [I_0 \dot{\theta}^2 + I_0 \dot{\psi}^2 \sin^2 \theta + I (\dot{\psi} \cos \theta + \dot{\phi})^2] - mgl \cos \theta$$



The expression of the Lagrangian in terms of the generalized coordinates is presented in the slide above.

(Refer Slide Time: 17:40)

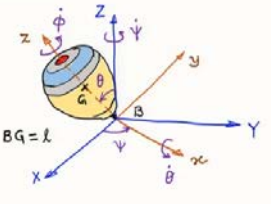

**Top dynamics**

$$\mathcal{L} = \frac{1}{2} [I_0 \dot{\theta}^2 + I_0 \dot{\psi}^2 \sin^2 \theta + I (\dot{\psi} \cos \theta + \dot{\phi})^2] - mgl \cos \theta$$

Cyclic coordinates:  $(\psi, \phi)$

$$\Rightarrow p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_0 \dot{\psi} \sin^2 \theta + I (\dot{\psi} \cos \theta + \dot{\phi}) \cos \theta = a \quad (\text{constant})$$

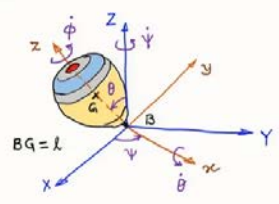
$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I (\dot{\psi} \cos \theta + \dot{\phi}) = b \quad (\text{constant})$$

$$\Rightarrow \dot{\psi} = \frac{a - b \cos \theta}{I_0 \sin^2 \theta}$$



The cyclic coordinates are noted in the slide above, and the constants of motion are presented.

(Refer Slide Time: 19:22)

**Top dynamics**



$$\mathcal{L} = \frac{1}{2} [\Gamma_0 \dot{\theta}^2 + \Gamma_0 \dot{\psi}^2 s^2 \theta + I(\dot{\psi} c \theta + \dot{\phi})^2] - mgl c \theta$$

Explicit time independence of Lagrangian:  $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \mathcal{H} = \text{constant}$

$$\mathcal{H} = \vec{p} \cdot \vec{\dot{q}} - \mathcal{L} = \frac{1}{2} [\Gamma_0 \dot{\theta}^2 + \Gamma_0 \dot{\psi}^2 s^2 \theta + I(\dot{\psi} c \theta + \dot{\phi})^2] + mgl c \theta = E$$

Mechanical Energy

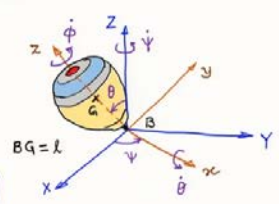
$$\Rightarrow E = \frac{1}{2} \left[ I_0 \dot{\theta}^2 + \frac{(a - b c \theta)^2}{I_0 s^2 \theta} + \frac{b^2}{I} \right] + mgl c \theta = \text{constant} \quad \left( \dot{\psi} = \frac{a - b c \theta}{I_0 s^2 \theta} \right)$$

$$\Rightarrow \frac{1}{2} I_0 \dot{\theta}^2 + \frac{(a - b c \theta)^2}{2 I_0 s^2 \theta} + mgl c \theta = E - \frac{b^2}{2 I} = E' \quad (\text{constant})$$

The next thing we notice is the Lagrangian is explicit time independent. This implies the Hamiltonian, which is the total mechanical energy in this case, is a conserved quantity.

(Refer Slide Time: 21:16)

**Top dynamics**



Governing equations (Integrals of motion)


$$I_0 \dot{\psi} s^2 \theta + I(\dot{\psi} c \theta + \dot{\phi}) c \theta = a \quad (\text{constant}) \quad - (1)$$

$$I(\dot{\psi} c \theta + \dot{\phi}) = b \quad (\text{constant}) \quad - (2)$$

$$\frac{1}{2} I_0 \dot{\theta}^2 + \frac{(a - b c \theta)^2}{2 I_0 s^2 \theta} + mgl c \theta = E' \quad (\text{constant}) \quad - (3)$$

Solve to determine motion from an initial condition  
( $\psi_0, \theta_0, \phi_0, \dot{\psi}_0, \dot{\theta}_0, \dot{\phi}_0$ )

Let  $u = c \theta$  in (3)

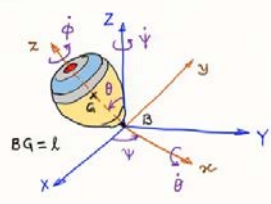

$$\Rightarrow \frac{1}{2} \dot{u}^2 + \frac{(a - b u)^2}{2 I_0 s^2 \theta} + \left[ \frac{mgl}{I_0} u - \frac{E'}{I_0} \right] (1 - u^2) = 0 \quad - (4)$$


Finally the integrals of motion are noted in the above slide.

(Refer Slide Time: 23:20)

### Top dynamics

$$\underbrace{\frac{1}{2} \dot{u}^2}_{KE} + \underbrace{\frac{(a-bu)^2}{2I_0^2} + \left[ \frac{mgl}{I_0} u - \frac{E'}{I_0} \right] (1-u^2)}_{\text{Effective PE: } V_E(u)} = 0 \quad u = \cos \theta$$

$$V_E(u) = -\frac{mgl}{I_0} u^3 + \left( \frac{E'}{I_0} + \frac{b^2}{2I_0^2} \right) u^2 + \left( \frac{mgl}{I_0} - \frac{ab}{I_0^2} \right) u + \frac{a^2}{2I_0^2} - \frac{E'}{I_0}$$



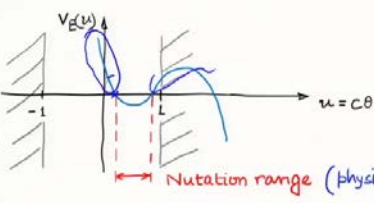
Using the integrals, one can reduce the dynamics of the top to a single dynamical equation, as shown in the slide above. Using the qualitative features of the effective potential energy expression, we study the dynamics of the top in the following 2 slides.

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### Top dynamics

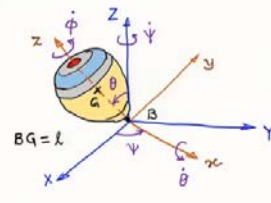

$$\underbrace{\frac{1}{2} \dot{u}^2}_{KE} + \underbrace{\frac{(a-bu)^2}{2I_0^2} + \left[ \frac{mgl}{I_0} u - \frac{E'}{I_0} \right] (1-u^2)}_{\text{Effective PE: } V_E(u)} = 0 \quad \Rightarrow \quad V_E(u) > 0$$

$$V_E(u) = -\frac{mgl}{I_0} u^3 + \left( \frac{E'}{I_0} + \frac{b^2}{2I_0^2} \right) u^2 + \left( \frac{mgl}{I_0} - \frac{ab}{I_0^2} \right) u + \frac{a^2}{2I_0^2} - \frac{E'}{I_0} \quad \left( \lim_{u \rightarrow \pm \infty} V_E(u) = \mp \infty \right)$$



$$V_E(\pm 1) = \frac{(a \mp b)^2}{2I_0^2} \geq 0$$

$$V_E(0) = \frac{a^2}{2I_0^2} - \frac{E'}{I_0}$$

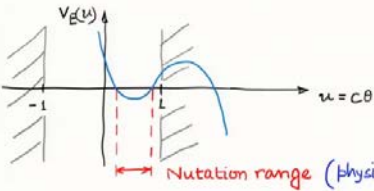



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### Top dynamics

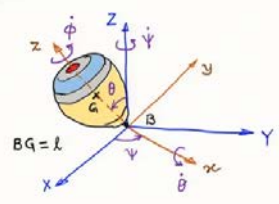
$$\underbrace{\frac{1}{2} \dot{u}^2}_{KE} + \underbrace{\left( \frac{(a-bu)^2}{2I_0^2} + \left[ \frac{mg\ell}{I_0} u - \frac{E'}{I_0} \right] (1-u^2) \right)}_{\text{Effective PE: } V_E(u)} = 0$$

$$V_E(u) = -\frac{mg\ell}{I_0} u^3 + \left( \frac{E'}{I_0} + \frac{b^2}{2I_0^2} \right) u^2 + \left( \frac{mg\ell}{I_0} - \frac{ab}{I_0^2} \right) u + \frac{a^2}{2I_0^2} - \frac{E'}{I_0} \quad \left( \lim_{u \rightarrow \pm\infty} V_E(u) = \mp\infty \right)$$



$$V_E(\pm 1) = \frac{(a \mp b)^2}{2I_0^2} \geq 0$$

$$V_E(0) = \frac{a^2}{2I_0^2} - \frac{E'}{I_0}$$



$BQ = \ell$

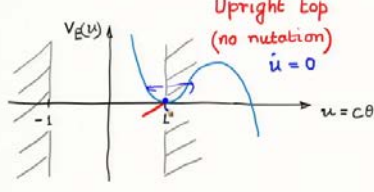
The condition of a sleeping top is presented in the following 2 slides.

(Refer Slide Time: 32:52)

### Top dynamics : sleeping top

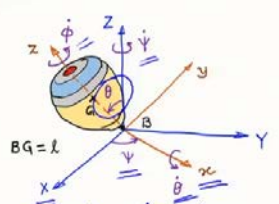
$$\underbrace{\frac{1}{2} \dot{u}^2}_{KE} + \underbrace{\left( \frac{(a-bu)^2}{2I_0^2} + \left[ \frac{mg\ell}{I_0} u - \frac{E'}{I_0} \right] (1-u^2) \right)}_{\text{Effective PE: } V_E(u)} = 0$$

$$V_E(u) = -\frac{mg\ell}{I_0} u^3 + \left( \frac{E'}{I_0} + \frac{b^2}{2I_0^2} \right) u^2 + \left( \frac{mg\ell}{I_0} - \frac{ab}{I_0^2} \right) u + \frac{a^2}{2I_0^2} - \frac{E'}{I_0} \quad \left( \lim_{u \rightarrow \pm\infty} V_E(u) = \mp\infty \right)$$



$$V_E(\pm 1) = \frac{(a \mp b)^2}{2I_0^2} \geq 0$$

$$V_E(0) = \frac{a^2}{2I_0^2} - \frac{E'}{I_0}$$



$BQ = \ell$

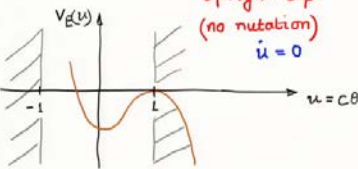
$\hat{x} = \hat{Z} \times \hat{x}$

(Refer Slide Time: 32:44)

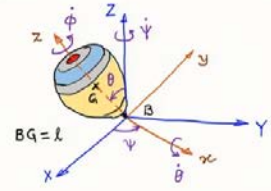
**Top dynamics: sleeping top**

$$\underbrace{\frac{1}{2} \dot{u}^2}_{KE} + \underbrace{\left( \frac{(a-bu)^2}{2I_0^2} + \left[ \frac{mg\ell}{I_0} u - \frac{E'}{I_0} \right] (1-u^2) \right)}_{\text{Effective PE: } V_E(u)} = 0$$

$$V_E(u) = -\frac{mg\ell}{I_0} u^3 + \left( \frac{E'}{I_0} + \frac{b^2}{2I_0^2} \right) u^2 + \left( \frac{mg\ell}{I_0} - \frac{ab}{I_0^2} \right) u + \frac{a^2}{2I_0^2} - \frac{E'}{I_0} \quad \left( \lim_{u \rightarrow \pm\infty} V_E(u) = \mp\infty \right)$$



Upright top  
(no nutation)  
 $\dot{u}=0$



$BG=\ell$

$$V_E(\pm 1) = \frac{(a \mp b)^2}{2I_0^2} = 0, \quad \frac{2a^2}{I_0^2} \quad (a=b)$$

$$V_E(0) = \frac{a^2}{2I_0^2} - \frac{E'}{I_0}$$


$$V'(\pm 1) = 0 \Rightarrow E = \frac{1}{2} I \dot{\phi}^2 + mg\ell \quad (\text{upright top})$$

$$V''(\pm 1) < 0 \Rightarrow \text{unstable sleeping top}$$

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**Summary**

- Symmetry and conservation
- Integrals of motion
- Examples



The summary is presented in the slide above.