

Advanced Dynamics
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Lecture – 54
System with Constraints – III

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Overview

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier

We are going to continue our discussions on systems with constraints. We had started discussing problems with constraints and used the method of Lagrange multiplier to derive the equations of motion. And in the last lecture, I had also discussed the interpretation of Lagrange multiplier and how that is used in finding out the constraint forces.

In this lecture, I am going to take a few more examples and show you the full power of this method of Lagrange multiplier in determining constrained forces. Recall that in order to determine constrained forces, we have to first break the constraint, and then introduce the constraint as a mathematical condition.

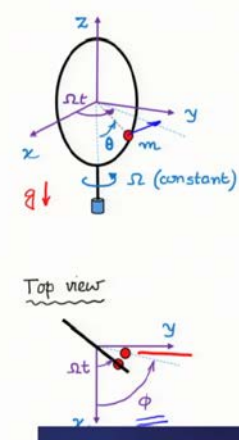
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Example 3:

Lateral reaction force on the bead on a rotating hoop

Physical coordinates: (θ, ϕ)

Constraint: $\phi - \Omega t = 0$





We will start with this example of bead on a hoop. The hoop is rotating at a constant angular velocity. I would like to find out the lateral reaction force on the bead from the hoop.

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Lagrangian: $\mathcal{L} = T - V$

$T = \frac{1}{2} m v_m^2$

$\vec{r}_m = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} - r \cos \theta \hat{k}$

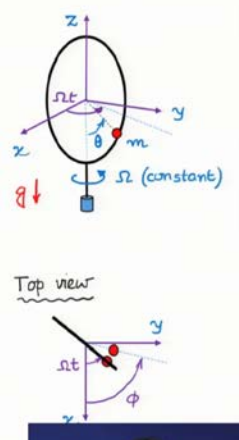
$\vec{v}_m = (r \dot{\theta} \cos \phi - r \dot{\phi} \sin \theta \sin \phi) \hat{i} + (r \dot{\theta} \sin \phi + r \dot{\phi} \sin \theta \cos \phi) \hat{j} + r \dot{\theta} \sin \theta \hat{k}$


$\Rightarrow v_m^2 = r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta$

$\Rightarrow T = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta)$

$V = -mg r \cos \theta$

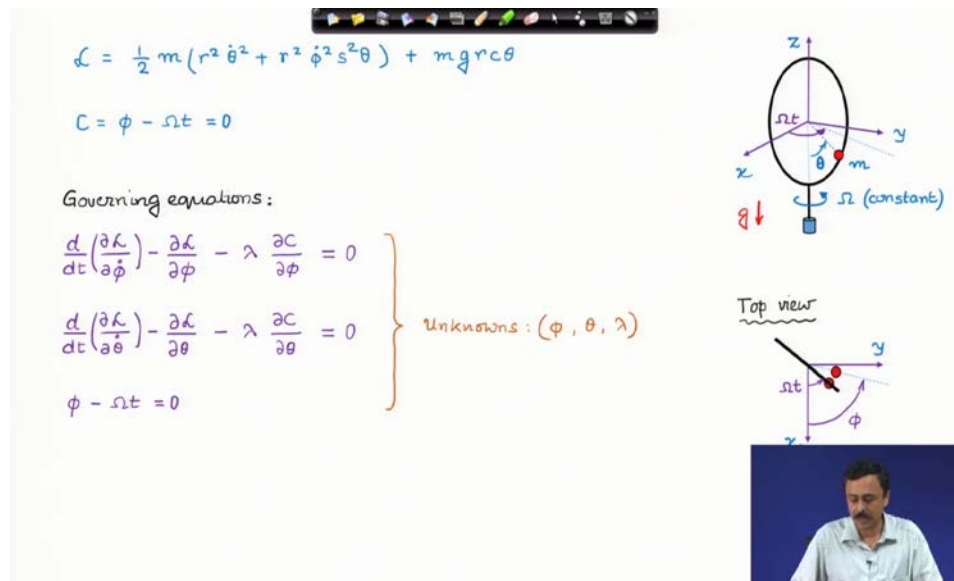
$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) + mg r \cos \theta$





We introduce a new coordinate ϕ as shown above. Thus, we break the constraint that the bead always remains on the hoop.

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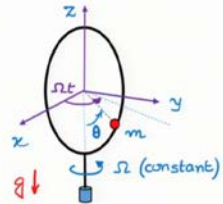


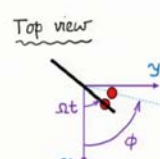
$$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) + m g r \cos \theta$$

$$C = \phi - \Omega t = 0$$

Governing equations:

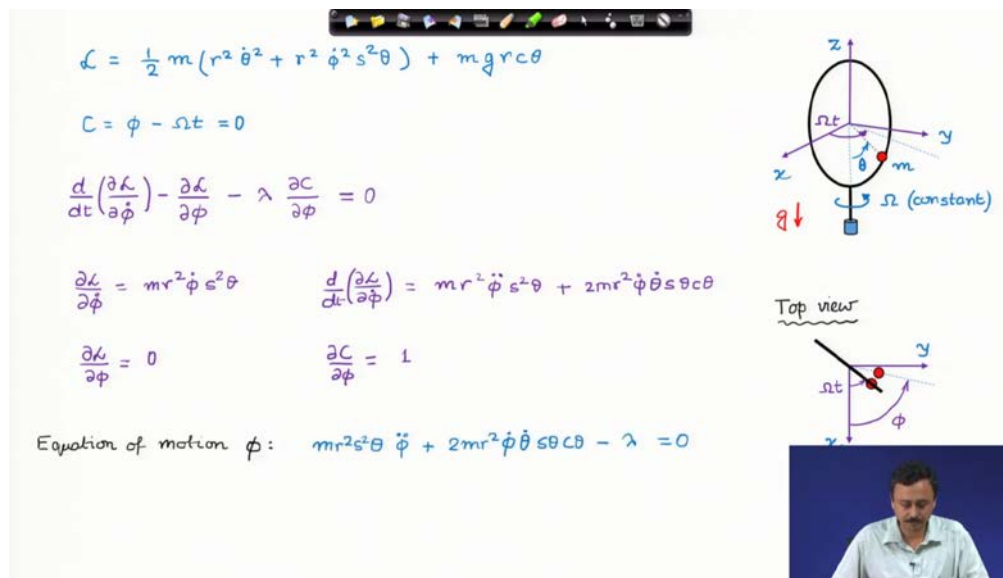
$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} - \lambda \frac{\partial C}{\partial \phi} &= 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} &= 0 \\ \phi - \Omega t &= 0 \end{aligned} \right\} \text{Unknowns: } (\phi, \theta, \lambda)$$





We introduce the constraint separately to enforce the condition that the bead actually moves on the hoop as shown above.

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$$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) + m g r \cos \theta$$

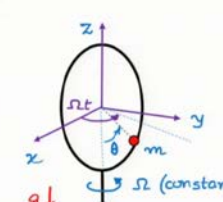
$$C = \phi - \Omega t = 0$$

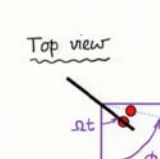
$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} - \lambda \frac{\partial C}{\partial \phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} \sin^2 \theta \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m r^2 \ddot{\phi} \sin^2 \theta + 2 m r^2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta$$

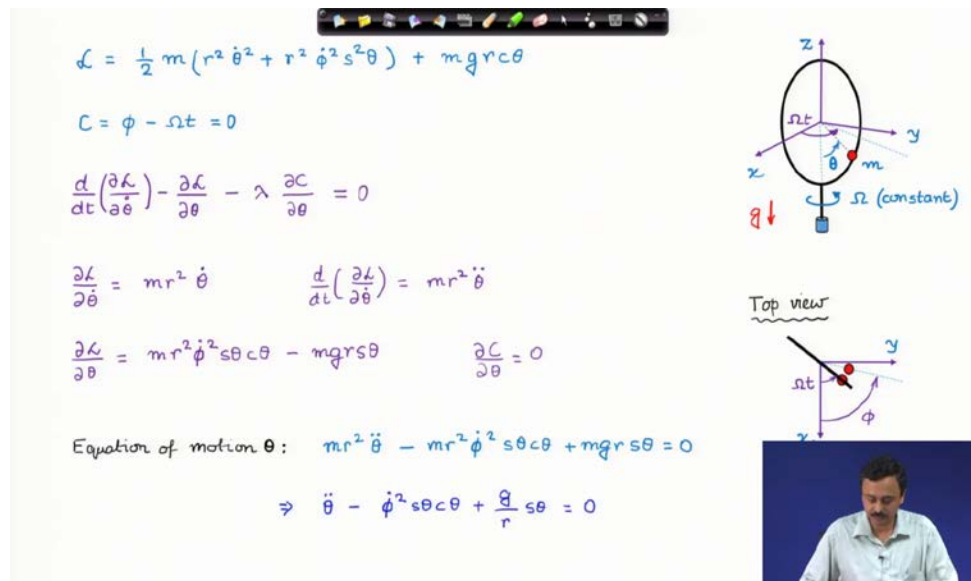
$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \frac{\partial C}{\partial \phi} = 1$$

Equation of motion ϕ : $m r^2 \sin^2 \theta \ddot{\phi} + 2 m r^2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta - \lambda = 0$





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$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) + m g r \cos \theta$$

$$C = \phi - \Omega t = 0$$

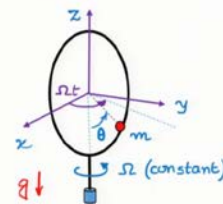
$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} = 0$$

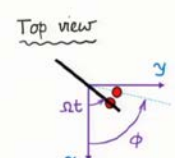
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m r^2 \ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \dot{\phi}^2 \sin \theta \cos \theta - m g r \sin \theta \quad \frac{\partial C}{\partial \theta} = 0$$

Equation of motion θ : $m r^2 \ddot{\theta} - m r^2 \dot{\phi}^2 \sin \theta \cos \theta + m g r \sin \theta = 0$

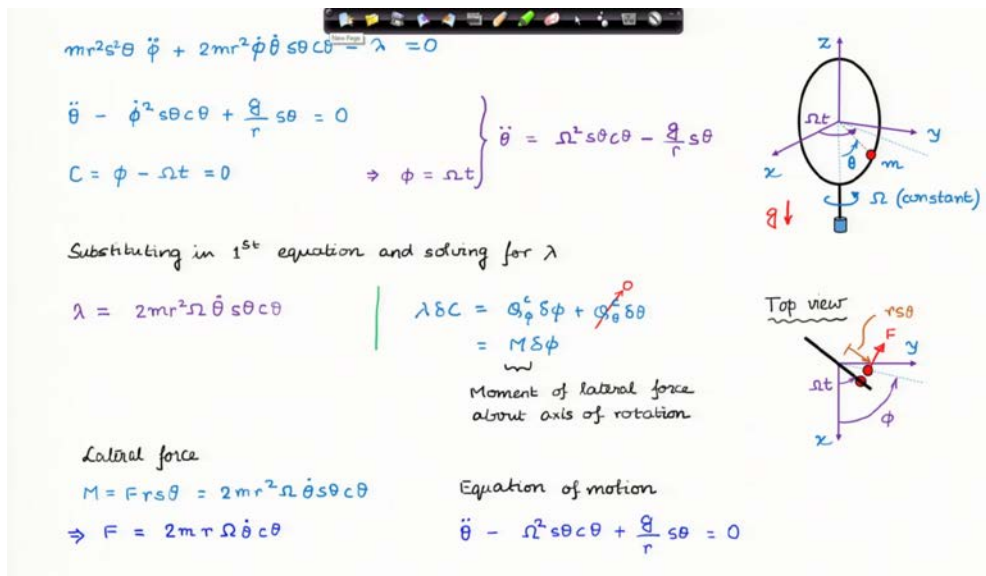
$$\Rightarrow \ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta + \frac{g}{r} \sin \theta = 0$$





The equations of motion are then derived as shown in the 2 slides above. These equations involve the Lagrange multiplier.

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$$m r^2 \sin^2 \theta \ddot{\phi} + 2 m r^2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta - \lambda = 0$$

$$\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta + \frac{g}{r} \sin \theta = 0$$

$$C = \phi - \Omega t = 0 \quad \Rightarrow \quad \phi = \Omega t$$

$$\ddot{\theta} = \Omega^2 \sin \theta \cos \theta - \frac{g}{r} \sin \theta$$

Substituting in 1st equation and solving for λ

$$\lambda = 2 m r^2 \Omega \dot{\theta} \sin \theta \cos \theta$$

$$\lambda \delta C = \mathcal{Q}_\phi \delta \phi + \mathcal{Q}_\theta \delta \theta$$

$$= M \delta \phi$$

Moment of lateral force about axis of rotation

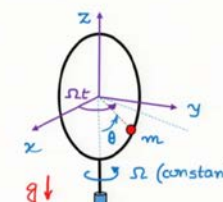
Lateral force

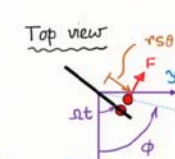
$$M = F r \sin \theta = 2 m r^2 \Omega \dot{\theta} \sin \theta \cos \theta$$

$$\Rightarrow F = 2 m r \Omega \dot{\theta} \sin \theta \cos \theta$$

Equation of motion

$$\ddot{\theta} - \Omega^2 \sin \theta \cos \theta + \frac{g}{r} \sin \theta = 0$$





The above slide shows how the Lagrange multiplier is used to determine the lateral force and obtain the equation of motion.

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Example 4:

Determination of contact friction in a rolling disc

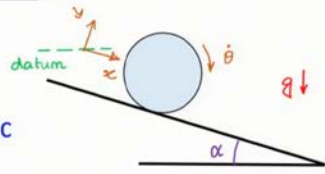

Physical coordinates: (x, θ)

Constraint: $x - r\theta = 0$

Lagrangian: $\mathcal{L} = T - V$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$V = -mgx \sin \alpha$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + mgx \sin \alpha$$



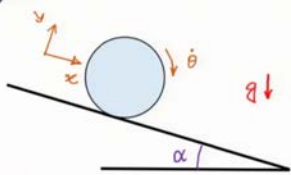

Now, we come to this rolling disc on an incline in a uniform gravitational field. We want to find out that friction force at the contact.

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$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + mgx \sin \alpha$$

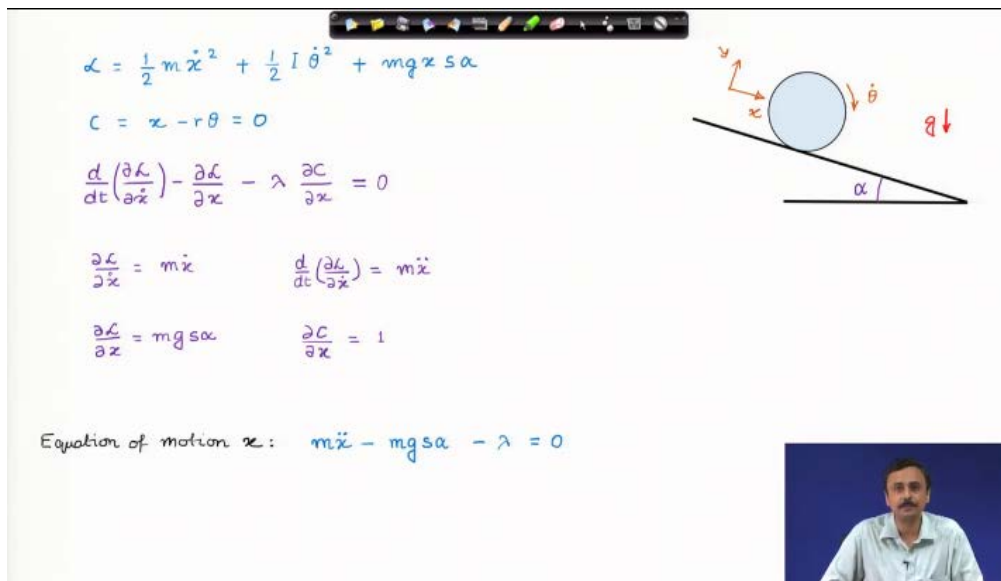
$$C = x - r\theta = 0$$

Governing equations:

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} - \lambda \frac{\partial C}{\partial x} &= 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} &= 0 \\ x - r\theta &= 0 \end{aligned} \right\} \text{Unknowns: } (x, \theta, \lambda)$$



In order to do this, we break the rolling constraint and introduce it as a mathematical condition.

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$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + mgx \sin \alpha$$

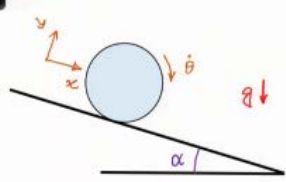

$$C = x - r\theta = 0$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} - \lambda \frac{\partial C}{\partial x} = 0$$

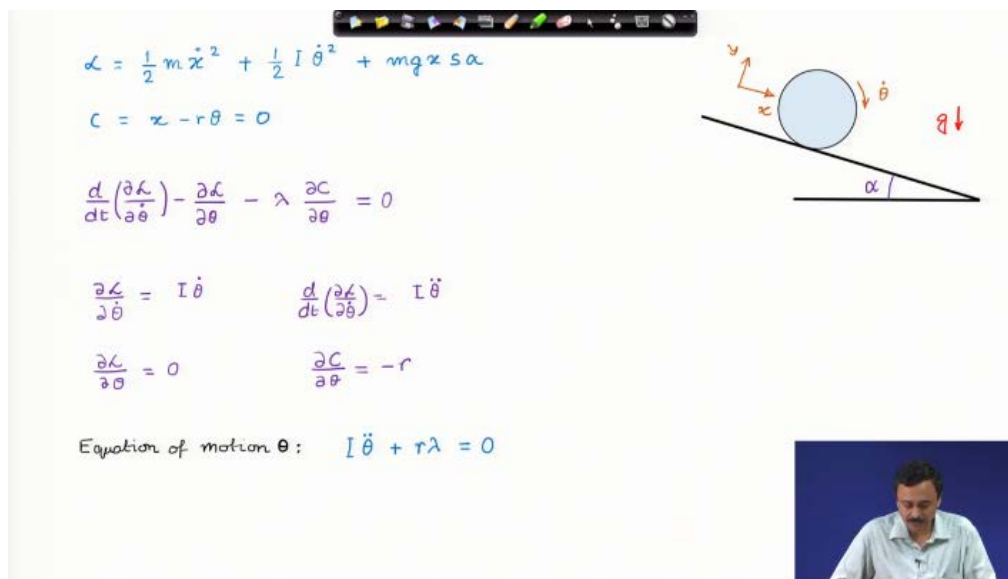
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = mg \sin \alpha \quad \frac{\partial C}{\partial x} = 1$$

Equation of motion x : $m\ddot{x} - mg \sin \alpha - \lambda = 0$

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Handwritten slide content for slide 21:19:

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + mgx \sin \alpha$$

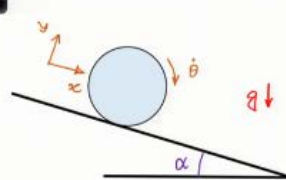

$$C = x - r\theta = 0$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = I\ddot{\theta}$$

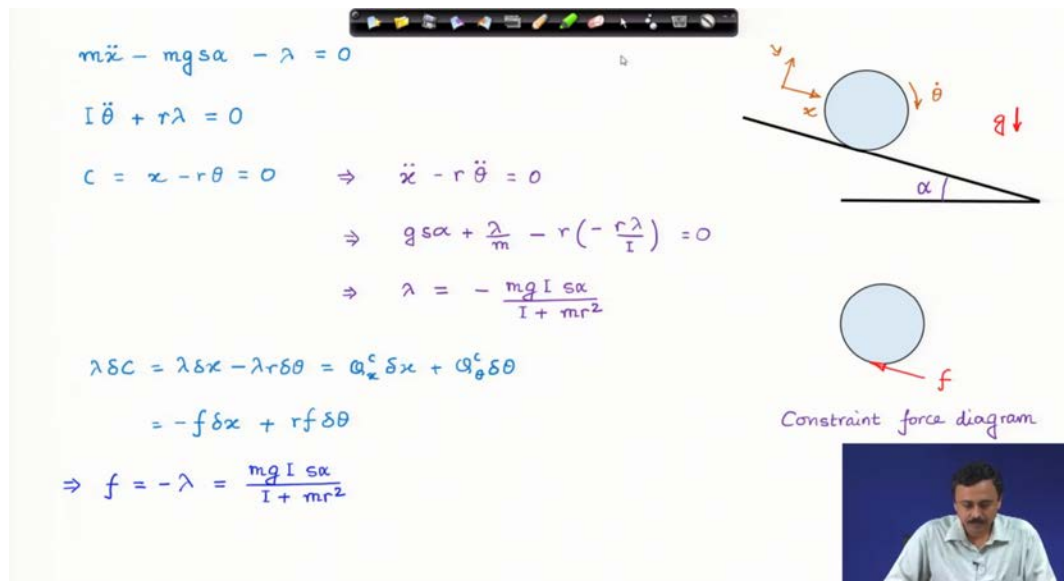
$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \frac{\partial C}{\partial \theta} = -r$$

Equation of motion θ : $I\ddot{\theta} + r\lambda = 0$

The equations of motion are derived in the above 2 slides. These equations involve the unknown Lagrange multiplier.

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Handwritten derivation of the friction force on a rolling sphere using the Lagrange multiplier method.

Equations shown:

$$m\ddot{x} - mg\sin\alpha - \lambda = 0$$

$$I\ddot{\theta} + r\lambda = 0$$

$$C = x - r\theta = 0 \Rightarrow \ddot{x} - r\ddot{\theta} = 0$$

$$\Rightarrow g\sin\alpha + \frac{\lambda}{m} - r\left(-\frac{r\lambda}{I}\right) = 0$$

$$\Rightarrow \lambda = -\frac{mgI\sin\alpha}{I + mr^2}$$

Virtual work calculation:

$$\lambda\delta C = \lambda\delta x - \lambda r\delta\theta = Q_x^c\delta x + Q_\theta^c\delta\theta$$

$$= -f\delta x + rf\delta\theta$$

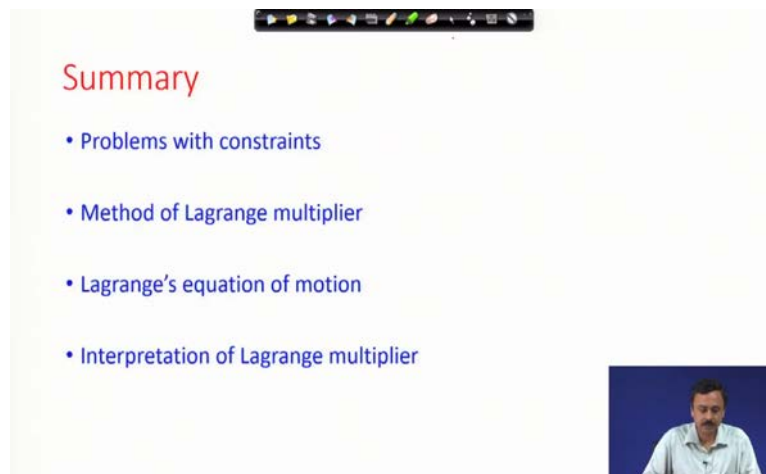
$$\Rightarrow f = -\lambda = \frac{mgI\sin\alpha}{I + mr^2}$$

Diagram of a sphere on an inclined plane with angle α . The sphere has radius r and moment of inertia I . The forces shown are gravity g acting vertically downwards, and a constraint force f acting horizontally to the right at the point of contact. The coordinate system (x, y) is shown with x along the incline and y perpendicular to it. The angle of rotation is θ .

Constraint force diagram

The above slide discusses the approach to determine the unknown friction force from the Lagrange multiplier.

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Summary

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier