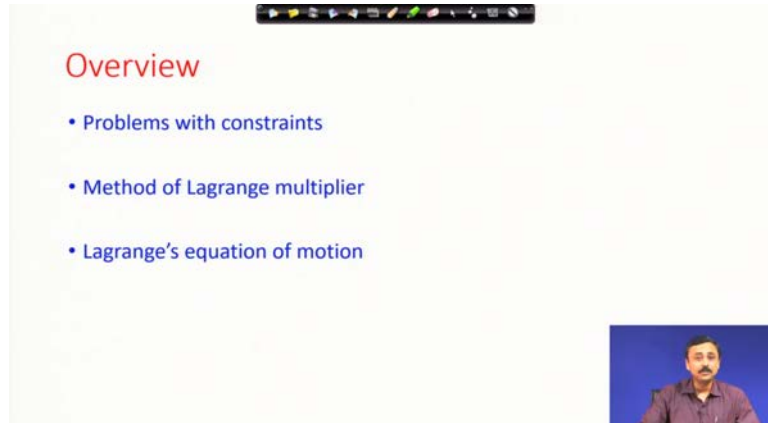


Advanced Dynamics
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Lecture – 52
System with Constraints – I

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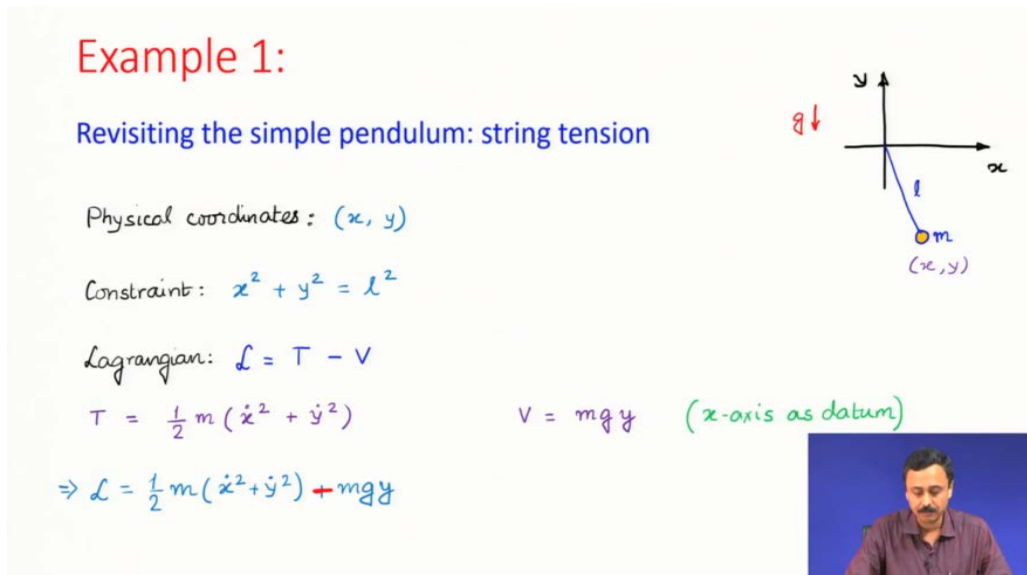
Overview

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion

A small video inset in the bottom right corner shows Prof. Anirvan Dasgupta speaking.

In this lecture, we are going to discuss systems with constraints. We are going to look at the method of Lagrange multipliers and derive the equation of motion in that scenario.

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Example 1:

Revisiting the simple pendulum: string tension

Physical coordinates: (x, y)

Constraint: $x^2 + y^2 = l^2$

Lagrangian: $\mathcal{L} = T - V$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$V = mgy$ (x -axis as datum)

$\Rightarrow \mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$

A diagram on the right shows a simple pendulum with a mass m at position (x, y) and string length l . A red arrow labeled g points downwards, indicating the direction of gravity.

A small video inset in the bottom right corner shows Prof. Anirvan Dasgupta speaking.

We revisit the example of a pendulum and treat it as a constrained problem as defined in the slide above.

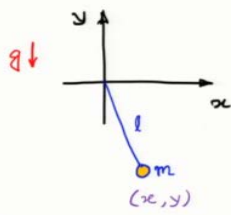

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Hamilton's principle $\delta \int_{t_0}^{t_1} \mathcal{L} dt = 0$ $\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$

$$\Rightarrow \int_{t_0}^{t_1} \delta \left[\frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy \right] dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} (m\dot{x} \delta \dot{x} + m\dot{y} \delta \dot{y} - mg \delta y) dt = 0$$

$$\Rightarrow \underbrace{m\dot{x} \delta x}_{=0} \Big|_{t_0}^{t_1} + \underbrace{m\dot{y} \delta y}_{=0} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} [-m\ddot{x} \delta x + (-m\ddot{y} - mg) \delta y] dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} [-m\ddot{x} \delta x + (-m\ddot{y} - mg) \delta y] dt = 0$$



We obtain the variation of the action as presented above. However, the variation of the two coordinates chosen are not independent due to the constraint, as noted in the slide below.

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$$\int_{t_0}^{t_1} [-m\ddot{x} \delta x + (-m\ddot{y} - mg) \delta y] dt = 0$$

Caution:
 δx and δy are not independent variations

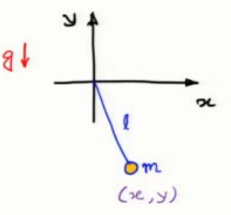

Constraint: $x^2 + y^2 = l^2$

$$\Rightarrow 2x \delta x + 2y \delta y = 0 \Rightarrow \boxed{x \delta x + y \delta y = 0}$$

Method of Lagrange multiplier:

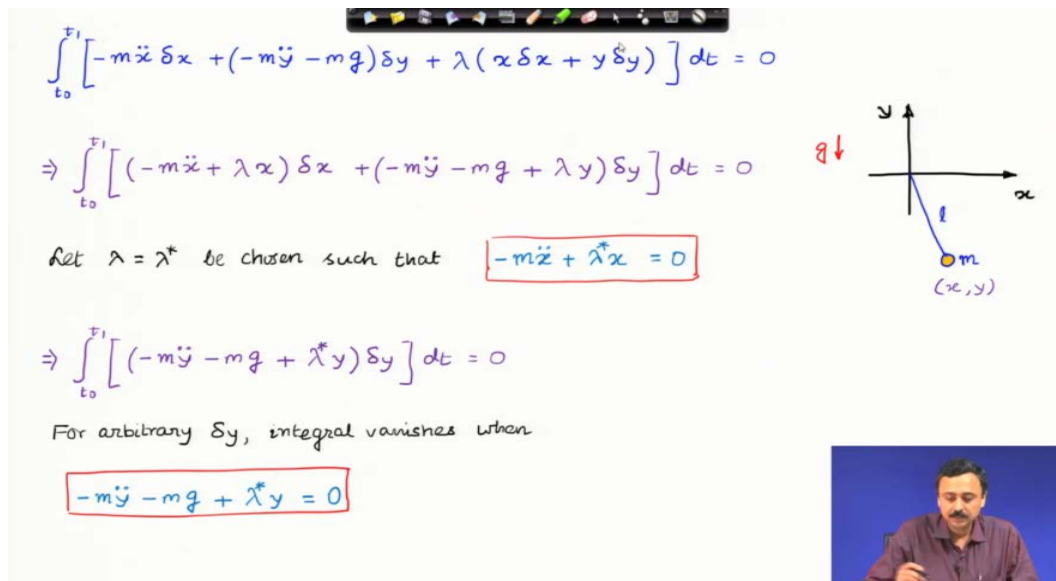
$$\int_{t_0}^{t_1} [-m\ddot{x} \delta x + (-m\ddot{y} - mg) \delta y + \lambda (x \delta x + y \delta y)] dt = 0$$

(Unknown)

We introduce the variation of the constraint using the Lagrange multiplier as demonstrated in the slide above.

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$$\int_{t_0}^{t_1} [-m\ddot{x} \delta x + (-m\ddot{y} - mg) \delta y + \lambda (x \delta x + y \delta y)] dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} [(-m\ddot{x} + \lambda x) \delta x + (-m\ddot{y} - mg + \lambda y) \delta y] dt = 0$$

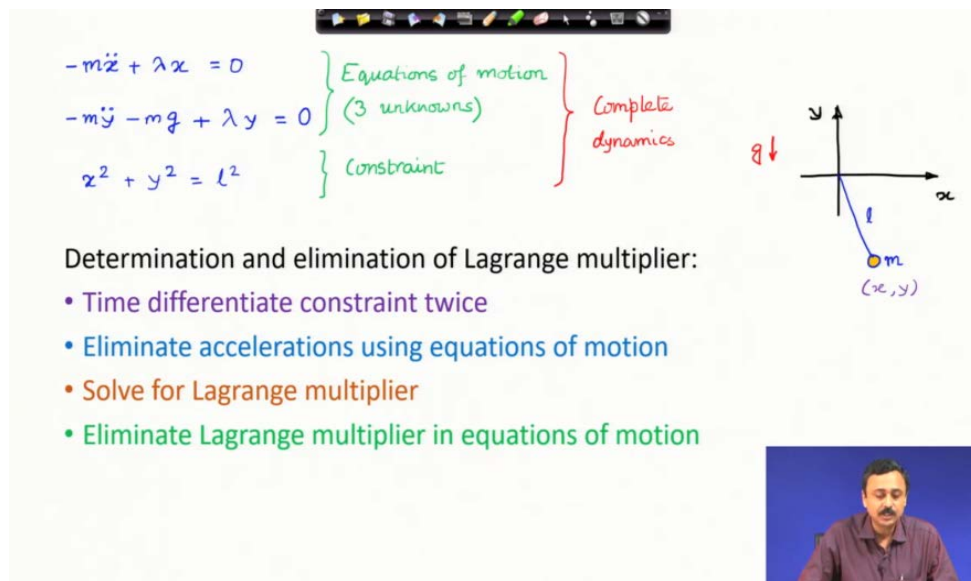
Let $\lambda = \lambda^*$ be chosen such that $-m\ddot{x} + \lambda^* x = 0$

$$\Rightarrow \int_{t_0}^{t_1} [(-m\ddot{y} - mg + \lambda^* y) \delta y] dt = 0$$

For arbitrary δy , integral vanishes when $-m\ddot{y} - mg + \lambda^* y = 0$

Using the Lagrange multiplier as an additional handle, we can now obtain the equations of motion as discussed in the slide above. However, the equations involve the unknown Lagrange multipliers.

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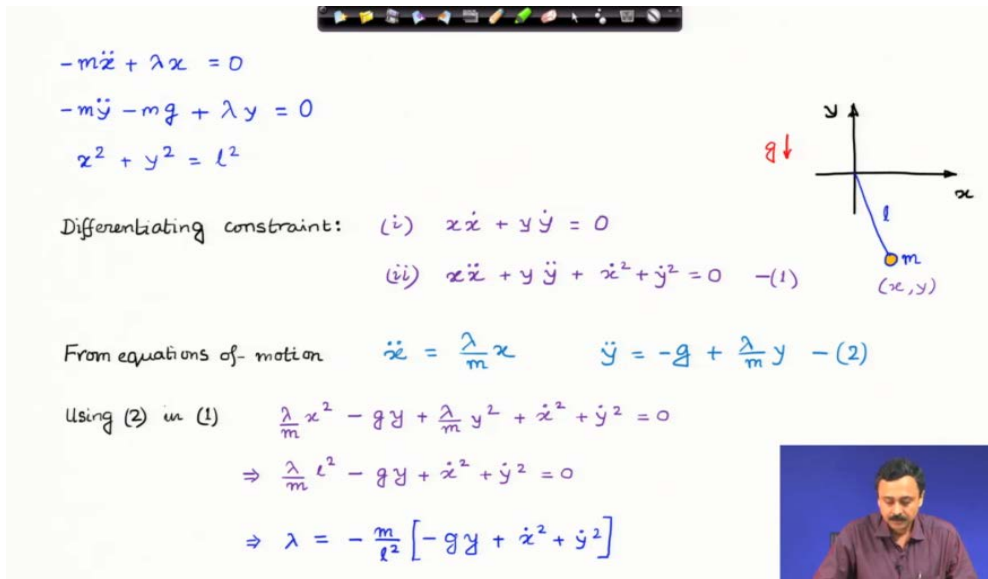
$$\begin{aligned} -m\ddot{x} + \lambda x &= 0 \\ -m\ddot{y} - mg + \lambda y &= 0 \\ x^2 + y^2 &= l^2 \end{aligned} \quad \left. \begin{array}{l} \text{Equations of motion} \\ (3 \text{ unknowns}) \\ \text{Constraint} \end{array} \right\} \text{Complete dynamics}$$

Determination and elimination of Lagrange multiplier:

- Time differentiate constraint twice
- Eliminate accelerations using equations of motion
- Solve for Lagrange multiplier
- Eliminate Lagrange multiplier in equations of motion

Finally, the steps to eliminate the Lagrange multiplier is presented in the slide above.

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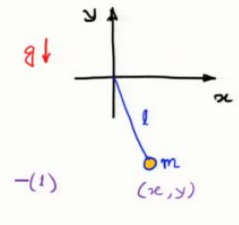



$-m\ddot{z} + \lambda x = 0$
 $-m\ddot{y} - mg + \lambda y = 0$
 $x^2 + y^2 = l^2$

Differentiating constraint: (i) $x\dot{x} + y\dot{y} = 0$
(ii) $x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0 \quad \text{---(1)}$

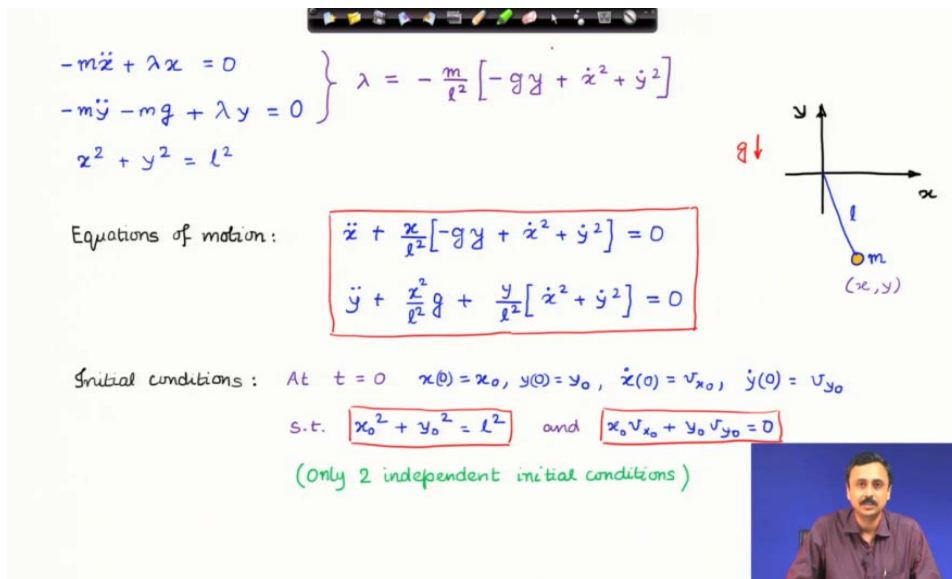
From equations of motion $\ddot{x} = \frac{\lambda}{m}x \quad \ddot{y} = -g + \frac{\lambda}{m}y \quad \text{---(2)}$

Using (2) in (1) $\frac{\lambda}{m}x^2 - gy + \frac{\lambda}{m}y^2 + \dot{x}^2 + \dot{y}^2 = 0$
 $\Rightarrow \frac{\lambda}{m}l^2 - gy + \dot{x}^2 + \dot{y}^2 = 0$
 $\Rightarrow \lambda = -\frac{m}{l^2}[-gy + \dot{x}^2 + \dot{y}^2]$

The Lagrange multiplier can be obtained as detailed above.

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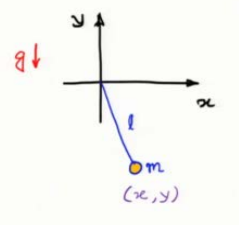



$-m\ddot{z} + \lambda x = 0$
 $-m\ddot{y} - mg + \lambda y = 0$
 $x^2 + y^2 = l^2$

Equations of motion: $\lambda = -\frac{m}{l^2}[-gy + \dot{x}^2 + \dot{y}^2]$

$\ddot{x} + \frac{x}{l^2}[-gy + \dot{x}^2 + \dot{y}^2] = 0$
 $\ddot{y} + \frac{y}{l^2}g + \frac{y}{l^2}[-\dot{x}^2 + \dot{y}^2] = 0$

Initial conditions: At $t = 0$ $x(0) = x_0, y(0) = y_0, \dot{x}(0) = v_{x_0}, \dot{y}(0) = v_{y_0}$
s.t. $x_0^2 + y_0^2 = l^2$ and $x_0 v_{x_0} + y_0 v_{y_0} = 0$
(Only 2 independent initial conditions)

Next, we eliminate the Lagrange multiplier as shown above. The initial conditions are also discussed.

(Refer Slide Time: 16:04)

Lagrange's equation with constraints

Coordinates: $\vec{r} = (r_1, \dots, r_m)$

Constraints: $C_1(\vec{r}, t) = 0, \dots, C_{m-n}(\vec{r}, t) = 0$ (Holonomic, rheonomic)

Hamilton's principle: $\delta A = 0 \Rightarrow \int_{t_0}^{t_1} \delta \mathcal{L}(\vec{r}, \dot{\vec{r}}, t) dt = 0$

$$\Rightarrow \int_{t_0}^{t_1} \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \right) + \frac{\partial \mathcal{L}}{\partial \vec{r}} \right] \cdot \delta \vec{r} dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \left[\left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \right) + \frac{\partial \mathcal{L}}{\partial r_1} \right) \delta r_1 + \dots + \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_m} \right) + \frac{\partial \mathcal{L}}{\partial r_m} \right) \delta r_m \right] dt = 0$$

Now, we generalize this approach as discussed in the following 3 slides.

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Lagrange's equation with constraints


$$\int_{t_0}^{t_1} \left[\left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \right) + \frac{\partial \mathcal{L}}{\partial r_1} \right) \delta r_1 + \dots + \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_m} \right) + \frac{\partial \mathcal{L}}{\partial r_m} \right) \delta r_m \right] dt = 0$$

Subject to $\delta C_1 = 0 \Rightarrow \left(\frac{\partial C_1}{\partial r_1} \delta r_1 + \dots + \frac{\partial C_1}{\partial r_m} \delta r_m = 0 \right) \lambda_1$

\vdots

$\delta C_{m-n} = 0 \Rightarrow \left(\frac{\partial C_{m-n}}{\partial r_1} \delta r_1 + \dots + \frac{\partial C_{m-n}}{\partial r_m} \delta r_m = 0 \right) \lambda_{m-n}$

Using $(m-n)$ Lagrange multipliers

$$\int_{t_0}^{t_1} \left[\sum_{i=1}^m \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} \right) \delta r_i + \sum_{j=1}^{m-n} \lambda_j \sum_{i=1}^m \frac{\partial C_j}{\partial r_i} \delta r_i \right] dt = 0$$


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Lagrange's equation with constraints

$$\int_{t_0}^{t_f} \left[\sum_{i=1}^m \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} \right) \delta r_i + \sum_{j=1}^{m-n} \lambda_j \sum_{i=1}^m \frac{\partial C_j}{\partial r_i} \delta r_i \right] dt = 0$$

$$\Rightarrow \int_{t_0}^{t_f} \sum_{i=1}^m \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} \right] \delta r_i dt = 0 \quad \text{--- (1)}$$

Choose λ_j $j = 1, \dots, m-n$ s.t. $-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} = 0 \quad i = 1, \dots, m-n$
(m-n equations)

Then integral (1) vanishes for arbitrary δr_i $i = m-n+1, \dots, m$ when

$$-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) + \frac{\partial \mathcal{L}}{\partial r_i} + \sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} = 0 \quad i = m-n+1, \dots, m$$
(n equations)



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Lagrange's equation with constraints

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) - \frac{\partial \mathcal{L}}{\partial r_i} - \sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} &= 0 \quad i = 1, \dots, m-n \\ &\text{(m-n equations)} \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i} \right) - \frac{\partial \mathcal{L}}{\partial r_i} - \sum_{j=1}^{m-n} \lambda_j \frac{\partial C_j}{\partial r_i} &= 0 \quad i = m-n+1, \dots, m \\ &\text{(n equations)} \end{aligned} \right\} \begin{aligned} &m \text{ equations of motion} \\ &(2m-n \text{ unknowns}) \end{aligned}$$

$$\left. \begin{aligned} C_1(\vec{r}, t) &= 0 \\ &\vdots \\ C_{m-n}(\vec{r}, t) &= 0 \end{aligned} \right\} m-n \text{ constraints}$$

Total $2m-n$ equations in $2m-n$ unknowns




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Lagrange's equation with constraints

Determination and elimination of Lagrange multiplier:

- Time differentiate constraint twice
- Eliminate accelerations using equations of motion
- Solve for Lagrange multiplier
- Eliminate Lagrange multiplier in equations of motion



Finally, we follow the steps that we have discussed for determination and elimination of the Lagrange multipliers, as presented above.

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Lagrange's equation with constraints

Setting up initial conditions:

- Use the constraint equations and its derivative

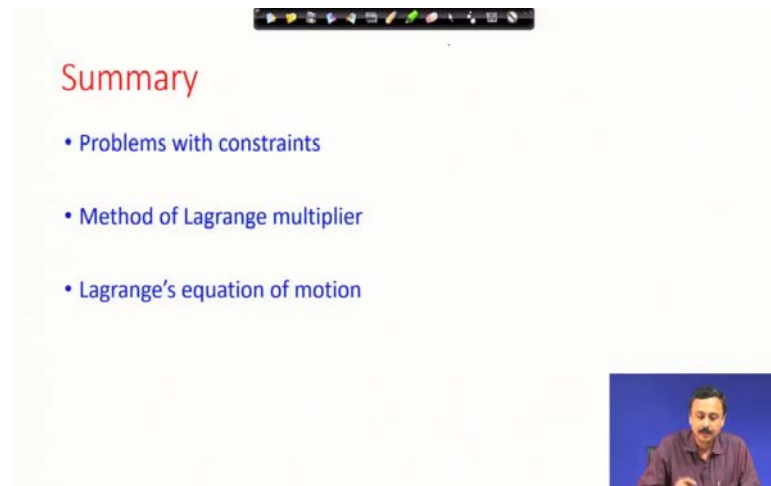
$$\underbrace{\vec{r}(0) = \vec{r}_0, \quad \dot{\vec{r}}(0) = \vec{v}_0}_{2m} \quad \text{s.t.} \quad \underbrace{\vec{C}(\vec{r}_0, 0) = 0}_{m-n} \quad \text{and} \quad \underbrace{\left. \frac{\partial \vec{C}}{\partial \vec{r}_i} \right|_{\vec{r}=\vec{r}_0} \cdot \vec{v}_0 = 0}_{m-n}$$

$\vec{r}_0'(t)$

$$\Rightarrow 2m - (2m - 2n) = 2n \text{ independent initial conditions}$$


The above slide discusses setting up of the initial conditions for the problem.

(Refer Slide Time: 29:28)



The image shows a presentation slide with a light yellow background. At the top center, there is a black toolbar with various icons. The word "Summary" is written in red at the top left. Below it, there is a bulleted list of three items in blue text. In the bottom right corner, there is a small video inset showing a man with a mustache, wearing a purple shirt, speaking against a blue background.

Summary

- Problems with constraints
- Method of Lagrange multiplier
- Lagrange's equation of motion