

Advanced Dynamics
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Lecture – 55
Systems with Constraints – IV

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Overview

- Problems with non-holonomic constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier

We will discuss systems with constraints a little further. And in this lecture, I am going to introduce non-holonomic constraints and discuss the derivation of the equation of motion using the method of Lagrange multipliers.

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Systems with non-holonomic constraints

- Non-integrable constraints: linear in coordinate time rates

$$\vec{C} = \underbrace{A(\vec{r}, t)}_{(m-n) \times m} \cdot \underbrace{\dot{\vec{r}}}_{m \times 1} + \underbrace{\vec{b}}_{(m-n) \times 1} = \vec{0}$$

$$C_j = \sum_{i=1}^m A_{ji} \dot{r}_i + b_j = 0 \quad \forall j = 1, \dots, m-n$$

$$\Rightarrow \sum_{i=1}^m A_{ji} dr_i + b_j dt = 0 \quad \forall j = 1, \dots, m-n$$

$$\delta C_j = \sum_{i=1}^m A_{ji} \delta r_i = 0 \quad \forall j = 1, \dots, m-n \quad \left\{ \begin{array}{l} \text{variation in the } C \\ \text{coordinate rates } (\dot{r}) \end{array} \right.$$

What are non-holonomic constraints? Constraints that come with derivatives of the coordinates and which we cannot integrate to obtain a constraint relating only the


coordinates. We consider non-holonomic constraints that are linear in the generalized velocities. The above slide shows how non-holonomic constraints relate the variations of the coordinates.

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Systems with non-holonomic constraints

Hamilton's principle: $\delta \mathcal{A} = 0 \Rightarrow \int_{t_0}^{t_1} \delta \mathcal{L}(\vec{r}, \dot{\vec{r}}, t) dt = 0$

$$\Rightarrow \int_{t_0}^{t_1} \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \right) + \frac{\partial \mathcal{L}}{\partial \vec{r}} \right] \cdot \delta \vec{r} dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \left[\left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \right) + \frac{\partial \mathcal{L}}{\partial r_1} \right) \delta r_1 + \dots + \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_m} \right) + \frac{\partial \mathcal{L}}{\partial r_m} \right) \delta r_m \right] dt = 0$$


The variational statement obtained from the Hamilton's principle is presented in the above slide.

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Systems with non-holonomic constraints


$$\int_{t_0}^{t_1} \left[\left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \right) + \frac{\partial \mathcal{L}}{\partial r_1} \right) \delta r_1 + \dots + \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_m} \right) + \frac{\partial \mathcal{L}}{\partial r_m} \right) \delta r_m \right] dt = 0$$

Subject to $\delta C_1 = 0 \Rightarrow A_{11} \delta r_1 + \dots + A_{1m} \delta r_m = 0$

⋮

$\delta C_{m-n} = 0 \Rightarrow A_{(m-n)1} \delta r_1 + \dots + A_{(m-n)m} \delta r_m = 0$

Using $(m-n)$ Lagrange multipliers

$$\int_{t_0}^{t_1} \left[\sum_{i=1}^m \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} \right) \delta r_i + \sum_{j=1}^{m-n} \lambda_j \sum_{i=1}^m A_{ji} \delta r_i \right] dt = 0$$


Next, we introduce the non-holonomic relations into the variational statement using the Lagrange multipliers, as shown above.

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Systems with non-holonomic constraints

$$\int_{t_0}^{t_1} \left[\sum_{i=1}^m \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} \right) \delta r_i + \sum_{j=1}^{m-n} \lambda_j \sum_{i=1}^m A_{ji} \delta r_i \right] dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \sum_{i=1}^m \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j A_{ji} \right] \delta r_i dt = 0 \quad (1)$$

Choose λ_j $j=1, \dots, m-n$ s.t. $-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j A_{ji} = 0 \quad i=1, \dots, m-n$
(m-n equations)

Then integral (1) vanishes for arbitrary δr_i $i = m-n+1, \dots, m$ when

$$-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j A_{ji} = 0 \quad i = m-n+1, \dots, m$$

(n equations)

Using the arguments discussed previously, we obtain the equations of motion as presented in the slide above.

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Systems with non-holonomic constraints

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) - \frac{\partial \mathcal{L}}{\partial r_i} - \sum_{j=1}^{m-n} \lambda_j A_{ji} = 0 \quad i=1, \dots, m-n$$


(m-n equations)

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) - \frac{\partial \mathcal{L}}{\partial r_i} - \sum_{j=1}^{m-n} \lambda_j A_{ji} = 0 \quad i = m-n+1, \dots, m$$

(n equations)

$$\left. \begin{array}{l} \sum_{i=1}^m A_{1i} \dot{r}_i + b_1 = 0 \\ \vdots \\ \sum_{i=1}^m A_{(m-n)i} \dot{r}_i + b_{(m-n)} = 0 \end{array} \right\} \begin{array}{l} m \text{ equations of motion} \\ (2m-n \text{ unknowns}) \\ m-n \text{ constraints} \end{array}$$

Total $2m-n$ equations in $2m-n$ unknowns



The complete set of equations involve the obtained equations of motion and the non-holonomic constraints as presented above.

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Systems with non-holonomic constraints

Determination and elimination of Lagrange multiplier:

- Time differentiate constraint once

$$C_j = \sum_{i=1}^m A_{ji} \dot{r}_i + b_j = 0 \quad \Rightarrow \quad \sum_{i=1}^m A_{ji} \ddot{r}_i + \sum_{k=1}^m \sum_{l=1}^m D_{jkl} \dot{r}_i \dot{r}_k + \sum_{k=1}^m \frac{\partial b_j}{\partial r_k} \dot{r}_k = 0 \quad \forall j$$

- Eliminate accelerations using equations of motion
- Solve for Lagrange multiplier
- Eliminate Lagrange multiplier in equations of motion



We eliminate the Lagrange multipliers following the steps outline in the above slide.

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Systems with non-holonomic constraints

Setting up initial conditions:

- Use the constraint equations and its derivative

$$\underbrace{\vec{r}(0) = \vec{r}_0, \quad \dot{\vec{r}}(0) = \dot{\vec{r}}_0}_{2m} \quad \text{s.t.} \quad \underbrace{\vec{C}(\vec{r}_0, \dot{\vec{r}}_0, 0) = 0}_{m-n}$$

$$\Rightarrow C_j = \sum_{i=1}^m A_{ji}(\vec{r}_0, 0) v_{i0} + b_j(\vec{r}_0, 0) = 0 \quad \forall j$$

$$\Rightarrow 2m - (m - n) = \underline{(m+n) \text{ independent initial conditions}}$$


The above slide discusses the setting-up of the initial conditions.

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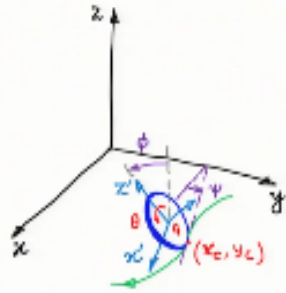

Example:

Dynamics of a rolling coin

Coordinates: $(x_c, y_c, \psi, \phi, \theta)$
 Contact point: (x_c, y_c)
 Orientation: ψ, ϕ, θ
 Spin: $\dot{\phi}, \dot{\theta}$

Constraints:

- Always in ground contact $\Rightarrow z_c = r \cos \phi$
- No-slip constraint $\Rightarrow \dot{x}_c - r \dot{\theta} \sin \psi = 0$
 $\dot{y}_c - r \dot{\phi} \cos \psi = 0$ } cannot be integrated

Now, let us look at this example of the dynamics of a rolling coin.

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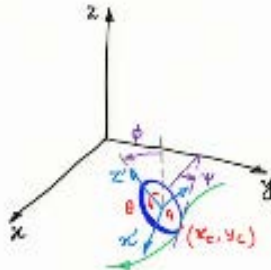

Velocity of center of mass (x-y-z frame)

$$\begin{Bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{z}_c \end{Bmatrix} = \begin{Bmatrix} \dot{x}_c + r \dot{\psi} \sin \phi \\ \dot{y}_c - r \dot{\psi} \cos \phi \\ r \dot{\phi} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{z}_c \end{Bmatrix} = \begin{Bmatrix} \dot{x}_c + r \dot{\psi} \sin \phi + r \dot{\phi} \sin \psi \\ \dot{y}_c + r \dot{\psi} \cos \phi - r \dot{\phi} \cos \psi \\ -r \dot{\phi} \sin \phi \end{Bmatrix}$$

Translational KE

$$T_T = \frac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2)$$

$$= \frac{1}{2} m [\dot{x}_c^2 + \dot{y}_c^2 + r^2 \dot{\psi}^2 \sin^2 \phi + r^2 \dot{\phi}^2 + 2r \dot{x}_c \dot{\psi} \sin \phi - 2r \dot{y}_c \dot{\psi} \cos \phi + 2r \dot{\phi} \dot{x}_c \sin \psi - 2r \dot{\phi} \dot{y}_c \cos \psi]$$



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Angular velocity ($x'-y'-z'$ frame)

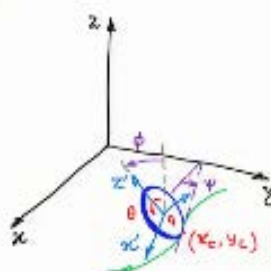

$$\vec{\omega}' = \dot{\theta} \hat{j}' + \dot{\phi} \hat{i}' + \dot{\psi} (c\phi \hat{k}' + s\phi \hat{j}')$$

$$I_G = \begin{bmatrix} \frac{1}{4}mr^2 & 0 & 0 \\ 0 & \frac{1}{2}mr^2 & 0 \\ 0 & 0 & \frac{1}{4}mr^2 \end{bmatrix}$$

Rotational KE

$$T_R = \frac{1}{2} \vec{\omega}' \cdot I_G \vec{\omega}'$$

$$= \frac{1}{2} [\dot{\phi} \hat{i}' + (\dot{\theta} + \dot{\psi} s\phi) \hat{j}' + \dot{\psi} c\phi \hat{k}'] \cdot \left[\frac{1}{4}mr^2 \dot{\phi} \hat{i}' + \frac{1}{2}mr^2 (\dot{\theta} + \dot{\psi} s\phi) \hat{j}' + \frac{1}{2}mr^2 \dot{\psi} c\phi \hat{k}' \right]$$

$$= \frac{1}{8}mr^2 [\dot{\phi}^2 + 2(\dot{\theta} + \dot{\psi} s\phi)^2 + \dot{\psi}^2 c^2\phi]$$



The translational and rotational kinetic energy terms are presented in the 2 slides above.

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$$T_T = \frac{1}{2} m [\dot{x}_c^2 + \dot{y}_c^2 + r^2 \dot{\psi}^2 s^2\phi + r^2 \dot{\phi}^2 + 2r\dot{x}_c \dot{\psi} s\phi - 2r\dot{y}_c \dot{\psi} c\phi + 2r\dot{x}_c \dot{\phi} s\phi - 2r\dot{y}_c \dot{\phi} c\phi]$$

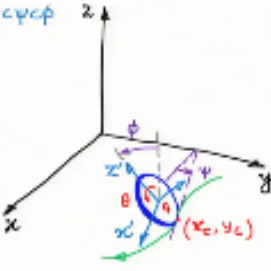

$$T_R = \frac{1}{8}mr^2 [\dot{\phi}^2 + 2(\dot{\theta} + \dot{\psi} s\phi)^2 + \dot{\psi}^2 c^2\phi]$$

$$V = mgrc\phi$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} m [\dot{x}_c^2 + \dot{y}_c^2 + r^2 \dot{\psi}^2 s^2\phi + r^2 \dot{\phi}^2 + 2r\dot{x}_c \dot{\psi} s\phi - 2r\dot{y}_c \dot{\psi} c\phi + 2r\dot{x}_c \dot{\phi} s\phi - 2r\dot{y}_c \dot{\phi} c\phi] + \frac{1}{8}mr^2 [\dot{\phi}^2 + 2(\dot{\theta} + \dot{\psi} s\phi)^2 + \dot{\psi}^2 c^2\phi] - mgrc\phi$$

constraints:
$$\begin{cases} \dot{x}_c - r\dot{\theta} s\psi = 0 \\ \dot{y}_c - r\dot{\theta} c\psi = 0 \end{cases}$$

The Lagrangian and the 2 non-holonomic constraints are presented in the above slide.

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Lagrangian:

$$\mathcal{L} = \frac{1}{2} m \left[\dot{x}_c^2 + \dot{y}_c^2 + r^2 \dot{\psi}^2 \sin^2 \phi + r^2 \dot{\phi}^2 + 2r\dot{x}_c \dot{\psi} \sin \psi \phi - 2r\dot{y}_c \dot{\phi} \cos \psi \phi \right] + \frac{1}{8} m r^2 \left[\dot{\phi}^2 + 2(\dot{\theta} + \dot{\psi} \cos \phi)^2 + \dot{\psi}^2 \sin^2 \phi \right] - m g r \cos \phi$$

constraints: $\begin{cases} \dot{x}_c - r \dot{\theta} \cos \psi = 0 \\ \dot{y}_c - r \dot{\theta} \sin \psi = 0 \end{cases}$

Variational problem:

$$\int_{t_0}^{t_1} \left[\delta \mathcal{L} + \lambda_1 (\delta x_c - r \cos \psi \delta \theta) + \lambda_2 (\delta y_c - r \sin \psi \delta \theta) \right] dt = 0$$

Complex equations of motion!

We can derive the equations of motion but they are nonlinear, coupled and extremely complex.

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Interpretation of Lagrange multipliers

Variational problem:

$$\int_{t_0}^{t_1} \left[\delta \mathcal{L} + \lambda_1 (\delta x_c - r \cos \psi \delta \theta) + \lambda_2 (\delta y_c - r \sin \psi \delta \theta) \right] dt = 0$$

Virtual work done by constraint forces

$$\delta W^c = (\vec{F}_t + \vec{F}_n) \cdot (\delta x_c \hat{i} + \delta y_c \hat{j}) + r F_t \delta \theta$$

$$\vec{F}_t = F_t (\cos \psi \hat{i} + \sin \psi \hat{j}) \quad \vec{F}_n = F_n (-\sin \psi \hat{i} + \cos \psi \hat{j})$$

$$\Rightarrow \delta W^c = (F_t \cos \psi - F_n \sin \psi) \delta x_c + (F_t \sin \psi + F_n \cos \psi) \delta y_c + r F_t \delta \theta$$

$$= \lambda_1 \delta x_c + \lambda_2 \delta y_c - r (\lambda_1 \cos \psi + \lambda_2 \sin \psi) \delta \theta$$

$$\begin{cases} F_t \cos \psi - F_n \sin \psi = \lambda_1 \\ F_t \sin \psi + F_n \cos \psi = \lambda_2 \end{cases} \Rightarrow \begin{cases} F_t = \frac{\lambda_1 \cos \psi + \lambda_2 \sin \psi}{\cos^2 \psi + \sin^2 \psi} \\ F_n = \frac{\lambda_2 \cos \psi - \lambda_1 \sin \psi}{\cos^2 \psi + \sin^2 \psi} \end{cases}$$

The interpretation of the Lagrange multipliers is presented in the above slide.

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Lagrangian:

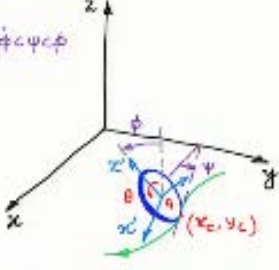

$$\mathcal{L} = \frac{1}{2} m \left[\dot{x}_c^2 + \dot{y}_c^2 + r^2 \dot{\psi}^2 \sin^2 \phi + r^2 \dot{\phi}^2 + 2r\dot{x}_c \dot{\psi} \cos \psi \phi - 2r\dot{y}_c \dot{\psi} \sin \psi \phi + 2r\dot{x}_c \dot{\phi} \sin \psi \phi - 2r\dot{y}_c \dot{\phi} \cos \psi \phi \right] + \frac{1}{2} m r^2 \left[\dot{\phi}^2 + 2(\dot{\theta} + \dot{\psi} \cos \phi)^2 + \dot{\psi}^2 \sin^2 \phi \right] - mgr \cos \phi$$

constraints: $\begin{cases} \dot{x}_c - r \dot{\theta} \cos \psi = 0 \\ \dot{y}_c - r \dot{\theta} \sin \psi = 0 \end{cases}$

Reduced Lagrangian: (Using constraints to eliminate \dot{x}_c, \dot{y}_c in \mathcal{L})

$$\mathcal{L} = \frac{1}{2} m r^2 \left[\frac{3}{2} \dot{\theta}^2 - \frac{5}{4} \dot{\phi}^2 + \frac{3}{2} \dot{\psi}^2 \sin^2 \phi + \frac{1}{4} \dot{\psi}^2 \cos^2 \phi + 3\dot{\theta} \dot{\psi} \sin \phi \right] - mgr \cos \phi$$

- θ, ϕ, ψ are independent coordinates
- Constraint forces cannot be determined

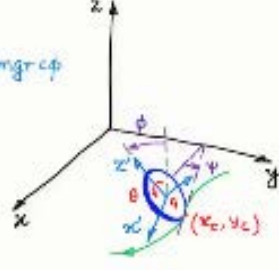

In the above slide, we obtain a reduced Lagrangian involving only the orientational coordinates. This is done by using the non-holonomic constraints to eliminate the contact point coordinate rates.

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Reduced Lagrangian:

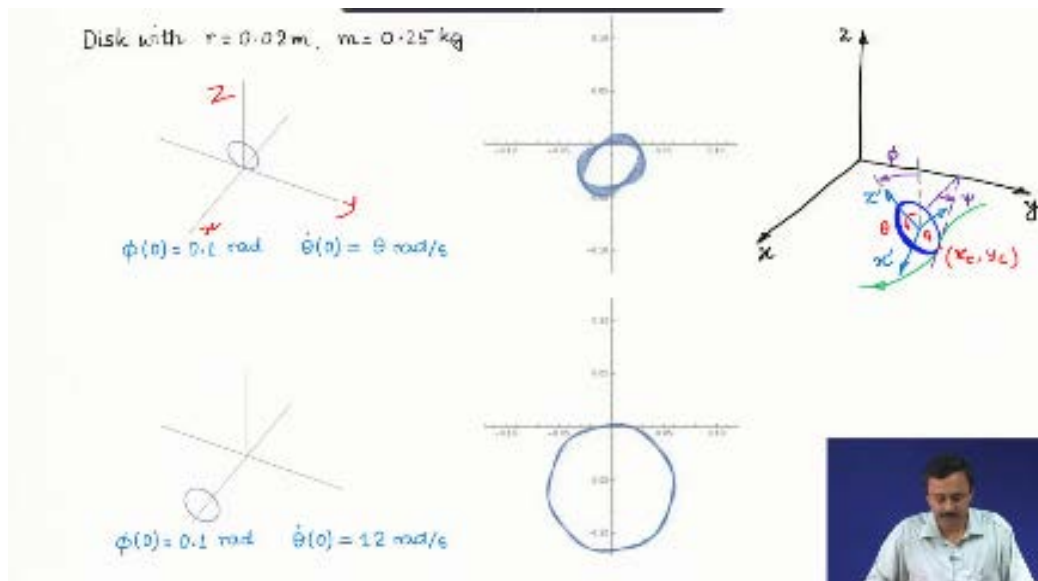
$$\mathcal{L} = \frac{1}{2} m r^2 \left[\frac{3}{2} \dot{\theta}^2 - \frac{5}{4} \dot{\phi}^2 + \frac{3}{2} \dot{\psi}^2 \sin^2 \phi + \frac{1}{4} \dot{\psi}^2 \cos^2 \phi + 3\dot{\theta} \dot{\psi} \sin \phi \right] - mgr \cos \phi$$

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} &= 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} &= 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) - \frac{\partial \mathcal{L}}{\partial \psi} &= 0 \end{aligned} \right\}$$

$$\begin{aligned} \dot{x}_c - r \dot{\theta} \cos \psi &= 0 \\ \dot{y}_c - r \dot{\theta} \sin \psi &= 0 \end{aligned}$$



The equations of motion are now obtained, and the evolution of the contact point is obtained by integrating (numerically) the non-holonomic constraints.

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Snapshots of two simulations are presented in the above slide.

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Summary

- Problems with non-holonomic constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier