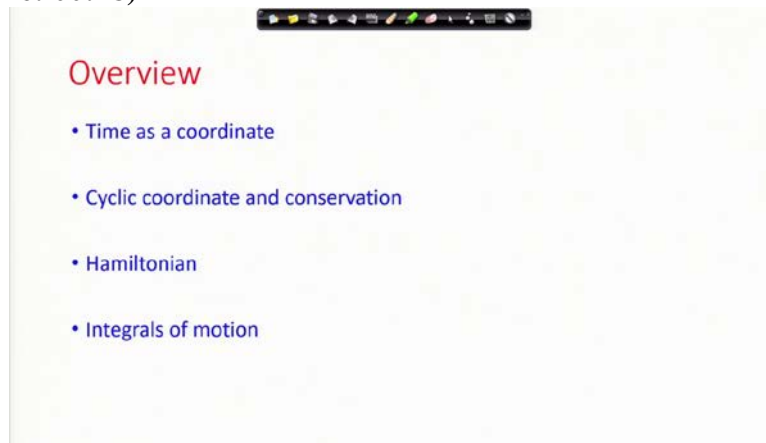


Advanced Dynamics
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Lecture - 58
Symmetries and Conservation Laws - III

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We discuss further on symmetries and the associated conservation laws. The overview of the lecture is shown above. We will consider now time as a coordinate and discuss the consequences of time as a cyclic coordinate.

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Time as a coordinate in Hamilton's principle

$$\delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) dt = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = \vec{0}$$

Let $t = t(\tau)$ (smooth, monotonic function of parameter τ)

$$dt = t' d\tau \quad t_0 = t(\tau_0), \quad t_1 = t(\tau_1) \quad \dot{\vec{q}} = \frac{d\vec{q}}{dt} \frac{d\tau}{dt} = \frac{d\vec{q}}{d\tau} \left(\frac{1}{t'} \right) = \frac{\vec{q}'}{t'}$$

$$\delta \int_{\tau_0}^{\tau_1} \underbrace{\left[\mathcal{L} \left(\vec{q}, \frac{\vec{q}'}{t'}, t \right) t' \right]}_{\tilde{\mathcal{L}}} d\tau = 0 \quad \Rightarrow \quad \tilde{\mathcal{L}}(\vec{q}, t, \vec{q}', t') = \mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) t'$$

We start with introducing time as a coordinate in Hamilton's principle. Consider the parameterization of time by another parameter τ as shown in the slide above.

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Time as a coordinate in Hamilton's principle

$$\delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) t' dt = 0 \quad \text{Let } \tilde{\mathcal{L}}(\vec{q}, t, \vec{q}', t') = \mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) t'$$

Note: Structure of Lagrange's equation of motion remains the same

$$\mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) = \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}} = \frac{\partial \mathcal{L}}{\partial \vec{q}} t' \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t} t'$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}'} = \left(\frac{1}{t'} \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) t' = \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t'} = \mathcal{L} + t' \frac{\partial \mathcal{L}}{\partial t} = \mathcal{L} + t' \left(-\frac{\vec{q}'}{t'^2} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) = \mathcal{L} - \dot{\vec{q}} \cdot \vec{p}$$

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Time as a coordinate in Hamilton's principle

$$\tilde{\mathcal{L}}(\vec{q}, t, \vec{q}', t') = \mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) t' \quad \left| \quad \mathcal{L}(\vec{q}, \frac{\vec{q}'}{t'}, t) = \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) \right.$$

Equations of motion:

$$\frac{d}{dt} \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}'} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}} = 0 \quad \left| \quad \begin{array}{l} \frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}} = \frac{\partial \mathcal{L}}{\partial \vec{q}} t' \quad \left| \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \vec{q}'} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \quad \left| \quad \frac{d}{dt} = t' \frac{d}{dt'} \right. \\ \frac{\partial \tilde{\mathcal{L}}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t} t' \quad \left| \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t'} = \mathcal{L} - \dot{\vec{q}} \cdot \vec{p} \right. \end{array} \right.$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0$$

$$\frac{d}{dt} \left(\frac{\partial \tilde{\mathcal{L}}}{\partial t'} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial t} = 0$$

$$\Rightarrow t' \frac{d}{dt} (\mathcal{L} - \dot{\vec{q}} \cdot \vec{p}) - \frac{\partial \mathcal{L}}{\partial t} t' = 0$$

$$\Rightarrow \underline{\frac{d}{dt} (\mathcal{L} - \dot{\vec{q}} \cdot \vec{p}) - \frac{\partial \mathcal{L}}{\partial t} = 0}$$

The Lagrange's equations of motion are derived in detail in the 2 slides above. We obtain a new equation corresponding to the coordinate t .

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Momentum conjugate to time: Hamiltonian

$$\frac{d}{dt} \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{t}} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial t} = 0$$

$$p_t = \mathcal{L} - \vec{p} \cdot \vec{a}_t$$

(Momentum conjugate of time)

Define $\mathcal{H} = -p_t = \vec{p} \cdot \vec{a}_t - \mathcal{L}$ as Hamiltonian

- Hamiltonian is the negative of momentum conjugate of time (as a coordinate)

We concentrate on this new equation, as shown in the slide above and define the Hamiltonian as negative of the momentum conjugate to the time coordinate.

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Time as a cyclic coordinate

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{t}} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial t} &= 0 \\ \underbrace{-\mathcal{H}}_{\text{No explicit time dependence}} &\quad \underbrace{t' \frac{\partial \mathcal{L}}{\partial t}}_{\Rightarrow \frac{\partial \mathcal{L}}{\partial t} = 0} = 0 \end{aligned} \right\} \Rightarrow t' \frac{d}{dt} (\mathcal{L} - \vec{p} \cdot \vec{a}_t) = 0$$

$$\Rightarrow \frac{d}{dt} (-\mathcal{H}) = 0$$

$$\Rightarrow \mathcal{H} = \vec{p} \cdot \vec{a}_t - \mathcal{L} \text{ is a constant of motion (no external force)}$$

- Time as a cyclic coordinate
- Lagrangian is not EXPLICITLY time dependent
- Hamiltonian is conserved



Next, we consider the situation where time is a cyclic coordinate. If the system has no external forcing, and time is a cyclic coordinate, then the momentum conjugate to time, the Hamiltonian, is conserved.

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Hamiltonian

$$\mathcal{L} = \mathcal{L}(\vec{a}_r, \dot{\vec{a}}_r, t)$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial \vec{a}_r} \cdot \dot{\vec{a}}_r + \frac{\partial \mathcal{L}}{\partial \dot{\vec{a}}_r} \cdot \ddot{\vec{a}}_r + \frac{\partial \mathcal{L}}{\partial t}$$

$$\Rightarrow \frac{d\mathcal{L}}{dt} = \left[\underbrace{\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{a}}_r} \right)}_{\vec{p}} - \vec{Q} \right] \cdot \dot{\vec{a}}_r + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{a}}_r} \cdot \ddot{\vec{a}}_r}_{\vec{F}} + \frac{\partial \mathcal{L}}{\partial t} \quad \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{a}}_r} \right) - \frac{\partial \mathcal{L}}{\partial \vec{a}_r} = \vec{Q} \right)$$

$$\Rightarrow \frac{d\mathcal{L}}{dt} = \frac{d}{dt} (\vec{p} \cdot \dot{\vec{a}}_r) - \vec{Q} \cdot \dot{\vec{a}}_r + \frac{\partial \mathcal{L}}{\partial t} \Rightarrow \boxed{\frac{d}{dt} (\vec{p} \cdot \dot{\vec{a}}_r - \mathcal{L}) = \frac{d\mathcal{H}}{dt} = \vec{Q} \cdot \dot{\vec{a}}_r - \frac{\partial \mathcal{L}}{\partial t}}$$

- No external force, and Lagrangian is not EXPLICITLY time dependent
- **Hamiltonian is conserved**

An alternative approach presented in the slide above.

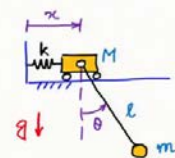
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Structure of Lagrangian

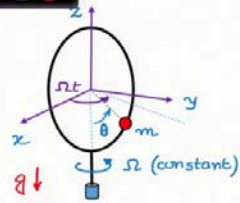
$$\mathcal{L} = \mathcal{L}(\vec{a}_r, \dot{\vec{a}}_r, t) = T - V$$

$$= T_2 + T_1 + T_0 - V$$

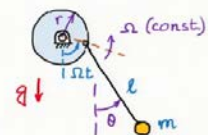
T_n : homogeneous of degree n in generalized velocities



$$T = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (\ell^2 \dot{\theta}^2 + 2\ell \dot{x} \dot{\theta} \cos \theta)$$



$$T = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta)$$



$$T_0 + T_2 + T_1$$

$$T = \frac{1}{2} m [r^2 \Omega^2 + \ell^2 \dot{\theta}^2 + 2r\ell \Omega \dot{\theta} \cos(\theta - \Omega t)]$$

In the above slide, we discuss the structure of the Lagrangian. It is observed that the kinetic energy term may be decomposed as $T = T_0 + T_1 + T_2$, where T_0 , T_1 , and T_2 are, respectively, terms that are independent, linear and quadratic in the generalized velocities.

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Hamiltonian

$$\mathcal{L} = \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) = T(\vec{q}, \dot{\vec{q}}, t) - V(\vec{q}, t) \quad (\text{velocity independent potential})$$

$$= T_1 + T_2 + T_3 - V$$

$$\mathcal{H} = T_2 - T_3 + V$$

- If $\mathcal{L} = T_2 - V \Rightarrow \mathcal{H} = T_2 + V = T + V = E$ (Total mechanical energy)
 $T = T_2$ ($\mathcal{H} = E$ if $T = T_2$ and $\frac{\partial V}{\partial \vec{q}} = 0$)
- If $\frac{\partial \mathcal{L}}{\partial t} = 0$ and $\vec{q} = \vec{0} \Rightarrow \mathcal{H}$ is conserved (may not be total mechanical energy)
 No explicit time dependence $\mathcal{L} = \mathcal{L}(\vec{q}, \dot{\vec{q}})$




Based on the structure of the Lagrangian, an interpretation of the Hamiltonian is provided in the above slide.

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Infinitesimal symmetries of the Lagrangian

- Importance of infinitesimal symmetries and conserved quantities: helps us integrate the equations of motion (order of derivative one less)
- Integrals of motion

Example: Use of linear and angular momentum, and energy conservation,
 Central force motion (angular momentum and energy conservation)



The reasons why we are interested in infinitesimal symmetries of the Lagrangian are presented in the above slide.

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The image shows a presentation slide with a light yellow background. At the top, there is a black toolbar with various icons. The word "Summary" is written in red at the top left. Below it, there is a bulleted list of topics in blue text. In the bottom right corner, there is a small video inset showing a man with a mustache, wearing a light blue shirt, speaking against a blue background.

Summary

- Time as a coordinate
- Cyclic coordinate and conservation
- Hamiltonian
- Integrals of motion

The summary is provided in the slide above.