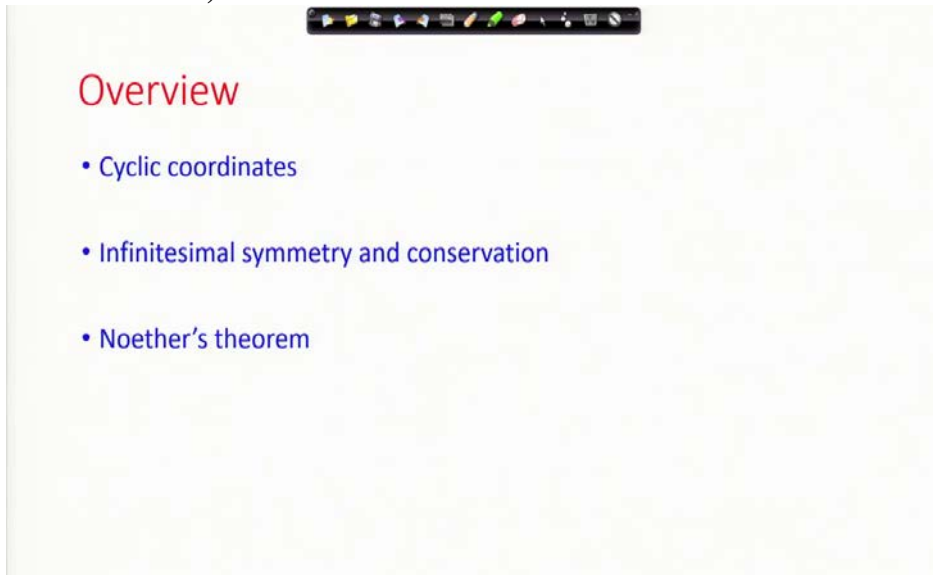


**Advanced Dynamics**  
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**Lecture - 57**  
**Symmetries and Conservation Laws - II**

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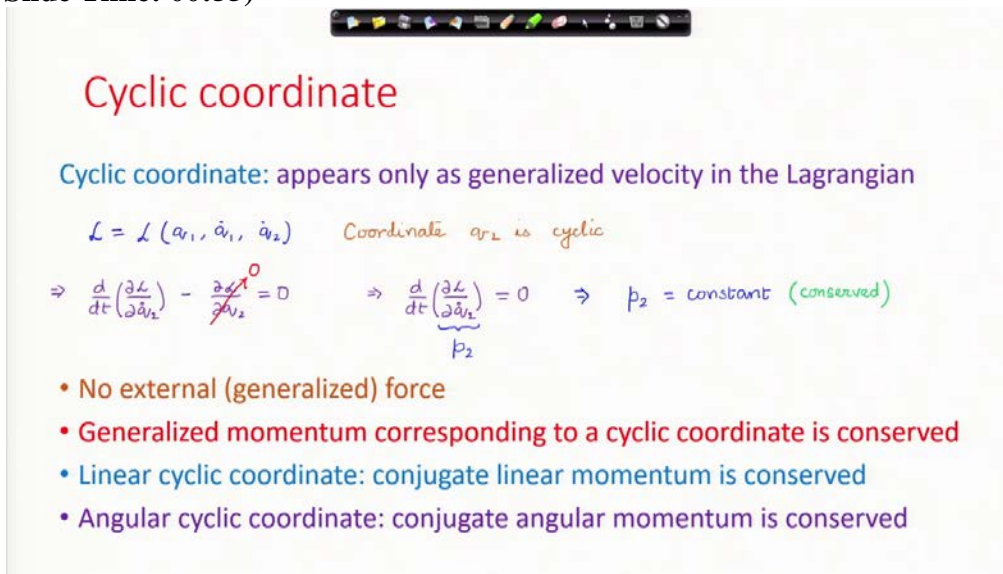


**Overview**

- Cyclic coordinates
- Infinitesimal symmetry and conservation
- Noether's theorem

In this lecture, I am going to continue further with symmetries and conservation laws. This is the overview of what we are going to discuss in this lecture, we are going to introduce this concept of cyclic coordinates. Then we are going to discuss infinitesimal symmetry and the corresponding conservation which goes by the name Noether's theorem.

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**Cyclic coordinate**

**Cyclic coordinate:** appears only as generalized velocity in the Lagrangian

$L = L(q_1, \dot{q}_1, \dot{q}_2)$       Coordinate  $q_2$  is cyclic

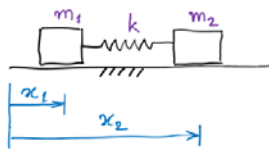
$\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) - \frac{\partial L}{\partial q_2} = 0$        $\Rightarrow \frac{d}{dt} \underbrace{\left( \frac{\partial L}{\partial \dot{q}_2} \right)}_{p_2} = 0 \quad \Rightarrow \quad p_2 = \text{constant (conserved)}$

- No external (generalized) force
- Generalized momentum corresponding to a cyclic coordinate is conserved
- Linear cyclic coordinate: conjugate linear momentum is conserved
- Angular cyclic coordinate: conjugate angular momentum is conserved

First we are going to look at cyclic coordinates. The concept is explained in the above slide. The consequences are also discussed.

## Cyclic coordinate: examples

- Blocks on a frictionless horizontal surface connected by a spring:



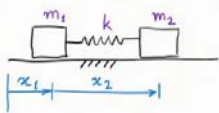
$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1 - l)^2$$

- We do not see any cyclic coordinate here

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**Cyclic coordinate: examples**

- Blocks on a frictionless horizontal surface connected by a spring:



$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 - \frac{1}{2} k (x_2 - l)^2 \quad (x_1 \text{ cyclic})$$

- Choice of coordinates might hide cyclic coordinates
- We need to study infinitesimal symmetries of the Lagrangian

In the 2 slides above, we observe that choice of coordinates can hide cyclic coordinates. This is investigated using the concept of symmetry of the Lagrangian.

## Infinitesimal symmetries of the Lagrangian

### Infinitesimal (smooth) transformation of Lagrangian

$$\left. \frac{d}{d\sigma} \left[ \tilde{\mathcal{L}}(\tilde{\mathbf{z}}(t, \sigma), \dot{\tilde{\mathbf{z}}}(t, \sigma), t) \right] \right|_{\sigma=0} = 0$$

$$\Rightarrow \left[ \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{\mathbf{z}}} \cdot \tilde{\mathbf{z}}'(t, \sigma) + \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{\mathbf{z}}}} \cdot \dot{\tilde{\mathbf{z}}}'(t, \sigma) \right]_{\sigma=0} = 0 \quad \left( \tilde{\mathbf{z}}' = \frac{\partial \tilde{\mathbf{z}}}{\partial \sigma} \right)$$

Lagrange's equation:  $\frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{\mathbf{z}}}} \right) = \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{\mathbf{z}}}$  Also,  $\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{\mathbf{z}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$   $\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{\mathbf{z}}}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}$  at  $\sigma = 0$

$$\Rightarrow \left[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \cdot \tilde{\mathbf{z}}'(t, \sigma) + \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \dot{\tilde{\mathbf{z}}}'(t, \sigma) \right]_{\sigma=0} = 0$$

## Infinitesimal symmetries of the Lagrangian

### Infinitesimal (smooth) transformation of Lagrangian

$$\left[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \cdot \tilde{\mathbf{z}}'(t, \sigma) + \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \dot{\tilde{\mathbf{z}}}'(t, \sigma) \right]_{\sigma=0} = 0 \quad \Rightarrow \quad \left[ \frac{d\vec{\mathbf{p}}}{dt} \cdot \tilde{\mathbf{z}}' + \vec{\mathbf{p}} \cdot \frac{d\tilde{\mathbf{z}}'}{dt} \right]_{\sigma=0} = 0$$

$$\Rightarrow \frac{d}{dt} \left[ \vec{\mathbf{p}} \cdot \tilde{\mathbf{z}}' \right]_{\sigma=0} = 0 \quad \left( \vec{\mathbf{p}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)$$

$$\Rightarrow \underbrace{\vec{\mathbf{p}} \cdot \tilde{\mathbf{z}}'}_{\sigma=0} \text{ is conserved}$$

Infinitesimal generator  
of the transformation

$$\begin{cases} \tilde{\mathbf{z}} = \mathbf{q} + \sigma \vec{\mathbf{b}} \quad (\vec{\mathbf{b}}: \text{constant}) \\ \frac{d\tilde{\mathbf{z}}}{d\sigma} = \vec{\mathbf{b}} \quad (\text{Translation direction}) \end{cases}$$

The concept of infinitesimal symmetry of the Lagrangian is explained in the 2 slides above.

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## Infinitesimal symmetries of the Lagrangian

**Example: cyclic coordinate**


$\mathcal{L} = \mathcal{L}(q_1, \dot{q}_1, \dot{q}_2)$     Coordinate  $q_2$  is cyclic

Let  $\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \sigma \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow \left. \frac{d\vec{z}}{d\sigma} \right|_{\sigma=0} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

Symmetry transformation      Infinitesimal generator of symmetry transformation

$\vec{p} = \begin{Bmatrix} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \\ \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \end{Bmatrix}$

Conserved quantity:  $\vec{p} \cdot \left. \frac{d\vec{z}}{d\sigma} \right|_{\sigma=0} = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = p_2$



The application of infinitesimal symmetry in the case of cyclic coordinate is explained in the above slide.

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## Cyclic coordinate: examples

**Example: spring connected blocks**

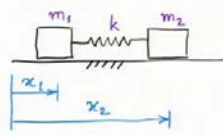

$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1 - l)^2$

$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \sigma \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$     Symmetry transformation

$\left. \frac{d\vec{z}}{d\sigma} \right|_{\sigma=0} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$     Infinitesimal generator

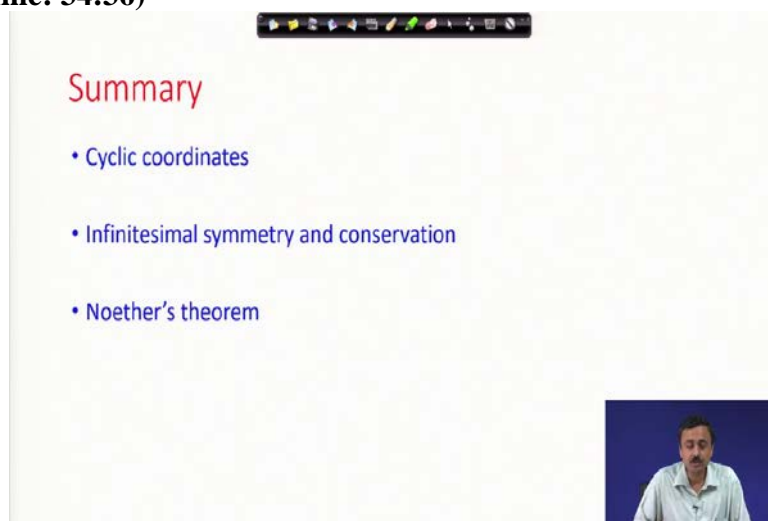
$\vec{p} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} m_1 \dot{x}_1 \\ m_2 \dot{x}_2 \end{Bmatrix}$

Conserved quantity:  $\vec{p} \cdot \left. \frac{d\vec{z}}{d\sigma} \right|_{\sigma=0} = m_1 \dot{x}_1 + m_2 \dot{x}_2$     Total linear momentum

Now, we come back to that example, with our original scheme of coordinate position where we do not see that symmetry in a certain coordinatization scheme. It is shown how the concept of infinitesimal symmetry can be used to uncover the conserved quantity.

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Summary

- Cyclic coordinates
- Infinitesimal symmetry and conservation
- Noether's theorem

The slide is a presentation slide with a light yellow background. At the top, there is a black toolbar with various icons. The title 'Summary' is written in red. Below it, there is a bulleted list of three items: 'Cyclic coordinates', 'Infinitesimal symmetry and conservation', and 'Noether's theorem'. In the bottom right corner, there is a small video inset showing a man with a beard and mustache, wearing a light blue shirt, speaking.

The above slide summarizes the discussions.