

Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture – 51
Hamilton's Principle and Lagrange's Equation of Motion – IV

(Refer Slide Time: 00:16)

Overview

- Hamilton-Ostrogradski principle for dynamical paths
- Lagrange's equation of motion

In this lecture, we are going to continue with our discussions on Hamilton-Ostrogradski principle which we had started off in our previous lecture. We are going to discuss further on Hamilton-Ostrogradski principle for dynamical paths and look at deriving the equations with external forces.

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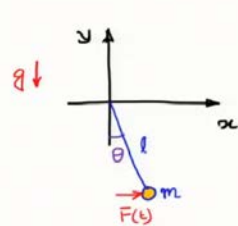
Example 1:

Simple pendulum with horizontal forcing

Generalized coordinate: θ

Lagrange's Equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q$

$\mathcal{L} = T - V \quad T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2 \quad V = -mgl \cos \theta \quad (x\text{-axis as datum})$



$\Rightarrow \mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta} \quad \frac{\partial \mathcal{L}}{\partial \theta} = -mgl \sin \theta$

Generalized force: $\delta W = \hat{F} \cdot (\delta x \hat{i} + \delta y \hat{j}) = F \delta x = F \delta (l \sin \theta) = Fl \cos \theta \delta \theta = Q \delta \theta$

$\Rightarrow Q = Fl \cos \theta$

We start with an example of a simple pendulum with horizontal forcing as shown in the above slide. The generalized force on the system is derived above.

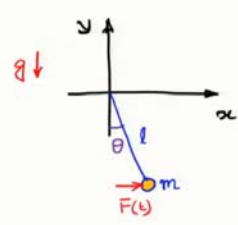
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Lagrange's Equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q$

$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta}$ $\frac{\partial \mathcal{L}}{\partial \theta} = -mgl \sin \theta$ $Q = F(t) l \cos \theta$

Equation of motion:

$$\Rightarrow m l^2 \ddot{\theta} + mgl \sin \theta = F(t) l \cos \theta$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l} \sin \theta = \frac{F(t)}{ml} \cos \theta$$


Finally, we obtain the equation as presented above.

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Example 2:

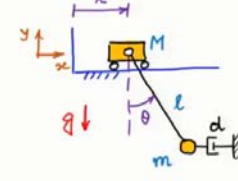

Horizontally damped simple pendulum on a cart

Generalized coordinates: (θ, x)

Lagrange's Equations of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_{\theta}$

$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = Q_x$

$\mathcal{L} = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (l^2 \dot{\theta}^2 + 2l\dot{x}\dot{\theta} \cos \theta) + mgl \cos \theta$ (x -axis)

We move to the next example. Here we have a pendulum on a cart and the bob of the pendulum is connected to a viscous dashpot.

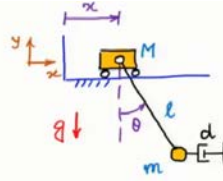

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Generalized forces :

$$\vec{r}_m = x\hat{i} + l(\sin\theta\hat{i} - \cos\theta\hat{j}) \Rightarrow \vec{v}_m = (\dot{x} + l\dot{\theta}\cos\theta)\hat{i} + l\dot{\theta}\sin\theta\hat{j}$$

$$\delta W = \underbrace{-d(\dot{x} + l\dot{\theta}\cos\theta)\hat{i}}_{F_d = -d\dot{v}_{mx}} \cdot \delta \vec{r}_m \quad \delta \vec{r}_m = (\delta x + l\delta\theta\cos\theta)\hat{i} + l\delta\theta\sin\theta\hat{j}$$

$$\Rightarrow \delta W = -d(\dot{x} + l\dot{\theta}\cos\theta)(\delta x + l\cos\theta\delta\theta) = Q_x\delta x + Q_\theta\delta\theta$$

$$\Rightarrow Q_x = -d(\dot{x} + l\dot{\theta}\cos\theta) \quad Q_\theta = -d(\dot{x} + l\dot{\theta}\cos\theta)l\cos\theta$$



The generalized forces are obtained in the above slide.

(Refer Slide Time: 10:23)

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta$$

$$Q_x = -d(\dot{x} + l\dot{\theta}\cos\theta) \quad Q_\theta = -d(\dot{x} + l\dot{\theta}\cos\theta)l\cos\theta$$

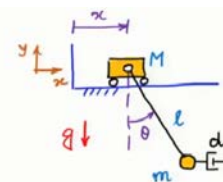

Equation of motion θ : $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} + m l \ddot{x} \cos\theta - m l \dot{x} \dot{\theta} \sin\theta \quad \frac{\partial \mathcal{L}}{\partial \theta} = -m l \dot{x} \dot{\theta} \sin\theta - m g l \sin\theta$$

$$\Rightarrow \underline{m l^2 \ddot{\theta} + m l \cos\theta \ddot{x} + m g l \sin\theta = -d(\dot{x} + l\dot{\theta}\cos\theta)l\cos\theta}$$

Equation of motion x : $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_x$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = (m+M)\ddot{x} + m l \ddot{\theta} \cos\theta - m l \dot{\theta}^2 \sin\theta \quad \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\Rightarrow \underline{(m+M)\ddot{x} + m l \cos\theta \ddot{\theta} - m l \dot{\theta}^2 \sin\theta = -d(\dot{x} + l\dot{\theta}\cos\theta)}$$



The equations of motion are derived in the above slide.

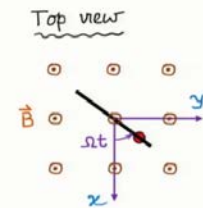
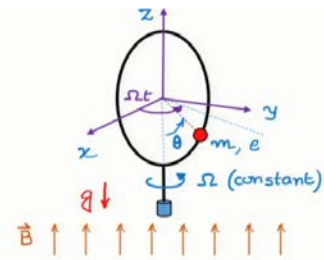
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Example 3:

Charged bead sliding on a uniformly rotating circular hoop in vertical uniform magnetic field

Generalized coordinate: θ

Lagrange's equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_{\theta}$



Next, we consider a charged bead sliding on a hoop which is rotating, as shown in the slide above.

(Refer Slide Time: 12:21)

$$\mathcal{L} = T - V$$

$$T = \frac{1}{2} m v_m^2$$

$$\vec{r}_m = r \sin \theta \cos \Omega t \hat{i} + r \sin \theta \sin \Omega t \hat{j} - r \cos \theta \hat{k}$$

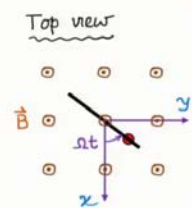
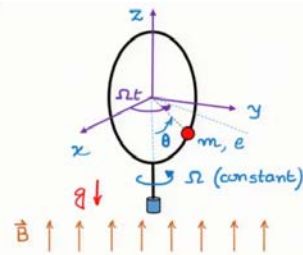
$$\vec{v}_m = (r \dot{\theta} \cos \theta \cos \Omega t - r \Omega \sin \theta \sin \Omega t) \hat{i} + (r \dot{\theta} \cos \theta \sin \Omega t + r \Omega \sin \theta \cos \Omega t) \hat{j} + r \dot{\theta} \sin \theta \hat{k}$$

$$\Rightarrow v_m^2 = r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta$$

$$\Rightarrow T = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta)$$

$$V = -mg r \cos \theta$$

$$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta) + mg r \cos \theta$$



The Lagrangian is presented above.

(Refer Slide Time: 13:17)

Generalized force:

$$\delta W = \vec{F} \cdot \delta \vec{r}_m$$

$$\vec{F} = e \vec{v}_m \times \vec{B} \quad \vec{B} = B \hat{k}$$

$$\vec{r}_m = r \sin \theta \cos \Omega t \hat{i} + r \sin \theta \sin \Omega t \hat{j} - r \cos \theta \hat{k}$$

$$\vec{v}_m = (r \dot{\theta} \cos \Omega t - r \Omega \sin \theta \sin \Omega t) \hat{i} + (r \dot{\theta} \sin \Omega t + r \Omega \sin \theta \cos \Omega t) \hat{j} + r \dot{\theta} \sin \theta \hat{k}$$

$$\Rightarrow \vec{F} = -e(r \dot{\theta} \cos \Omega t - r \Omega \sin \theta \sin \Omega t) \hat{j} + e(r \dot{\theta} \sin \Omega t + r \Omega \sin \theta \cos \Omega t) \hat{i}$$

$$\delta \vec{r}_m = (r \cos \theta \cos \Omega t \hat{i} + r \cos \theta \sin \Omega t \hat{j} + r \sin \theta \hat{k}) \delta \theta$$

$$\Rightarrow \delta W = e r^2 \Omega \sin \theta \cos \theta \delta \theta = Q_\theta \delta \theta \quad \Rightarrow Q_\theta = e r^2 \Omega \sin \theta \cos \theta$$

Top view

Using the Lorentz force, we determine the virtual work and obtain the generalized force on the bead.

(Refer Slide Time: 16:57)

Equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta$

$$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta) + m g r \cos \theta$$

$$Q_\theta = e r^2 \Omega \sin \theta \cos \theta$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m r^2 \ddot{\theta} \quad \frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \Omega^2 \sin \theta \cos \theta - m g r \sin \theta$$

$$\Rightarrow m r^2 \ddot{\theta} - m r^2 \Omega^2 \sin \theta \cos \theta + m g r \sin \theta = e r^2 \Omega \sin \theta \cos \theta$$

Top view

The equation of motion of the bead are then obtained as presented above.

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Summary

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion