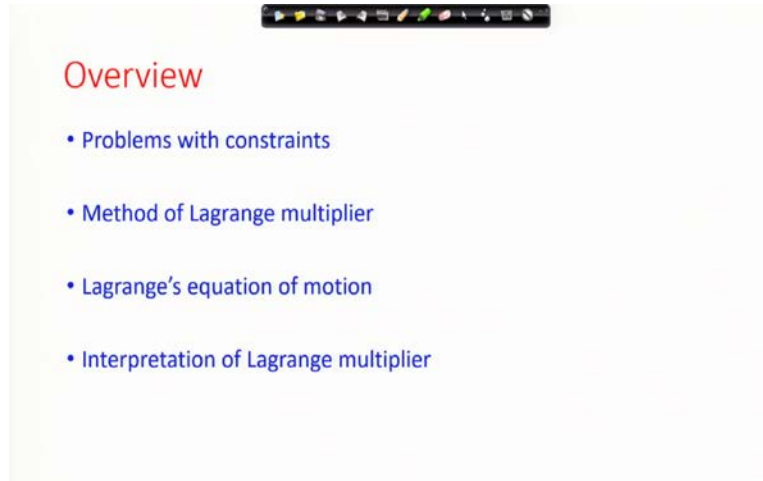


Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture – 53
System with Constraints - II

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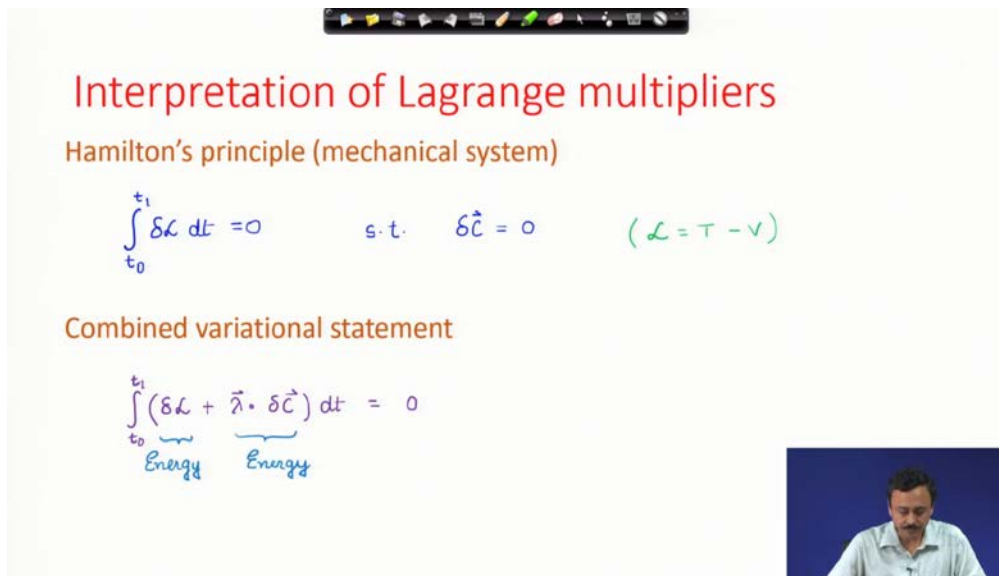


Overview

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier

In the last lecture, we had started discussing on systems with constraints. We will continue that in this lecture as well. We had looked at the method of Lagrange multiplier to deal with constraints in our Hamilton's principle and we have seen how to derive the equations of motion for systems with constraints. In this lecture, I am going to look at the interpretation of the Lagrange multiplier.

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Interpretation of Lagrange multipliers

Hamilton's principle (mechanical system)

$$\int_{t_0}^{t_1} \delta \mathcal{L} dt = 0 \quad \text{s.t.} \quad \delta \vec{C} = 0 \quad (\mathcal{L} = T - V)$$

Combined variational statement

$$\int_{t_0}^{t_1} \underbrace{(\delta \mathcal{L})}_{\text{Energy}} + \underbrace{\vec{\lambda} \cdot \delta \vec{C}}_{\text{Energy}} dt = 0$$


Let us go through a little bit of what we had discussed as presented in the slide above.


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Interpretation of Lagrange multipliers

Hamilton's principle (mechanical system)

$$\int_{t_0}^{t_1} (\underbrace{\delta \mathcal{L}}_{\text{Energy}} + \underbrace{\vec{\lambda} \cdot \delta \vec{C}}_{\text{Energy}}) dt = 0$$

$$\vec{\lambda} \cdot \delta \vec{C} = \sum_{j=1}^{m-n} \lambda_j \delta C_j = \sum_{j=1}^{m-n} \lambda_j \sum_{i=1}^m \frac{\partial C_j}{\partial r_i} \delta r_i \quad \left(\delta C_j = 0 \Rightarrow \frac{\partial C_j}{\partial r_1} \delta r_1 + \dots + \frac{\partial C_j}{\partial r_m} \delta r_m = 0 \right)$$

$$= \sum_{i=1}^m \left(\sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} \right) \delta r_i$$


All terms in the variational statement must have dimensions of energy, and from here we were going to interpret what is lambda.

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Interpretation of Lagrange multipliers

Hamilton's principle (mechanical system)

Virtual work done by generalized constraint forces

$$\underbrace{\vec{\lambda} \cdot \delta \vec{C}}_{\text{Energy}} = \sum_{i=1}^m \underbrace{\left(\sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} \right)}_{\substack{\text{Force in the } i^{\text{th}} \\ \text{coordinate} \\ \text{direction} \\ \text{Force/Moment} \\ \text{Generalized Force}}} \underbrace{\delta r_i}_{\substack{\text{Variational} \\ \text{displacement of} \\ i^{\text{th}} \text{ coordinate} \\ \text{Linear/Angular displacement}}} = \sum Q_i^c \delta r_i = \underbrace{\vec{Q}^c \cdot \delta \vec{r}}_{\substack{\text{Generalized constraint} \\ \text{forces}}}$$

- Generalized constraint forces maintain the constraints
- Determination of constraint force: define a constrained allow the constraint force to do virtual work



The above slide presents the interpretation of the Lagrange multipliers.

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Example 1:

Revisiting the simple pendulum: string tension

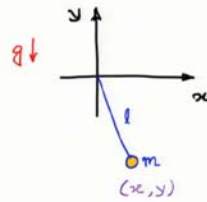
Physical coordinates: (x, y)

Constraint: $x^2 + y^2 = l^2$

Lagrangian: $\mathcal{L} = T - V$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$ $V = mgy$ (x -axis as datum)

$\Rightarrow \mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$



Let us revisit the pendulum problem that we had looked at in the last class.

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$$\begin{cases} -m\ddot{z} + \lambda x = 0 \\ -m\ddot{y} - mg + \lambda y = 0 \end{cases} \Rightarrow \lambda = -\frac{m}{l^2} [-gy + \dot{x}^2 + \dot{y}^2]$$

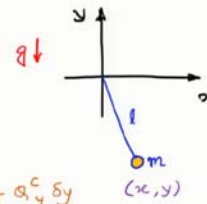

$x^2 + y^2 = l^2 \Rightarrow \delta C = x\delta x + y\delta y = 0$

Virtual work done by generalized constraint force

$$\lambda \delta C = -\frac{m}{l^2} [-gy + \dot{x}^2 + \dot{y}^2] (x\delta x + y\delta y) = \underbrace{\mathcal{Q}_x^c \delta x + \mathcal{Q}_y^c \delta y}_{\substack{\underbrace{(\mathcal{Q}_x^c \hat{i} + \mathcal{Q}_y^c \hat{j})}_{\vec{Q}^c} \cdot \underbrace{(\delta x \hat{i} + \delta y \hat{j})}_{\delta \vec{r}}}}$$

$$\Rightarrow \mathcal{Q}_x^c = -\frac{m}{l^2} x [-gy + \dot{x}^2 + \dot{y}^2]$$

$$\mathcal{Q}_y^c = -\frac{m}{l^2} y [-gy + \dot{x}^2 + \dot{y}^2]$$

In the above slide, we determine the generalized constraint forces for the pendulum.

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$\lambda \delta C = \vec{Q}^c \cdot \delta \vec{r}$

$Q_x^c = -\frac{m}{l^2} x [-g y + \dot{x}^2 + \dot{y}^2]$

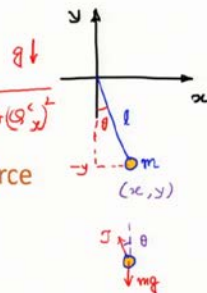

$Q_y^c = -\frac{m}{l^2} y [-g y + \dot{x}^2 + \dot{y}^2]$

$T = |\vec{Q}^c| = \sqrt{(Q_x^c)^2 + (Q_y^c)^2}$

Tension in the string maintains the constraint: **constraint force**

$\Rightarrow \vec{Q}^c = T(-\sin\theta \hat{i} + \cos\theta \hat{j}) = T\left(-\frac{x}{l} \hat{i} + \frac{y}{l} \hat{j}\right)$

$\Rightarrow T = \frac{m}{l} [-g y + \dot{x}^2 + \dot{y}^2] = \underbrace{mg\left(-\frac{y}{l}\right)}_{\text{projection of weight}} + \underbrace{m\left(\frac{\dot{x}^2 + \dot{y}^2}{l}\right)}_{\text{centrifugal force}}$

Using the generalized constraint forces, we determine the string tension.

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Example 2:

Pendulum on a driven cart: determination of driving force

Physical coordinates: (x, θ)

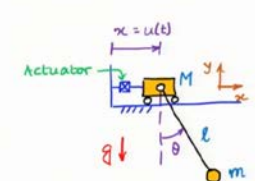

Constraint: $x - u(t) = 0$

Lagrangian: $\mathcal{L} = T - V$

$T = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta)$

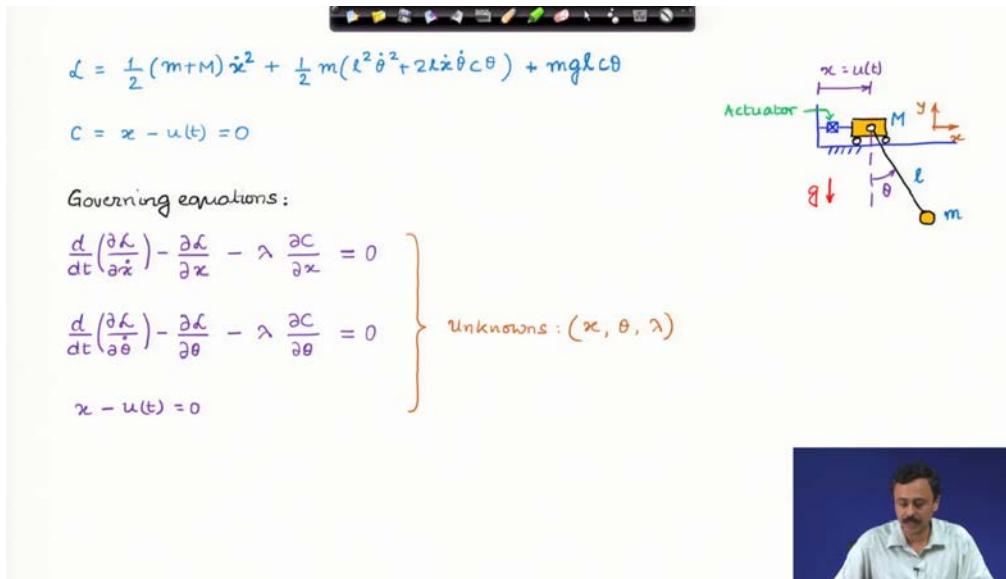
$V = -mgl\cos\theta$ (x -axis as datum)

$\Rightarrow \mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta$

Let us then proceed to our second example, pendulum on a driven cart. We would like to find out the force required to drive the cart for a desired motion $u(t)$ of the cart.

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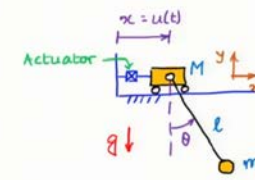



$$L = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (\dot{l}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta$$

$$C = x - u(t) = 0$$

Governing equations:

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} - \lambda \frac{\partial C}{\partial x} &= 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} &= 0 \\ x - u(t) &= 0 \end{aligned} \right\} \text{Unknowns: } (x, \theta, \lambda)$$

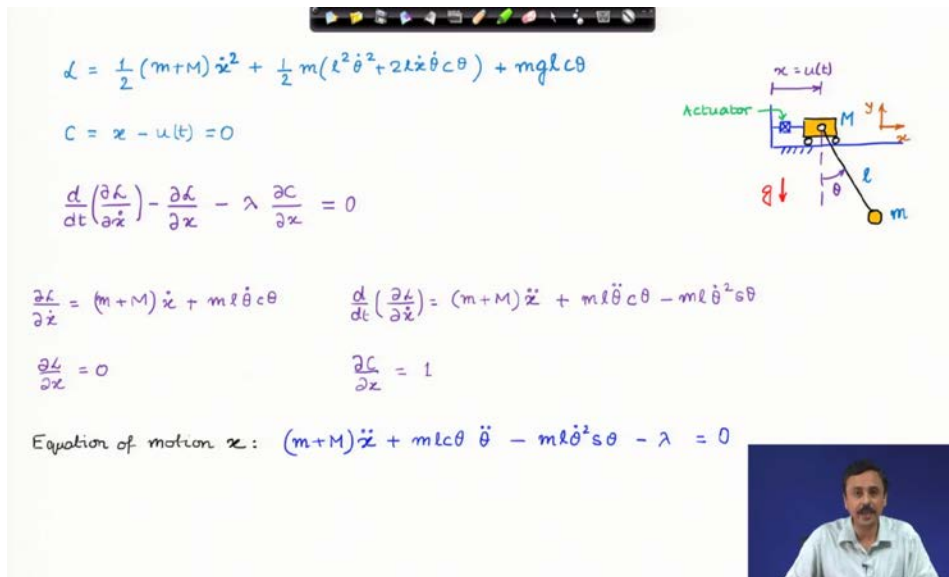




We pose it as a constrained problem as presented above.

The equations of motion are derived in the following 2 slides.

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$$L = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (\dot{l}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta$$

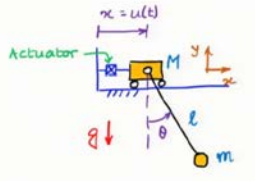
$$C = x - u(t) = 0$$


$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} - \lambda \frac{\partial C}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = (m+M) \dot{x} + m l \dot{\theta} \cos\theta \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = (m+M) \ddot{x} + m l \ddot{\theta} \cos\theta - m l \dot{\theta}^2 \sin\theta$$

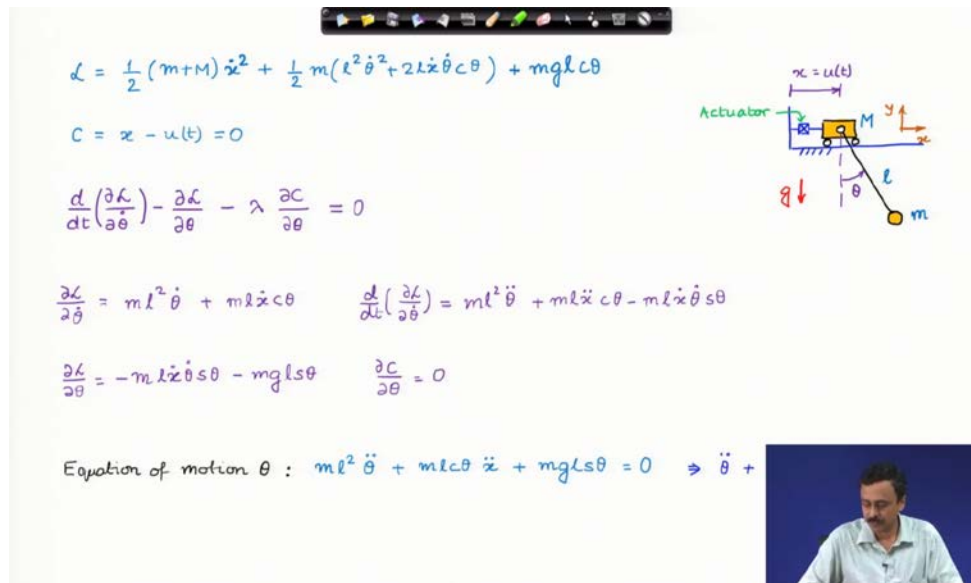
$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial C}{\partial x} = 1$$

Equation of motion x : $(m+M) \ddot{x} + m l \cos\theta \ddot{\theta} - m l \dot{\theta}^2 \sin\theta - \lambda = 0$





(Refer Slide Time: 18:10)



$$L = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(\ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta$$

$$C = x - u(t) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda \frac{\partial C}{\partial \theta} = 0$$

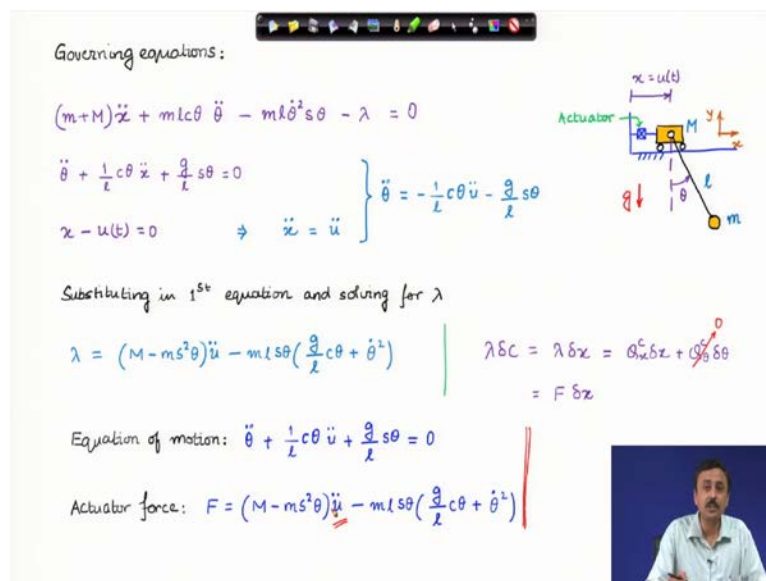
$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} + m\ell\dot{x}\cos\theta \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta} + m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta$$

$$\frac{\partial L}{\partial \theta} = -m\ell\dot{x}\dot{\theta}\sin\theta - mg\ell\sin\theta \quad \frac{\partial C}{\partial \theta} = 0$$

$$\text{Equation of motion } \theta : m\ell^2\ddot{\theta} + m\ell\cos\theta\ddot{x} + mg\ell\sin\theta = 0 \Rightarrow \ddot{\theta} +$$

Next, we determine and interpret the Lagrange multiplier.

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Governing equations:

$$(m+M)\ddot{x} + m\ell\cos\theta\ddot{\theta} - m\ell\dot{\theta}^2\sin\theta - \lambda = 0$$

$$\ddot{\theta} + \frac{1}{\ell}\cos\theta\ddot{x} + \frac{g}{\ell}\sin\theta = 0$$

$$x - u(t) = 0 \Rightarrow \ddot{x} = \ddot{u}$$

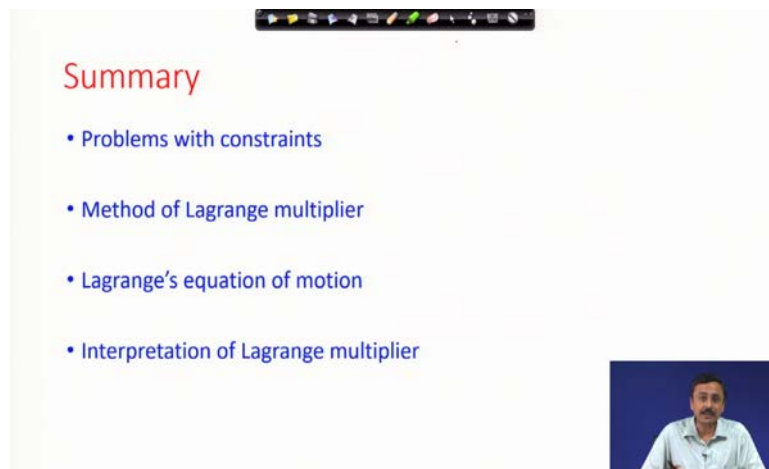
$$\left. \begin{aligned} \ddot{\theta} + \frac{1}{\ell}\cos\theta\ddot{x} + \frac{g}{\ell}\sin\theta &= 0 \\ \ddot{x} &= \ddot{u} \end{aligned} \right\} \Rightarrow \ddot{\theta} = -\frac{1}{\ell}\cos\theta\ddot{u} - \frac{g}{\ell}\sin\theta$$
 Substituting in 1st equation and solving for λ

$$\lambda = (M - m\sin^2\theta)\ddot{u} - m\ell\sin\theta\left(\frac{g}{\ell}\cos\theta + \dot{\theta}^2\right)$$

$$\lambda \delta C = \lambda \delta x = \theta_x^C \delta x + \theta_\theta^C \delta \theta = F \delta x$$
 Equation of motion: $\ddot{\theta} + \frac{1}{\ell}\cos\theta\ddot{u} + \frac{g}{\ell}\sin\theta = 0$
 Actuator force: $F = (M - m\sin^2\theta)\ddot{u} - m\ell\sin\theta\left(\frac{g}{\ell}\cos\theta + \dot{\theta}^2\right)$

Finally, we obtain the actuator force as calculated above.

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Summary

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion
- Interpretation of Lagrange multiplier